1. Simplify each of the following expressions. Use exact values.

   (a) $\cos(90° - x) =$ $\sin x$

   (b) $\sin\left(\alpha - \frac{\pi}{6}\right) = \sin\alpha\cos\left(\frac{\pi}{6}\right) - \cos\alpha\sin\left(\frac{\pi}{6}\right) =$ $\frac{\sqrt{3}}{2}\sin\alpha - \frac{1}{2}\cos\alpha$

2. Find the exact value for each of the following expression.

   (a) $\dfrac{\tan 72° + \tan 78°}{1 - \tan 72°\tan 78°} = \tan(72° + 78°) = \tan 150° =$ $-\frac{\sqrt{3}}{3}$

   (b) $\sin 15° = \sin(45° - 30°) =$ $\dfrac{\sqrt{6} - \sqrt{2}}{4}$

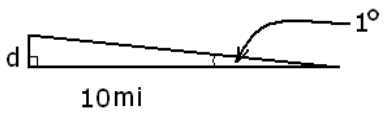
   (c) $\csc^2\left(-\frac{3\pi}{4}\right) - \cot^2\left(-\frac{3\pi}{4}\right) + \sec^2\left(-\frac{3\pi}{4}\right) - \tan^2\left(-\frac{3\pi}{4}\right) =$ 2

   (d) $\dfrac{\cos 21°}{\sin 69°} = \dfrac{\cos 21°}{\cos(90° - 69°)} = \dfrac{\cos 21°}{\cos 21°} =$ 1

   (e) $\cos 165° = \cos(120° + 45°) =$ $-\dfrac{\sqrt{6} + \sqrt{2}}{4}$

3. There is a 10 mile pathway toward the ocean, perpendicular to the beach. Two runners agree to race on this path. The first runner runs the path with perfect accuracy. The second runner misses the perpendicular path by 1° to the right. How far apart are the endpoints of the runners' paths? 0.174 55 mi

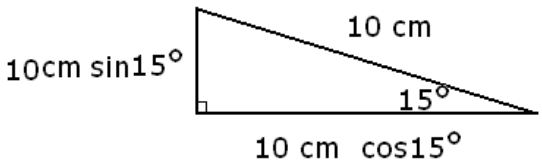
   Solution: The right path is perpendicular to the beach. Thus



$$\tan 1° = \frac{d}{10\,\text{mi}} \implies d = (10\,\text{mi})\tan 1° \cong 0.174\,55\,\text{mi}$$

4. A 12-sided polygon is inscribed in a circle whose radius is 10 cm. Find the area of the polygon.

   Solution: We need to find the area of the right triangloe shown below and then multiply it by 24. 300 cm^2



$$A = 24\,\frac{(10\ \text{cm})\sin 15°\,(10\ \text{cm})\cos 15°}{2} = 1200\sin 15°\cos 15°\ \text{cm}^2 = (2\sin 15°\cos 15°)\,600\ \text{cm}^2$$
$$= (\sin 30°)\,600\ \text{cm}^2 = \left(\frac{1}{2}\right)600\ \text{cm}^2 = 300\ \text{cm}^2$$

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5. Find the exact value of all solutions of each of the following equations. Present your answer in radians.

   (a) $\sin 2x = -\dfrac{1}{\sqrt{2}}$ $x = -\dfrac{\pi}{8} + k\pi, \quad x = -\dfrac{3\pi}{8} + k\pi, \quad k \in \mathbb{Z}$

   (b) $\cos 3x = \dfrac{\sqrt{3}}{2}$ $x = \pm\dfrac{\pi}{18} + \dfrac{2k\pi}{3}, \quad k \in \mathbb{Z}$

   (c) $\cos 2x = 2\sin^2 x - 2$ $\pm\dfrac{\pi}{3} + 2k\pi, \quad \pm\dfrac{2\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$

   (d) $2\cos 4x + 1 = 0$ $\pm\dfrac{\pi}{6} + \dfrac{k\pi}{2}, \quad k \in \mathbb{Z}$

   (e) $\cos^2 x + 4\cos x + 3 = 0$ $\pi + 2k\pi, \quad k \in \mathbb{Z}$

6. Consider the identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. Use this identity to find the formula for $\sin(\alpha-\beta)$.
   Solution: Substitute $-\beta$ into the formula.

$$\sin(\alpha-\beta) = \sin(\alpha+(-\beta)) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$

   Since we know that $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$, we now have

$$\sin(\alpha-\beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

7. Consider the identity $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. Use this identity to find the formula for $\cos(\alpha-\beta)$.
   Solution: Substitute $-\beta$ into the formula.

$$\cos(\alpha-\beta) = \cos(\alpha+(-\beta)) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$$

   Since we know that $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$, we now have

$$\cos(\alpha-\beta) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

8. The tangent formulas.

   (a) Use the identities $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ and $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ to derive the formula for $\tan(\alpha+\beta)$.
   Solution:

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

   We now divide both numerators and denominators by $\cos\alpha\cos\beta$.

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} = \frac{\dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\end{aligned}$$

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(b) Use the identity for $\tan(\alpha+\beta)$ to derive the formula for $\tan(\alpha-\beta)$.
Solution: We will use the fact that $\tan(-\beta) = -\tan\beta$

$$\tan(\alpha-\beta) = \tan(\alpha+(-\beta)) = \frac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\tan(-\beta)} = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

9. Suppose that $\cos\alpha = \frac{3}{5}$ and α is not in the first quadrant; $\sin\beta = -\frac{12}{13}$ and β is not in the fourth quadrant. Find the exact values for each of the following.

(a) $\sin 2\alpha = -\frac{24}{25}$
Solution: Since $\cos\alpha$ is positive and α is not in the first quadrant, α must be in the fourth quadrant, making $\sin\alpha$ negative. Then

$$\sin\alpha = -\sqrt{1-\cos^2\alpha} = -\sqrt{1-\left(\frac{3}{5}\right)^2} = -\sqrt{1-\frac{9}{25}} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

and $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{5}\cdot\frac{5}{3} = -\frac{4}{3}$.

Similarly, β is in the third quadrant and so $\sin\beta = -\frac{12}{13}$ $\cos\beta = -\frac{5}{13}$ $\tan\beta = \frac{12}{5}$. We are now ready to compute $\sin 2\alpha$.

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) = -\frac{24}{25}$$

(b) $\cos(\alpha-\beta) = \frac{33}{65}$
Solution:

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta = \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right)+\left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) = \frac{-15+48}{65} = \frac{33}{65}$$

(c) $\tan 2\beta = -\frac{120}{119}$
Solution:

$$\tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2\left(\frac{12}{5}\right)}{1-\left(\frac{12}{5}\right)^2} = \frac{\frac{24}{5}}{1-\frac{144}{25}} = \frac{\frac{24}{5}}{-\frac{119}{25}} = \frac{24}{5}\cdot\frac{-25}{119} = -\frac{120}{119}$$

(d) $\tan(\alpha+\beta) = \frac{16}{63}$
Solution:

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{-\frac{4}{3}+\frac{12}{5}}{1-\left(-\frac{4}{3}\right)\left(\frac{12}{5}\right)} = \frac{\frac{16}{15}}{1-\left(-\frac{16}{5}\right)} = \frac{\frac{16}{15}}{\frac{21}{5}} = \frac{16}{15}\cdot\frac{5}{21} = \frac{16}{63}$$

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10. What is the largest value that the expression $\sin x + \cos x$ takes? $\sqrt{2}$
Solution: Let us first find the largest value of the expression $(\sin x + \cos x)^2$.

$$(\sin x + \cos x)^2 = \sin x^2 + \cos x^2 + 2\sin x \cos x = 1 + 2\sin x \cos x = 1 + \sin 2x$$

$1 + \sin 2x$ clearly has a largest value, when $\sin 2x = 1$, the largest possible value for a sine value. If the largest possible value for $(\sin x + \cos x)^2$ is 2, then the largest possible value for $\sin x + \cos x$ is $\sqrt{2}$. If we solve the equation $\sin 2x = 1$, we get

$$\begin{aligned} \sin 2x &= 1 \\ 2x &= 90^\circ + k \cdot 360^\circ \\ x &= 45^\circ + k \cdot 180^\circ \end{aligned}$$

If $x = 45^\circ$, then $\sin x + \cos x = \sin 45^\circ + \cos 45^\circ = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2} = \sqrt{2}$, and so this possible largest value is achieved.

11. Prove each of the following identities.

(a) $2\dfrac{\cos^3 x + \sin^3 x}{\cos x + \sin x} = 2 - \sin 2x$
Solution: Recall that the sum of two cubes factors: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$.

$$\begin{aligned} \text{LHS} &= 2\frac{\cos^3 x + \sin^3 x}{\cos x + \sin x} = 2\frac{(\cos x + \sin x)\left(\cos^2 x - \cos x \sin x + \sin^2 x\right)}{\cos x + \sin x} \\ &= 2\left(\underbrace{\cos^2 x + \sin^2 x}_{1} - \cos x \sin x\right) = 2(1 - \sin x \cos x) = 2 - 2\sin x \cos x \\ &= 2 - \sin 2x = \text{RHS} \end{aligned}$$

(b) $\cos 2x + \sin 2x + 2\sin^2 x = (\sin x + \cos x)^2$
Solution:

$$\begin{aligned} \text{LHS} &= \cos 2x + \sin 2x + 2\sin^2 x = \underbrace{\left(\cos^2 x - \sin^2 x\right)}_{\cos 2x} + \underbrace{(2\sin x \cos x)}_{\sin 2x} + 2\sin^2 x \\ &= \cos^2 x + 2\sin x \cos x + \sin^2 x = (\sin x + \cos x)^2 = \text{RHS} \end{aligned}$$

(c) $\cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$
Solution: This is similar to the sum-formulas with tangent. This time, we divide by $\sin x \sin y$.

$$\begin{aligned} \text{LHS} &= \cot(x+y) = \frac{\cos(x+y)}{\sin(x+y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y} \\ &= \frac{\dfrac{\cos x \cos y - \sin x \sin y}{\sin x \sin y}}{\dfrac{\sin x \cos y + \cos x \sin y}{\sin x \sin y}} = \frac{\dfrac{\cos x \cos y}{\sin x \sin y} - \dfrac{\sin x \sin y}{\sin x \sin y}}{\dfrac{\sin x \cos y}{\sin x \sin y} + \dfrac{\cos x \sin y}{\sin x \sin y}} \\ &= \frac{\cot x \cot y - 1}{\cot y + \cot x} = \text{RHS} \end{aligned}$$

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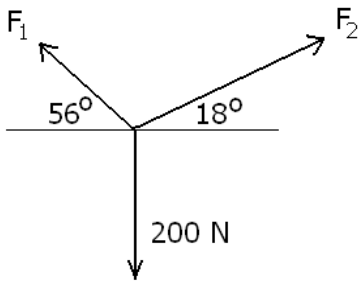
12. An object is held by two ropes as shown on the picture. Find the forces in the rope if the object weighs 200 N, and it is at rest. $F_1 = 197.876\,7$ N $\quad 116.345\,6$ N



Solution: we have two unknown quantites, F_1 and F_2. We will have to come up with two equations and then solve the system of equations. The first equation will express the fact that the object is not accelerating to the left or to the right, thus the horizontal component of all forces balance out each other.

$$F_1 \cos 56^\circ = F_2 \cos 18^\circ$$

The other equation will express the fact that the object is not accelerating upward or downward, thus the vertical component of all forces balance out each other. The vertical components of F_1 and F_2 are acting upward, and the weight of the object acts downward. The equation is then

$$200\,\text{N} = F_1 \sin 56^\circ + F_2 \sin 18^\circ$$

This is the system to solve. We will use substitution. Let us solve for F_2 in the first equation:

$$F_1 \cos 56^\circ = F_2 \cos 18^\circ \quad \Longrightarrow \quad F_2 = \frac{F_1 \cos 56^\circ}{\cos 18^\circ} = F_1 \frac{\cos 56^\circ}{\cos 18^\circ}$$

We substitute this into the second equation and solve for F_1:

$$\begin{aligned}
200\,\text{N} &= F_1 \sin 56^\circ + F_2 \sin 18^\circ \\
200\,\text{N} &= F_1 \sin 56^\circ + \left(F_1 \frac{\cos 56^\circ}{\cos 18^\circ}\right) \sin 18^\circ \\
200\,\text{N} &= F_1 \left(\sin 56^\circ + \frac{\cos 56^\circ \sin 18^\circ}{\cos 18^\circ}\right) \\
200\,\text{N} &= F_1 \left(\sin 56^\circ + \cos 56^\circ \tan 18^\circ\right) \\
\frac{200\,\text{N}}{\sin 56^\circ + \cos 56^\circ \tan 18^\circ} &= F_1
\end{aligned}$$

We can now also find F_2, since $F_2 = F_1 \dfrac{\cos 56^\circ}{\cos 18^\circ}$

$$F_2 = F_1 \frac{\cos 56^\circ}{\cos 18^\circ} = \left(\frac{200\,\text{N}}{\sin 56^\circ + \cos 56^\circ \tan 18^\circ}\right) \frac{\cos 56^\circ}{\cos 18^\circ}$$

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and so $F_1 = \dfrac{200\,\text{N}}{\sin 56^\circ + \cos 56^\circ \tan 18^\circ} \cong 197.876\,7\,\text{N}$ and

Note: Notice that the trigonometric expression for F_2 can be significantly simplified.

$$\begin{aligned} F_2 &= \left(\frac{200\,\text{N}}{\sin 56^\circ + \cos 56^\circ \tan 18^\circ}\right)\frac{\cos 56^\circ}{\cos 18^\circ} = \frac{200\,\text{N}\cos 56^\circ}{(\sin 56^\circ + \cos 56^\circ \tan 18^\circ)\cos 18^\circ} \\ &= \frac{200\,\text{N}\cos 56^\circ}{(\sin 56^\circ \cos 18^\circ + \cos 56^\circ \tan 18^\circ \cos 18^\circ)} = \frac{200\,\text{N}\cos 56^\circ}{(\sin 56^\circ \cos 18^\circ + \cos 56^\circ \sin 18^\circ)} \\ &= \frac{200\,\text{N}\cos 56^\circ}{\sin(56^\circ + 18^\circ)} = \frac{200\,\text{N}\cos 56^\circ}{\sin 74^\circ} = \frac{\cos 56^\circ}{\sin 74^\circ}200\,\text{N} \end{aligned}$$

By symmetry, we can suspect that the expression for F_1 can also be simplified.

$$\begin{aligned} F_1 &= \frac{200\,\text{N}}{\sin 56^\circ + \cos 56^\circ \tan 18^\circ} = \frac{200\,\text{N}}{\sin 56^\circ + \cos 56^\circ \dfrac{\sin 18^\circ}{\cos 18^\circ}} \\ &= \frac{200\,\text{N}}{\sin 56^\circ + \cos 56^\circ \dfrac{\sin 18^\circ}{\cos 18^\circ}} \cdot \frac{\cos 18^\circ}{\cos 18^\circ} = \frac{200\,\text{N}\cos 18^\circ}{(\sin 56^\circ \cos 18^\circ + \cos 56^\circ \sin 18^\circ)} \\ &= \frac{200\,\text{N}\cos 18^\circ}{\sin(56^\circ + 18^\circ)} = \frac{\cos 18^\circ}{\sin 74^\circ}200\,\text{N} \end{aligned}$$

Thus, the expressions we found for F_1 and F_2 have very nice, simplified forms:

$$F_1 = \frac{\cos 18^\circ}{\sin 74^\circ}200\,\text{N} \quad \text{and} \quad F_2 = \frac{\cos 56^\circ}{\sin 74^\circ}200\,\text{N}$$

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