Notes on Area

Definition 1 *The **area** of a geometric object is a measurement of its surface.*

While we could think about perimeter as a fencing problem, area can be thought of as follows. Suppose a geometric object is a room. How much rug do we need to buy to cover the entire room?

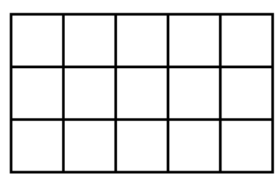
Understanding and remembering the area formulas are probably easier if we know how they were derived.

Definition 2 *The area of a* 1 *ft by* 1 *ft square is defined to be* 1 ft^2. *(Similar definitions can be formulated with* mi^2, cm^2, in^2, *etc.) The area of an object, in* ft^2, *is the number of* 1 *ft by* 1 *ft square needed to cover the object, cutting and pasting allowed.*

Area is not a length like perimeter. Area is always measured in ft^2, mi^2, cm^2, in^2, etc., and is usually denoted by A.

Theorem 1 *The area of a rectangle with sides* a *and* b *is* $A = ab$.

Proof. Consider rectangle with sides 3 m and 5 m. The area of this rectangle will be as many m^2 as many 1 m by 1 m square is needed to cover it. Once we place this grid on the rectangle, it is easy to see, just how many squares we need.



We used exactly 15 squares to cover the rectangle, and so the area is 15 m^2. ■

Mathematicians also proved that the formula is true even if the sides of the rectangle are not integers.

It is interesting to see that we basically counted how many meter^2 we have. A computation for the area, including the units is slightly different. Instead of counting meter^2, we literally multiply meter by meter.

$$A = ab = 3 \text{ m } (5 \text{ m}) = 15 \text{ m}^2$$

Area computation will always yield the right unit.

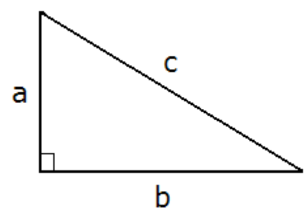
Example 1 *Find the area of a rectangle with sides* 13 *in and* 7 *in.*

Solution 1 *We apply the formula.*

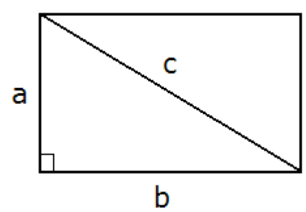
$$A = ab = 13 \ in \ (7 \ in) = 91 \ in^2$$

The following few area formulas will demonstrate how mathematicians work: we will use already proven results to come up with new formulas.

Theorem 2 *The area of a right triangle with sides a, b, and c (where c is the longest side) is* $A = \frac{ab}{2}$.



Proof. It is very easy to see that every right triangle is basically half of a rectangle. We can make a rectangle if we use two identical right triangles as shown on the picture below.



Since the rectangle's sides are a and b, its area is ab. The area of our triangle must be half of it. Thus

$$A = \frac{ab}{2}$$

■

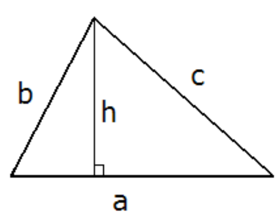
Notice that we never used the length of the longest side, c.

Example 2 *Find the area of the right triangle with sides 5 mi, 12 mi, and 13 mi long.*

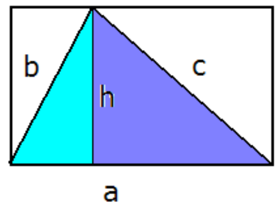
Solution 2 *It is important to know that the largest side, 13 mi long, is not needed for this computation. With labeling $a = 5$ mi and $b = 12$ mi, the area is*

$$A = \frac{ab}{2} = \frac{5\ mi\ (12\ mi)}{2} = 30\ mi^2$$

Theorem 3 *The area of a general triangle with sides a, b, c and height h as shown on the picture below is* $A = \frac{ah}{2}$.

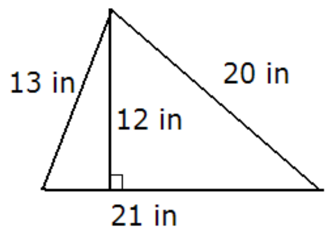


Proof. As before, we will use a previously obtained result. Since the general triangle no longer has a right angle, we create it by drawing in the altitude or height belonging to the side a. Now we split our triangle into two right triangles, and each of them is half of a rectangle.



Our triangle makes up for half of a rectangle, with sides a and h. Thus $A = \dfrac{ah}{2}$ ■

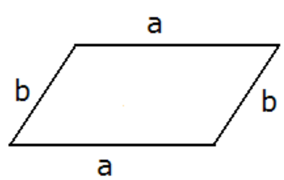
Example 3 *Find the area of the triangle shown on the picture below.*



Solution 3 *It is important to notice that we will not need all the information given. We apply the formula.*

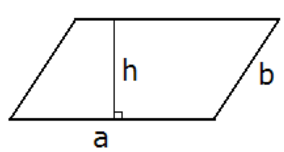
$$A = \frac{ah}{2} = \frac{21\ in\ (12\ in)}{2} = 126\ in^2$$

Definition 3 *A parallelogram is a four sided polygon with two pairs of parallel sides.*

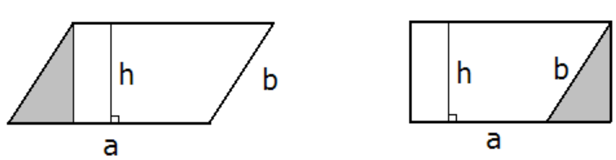


It is a proven fact that the opposite sides of a parallelogram are of equal length.

Theorem 4 *The are of a parallelogram with sides a, b and height h belonging to a is $A = ah$.*

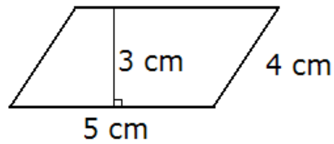


Proof. We will use (surprise, surprise!) a previously proven result. If we cut off a triangle and paste it back as show on the picture below, we obtain a rectangle.



Thus the area of the parallelogram equals to the area of a rectangle with sides a and h. ■

Example 4 *Find the area of the parallelogram shown on the picture below.*

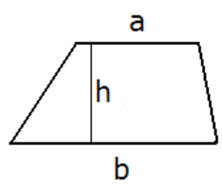


Solution 4 *We apply the formula.*

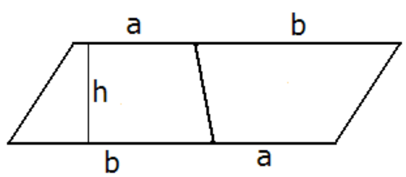
$$A = ah = 5\ cm\ (3\ cm) = 15\ cm^2$$

Definition 4 *A **trapezoid** is a four sided polygon with one pair of parallel sides.*

Theorem 5 *The area of a trapezoid, with sides and height labeled as on the picture below, is* $A = \frac{a+b}{2}h$.

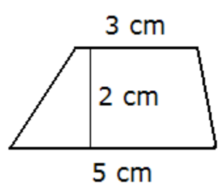


Proof. If we use two identical trapezoids, we can make a parallelogram as shown on the picture below.



We already know that the area of this parallelogram is $A = (a+b)\,h$. Since our trapezoid is exactly half of the parallelogram, its area is $A = \frac{(a+b)\,h}{2}$. ■

Example 5 *Find the area of the trapezoid shown on the picture below.*

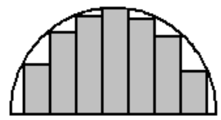


Solution 5 *We apply the formula*

$$A = \frac{a+b}{2}h = \frac{3\ cm\ + 5\ cm}{2}(2\ cm) = \frac{8\ cm}{2}(2\ cm) = 8\ cm^2$$

Theorem 6 *The area of a circle with radius* r *is* $A = \pi r^2$.

At this level of mathematics, we do not have the tools necessary to prove this. The proof actually uses (again) the formula for the area of a rectangle. Instead of a circle, we use just half of it, and then we multiply the result by 2. The basic idea is to first find an estimation for the area, using rectangles.



We CAN compute the grey area since it is composed of rectangles. But this is a crude underestimation of the actual area. The trick is that the more rectangles we use, the more accurate the approximation becomes. Using more and more rectangles to estimate the area of the semi-circle, the approximation becomes better and better.

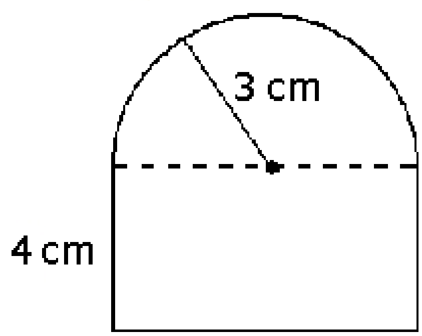
Calculus offers tools to find a unique number these approximations approach. This number is the area.

Example 6 *Find the area of a circle of radius* 7 *ft.*

Solution 6 *We apply the formula.*

$$A = \pi r^2 = \pi \left(7 \ ft\right)^2 = 153.94 \ ft^2$$

Example 7 *Find the area of the figure shown on the picture below.*



Solution 7 *The area is the sum of that of a rectangle and a semi-circle. We apply the appropriate formulas and then add the results. The bottom side of the rectangle is* 6 *cm since we can lay exactly two radii on it.*

$$\begin{aligned} A_{rectangle} &= ab = 4 \ cm \ (6 \ cm) = 24 \ cm^2 \\ A_{semicircle} &= \frac{\pi r^2}{2} = \frac{\pi \left(3 \ cm\right)^2}{2} = 14.137 \ cm^2 \\ A &= A_{rectangle} + A_{semicircle} = 24 \ cm^2 + 14.137 \ cm^2 = 38.137 \ cm^2 \end{aligned}$$