

1. (Hilbert) You are the manager of an infinitely long motel shown on the picture below. Your office is the room labeled M. Every room can accommodate one person, and so happens, every room is occupied. Every room is equipped with a speaker, so you can talk to every guest from your office. One rainy night a person arrives and needs a room. Make room for the new guest! (Nobody can board in the manager's office, and no room can be occupied by more than one person. Yet, it is possible to free up a room. How?)

M	1	2	3	4	5	...
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2. Let  $p$  be a prime number. Prove that with the exception of the first and last entry, every number in the  $p$ th row of the Pascal triangle is divisible by  $p$ .
3. There is a very difficult track in a mountain area, where the world record of driving a lap is 15 miles per hour average velocity. A race car driver announces that he intends to beat this record and run the course with an average velocity of 30 miles per hour. As he is driving, his progress is monitored. At the half of the track, his average velocity was 15 miles per hour. In spite of this, he arrives at the finish just a few seconds too late to have an average velocity of 30 miles per hour. His funeral was three days later. How did he die?
4. The **Fibonacci sequence**,  $F_n$  is 1, 1, 2, 3, 5, 8, 13, 21, ... Every element is obtained by adding the previous two elements. The recursive definition is

$$\begin{aligned} F_1 &= 1 \\ F_2 &= 1 \\ F_{n+2} &= F_n + F_{n+1} \quad \text{for all natural number } n, \text{ with } n \geq 1 \end{aligned}$$

Consider now another sequence,  $q_n$  that is formed by taking the ratios of consecutive elements of the Fibonacci sequence.

$$q_n = \frac{F_n}{F_{n+1}} \quad \text{for all natural number } n.$$

The first few elements of  $q_n$  are  $\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \dots$ . If we compute the decimal presentation of the first few elements of  $q_n$ , we observe a very interesting behavior, the numbers seem to approach one number. Find this number.

5. A trick: We start with any three-digit number. We first create a six-digit number by writing down the three-digit number twice. For example, if we start with 118, our six-digit number is 118118. No matter what number we started with, this number will be divisible by 7. Divide the six-digit number by 7. This number is divisible by 11. Divide. The result is divisible by 13. Divide. What do you notice? Can you explain it?