The deal with division by zero

The first commandment of mathematics is: *"Thy shall not divide by zero. Ever..."*

The expressions $\frac{3}{0}$ and $\frac{0}{3}$ look very similar, and yet they are very different. How can we remember which is which?

Method 1. If you don't remember which is which, just ask your calculator. If we punch in $\frac{0}{3}$, the calculator gives us the answer 0. But if we type in $\frac{3}{0}$, we get some kind of an error message Some calculator will say "domain error", others will say "error: division by zero"; but no calculator will let you go on thinking that there is a meaningful result to the operation $\frac{3}{0}$.

Method 2. You should know this one too, in case you are not allowed to use a calculator on an exam. The trick is to know that division is defined in terms of multiplication. Consider the easy division $\frac{10}{2}$.

$$\frac{10}{2} = 5 \quad \text{because the multiplication backwards works, i.e.} \quad 10 = 2 \cdot 5$$

$$\frac{0}{3} = \square$$

What could we enter into the empty box so that the multiplication backwards will work? Clearly the answer is zero. Indeed,

$$\frac{0}{3} = 0 \quad \text{because} \quad 3 \cdot 0 = 0$$

Consider now

$$\frac{3}{0} = \square$$

What could we enter into the empty box so that the multiplication backwards will work? No matter what number we would write, zero times it will be zero.

$$3 \neq 0 \cdot \square \text{ no matter what value we write in the box}$$

because $0 \cdot \square = 0$. Thus, it is impossible to enter a number into the box to make the multiplication backwards work.

Any time we are instructed to divide by zero, we need to write the final answer: undefined.

Example 1 $\dfrac{3-2(-2)}{3-2^2+1} = \dfrac{3-(-4)}{3-4+1} = \dfrac{7}{-1+1} = \dfrac{7}{0} = \text{undefined}$