1. Simplify each of the following.

a) $\cos\left(\frac{13\pi}{6}\right)$ b) $\csc\left(\frac{13\pi}{4}\right)$ c) $\cos(-105^\circ)$ d) $\tan 75^\circ$ e) $\tan 22.5^\circ$

f) $\dfrac{\csc 30^\circ + \csc 60^\circ + \csc 90^\circ}{\sec 0^\circ + \sec 30^\circ + \sec 60^\circ}$ h) $\cos 48^\circ \cos 3^\circ + \sin 48^\circ \sin 3^\circ$ i) $\dfrac{\tan\frac{2\pi}{15} + \tan\frac{\pi}{5}}{1 - \left(\tan\frac{2\pi}{15}\right)\left(\tan\frac{\pi}{5}\right)}$

g) $\sec^2\beta - \tan^2\beta$

j) $\cos\left(-\frac{13\pi}{3}\right) + \sin\frac{13\pi}{4} - \tan\frac{9\pi}{4} + \cos\left(\frac{13\pi}{3}\right)$

k) $\sin 120^\circ \cos 150^\circ - 3\tan 330^\circ - \tan 495^\circ$

l) $\sin 20^\circ \cos(-70^\circ) + \cos 380^\circ \cos(-20^\circ)$

m) $\cos 68^\circ \sin 8^\circ - \sin 68^\circ \cos 8^\circ$

n) $\cos\left(-\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right) - \tan(3\pi)$

o) $\sin 143^\circ \cos 82^\circ + \cos 143^\circ \sin 82^\circ$

2. Solve each of the following equations.

a) $\sin x + \cos x = 0$

b) $1 - \sin x = 2\cos^2 x$

c) $\cos x = \cos 2x$

d) $\tan^2 x = 3$

e) $\sin^2 x = \sin x$

f) $3\cos x + 2\cos^2 x = 2$

g) $\sin x = \sin 2x$

h) $\cos 2x + 5\cos x = -3$

i) $2 + 3\sin x = \cos 2x$

j) $2 + 3\sin 6x = \cos 12x$

k) $\sin 2x = 2\cos x$

l) $\sin 3x = \frac{\sqrt{3}}{2}$

m) $\cos x + \dfrac{\sin^2 x}{\cos x} + \sin x + \sin 2x = \dfrac{1}{\cos x}$

n) $2\sin x \cos x + \cos^2 x = \sin^2 x$

3. Compute the area of a regular $n-$sided polygon that is written inside a circle with radius R.

4. Simplify $(\sin 15^\circ)(2\cos 15^\circ)(2\cos 30^\circ)(2\cos 60^\circ)(2\cos 120^\circ)\ldots(2\cos 960^\circ)(2\cos 1920^\circ)$

5. a) Solve $\cos 5x = -\cos 10x$. You may present the solution in degrees.
   b) List all solutions that fall between 0° and 360°.

6. Consider the line $y = \frac{12}{5}x$. Find an equation of the line that bisects the smaller angle that is formed between the line and the positive part of the $x-$axis.

7. Simplify each of the following.
   a) $\sin(x + 180^\circ)$ b) $\cos(180^\circ - x)$ c) $\tan(90^\circ - x)$

8. Suppose that we draw both tangent lines to $(x-6)^2 + (y+3)^2 = 25$ from the point $P(-2, 2)$. Find an aproximate value for the angle formed by the two tangent lines.

9. Prove each of the following identities.

a) $\cot 2x = \dfrac{\cot^2 x - 1}{2\cot x}$

b) $4\sin^4 x = 1 - 2\cos 2x + \cos^2 2x$

c) $\cos 3x = 4\cos^3 x - 3\cos x$

d) $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

e) $\dfrac{\tan(45^\circ - x)}{\tan(45^\circ + x)} = \dfrac{1 - \sin 2x}{1 + \sin 2x}$

f) $\sin(30^\circ + \beta) + \sin(30^\circ - \beta) = \cos\beta$

g) $\dfrac{\cot x - \tan x}{\cot x + \tan x} = \cos 2x$

h) $\dfrac{\cos(\alpha - \beta)}{\sin\alpha \sin\beta} = \cot\alpha\cot\beta + 1$

i) $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$

j) $\tan^2 x = \dfrac{1-\cos 2x}{1+\cos 2x}$

10. Suppose that $\sin\alpha = \dfrac{3}{5}$ and α is not in the first quadrant; $\cos\beta = -\dfrac{7}{25}$ and β is not in the third quadrant. Find the exact value for each of the following.

a) $\sin 2\beta$ c) $\sin 2\alpha$ e) $\sin 3\alpha$ g) $\tan(\alpha+\beta)$

b) $\cos 2\beta$ d) $\cos 2\alpha$ f) $\cos(\alpha-\beta)$ h) $\tan(\alpha-\beta)$

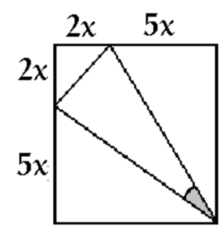
11. Find the exact value of $\tan\alpha$ if α is the acute angle formed by the lines $2x+3y=-12$ and $2x-7y=1$.

12. Find the exact value of $\tan x$ if $\tan y = \dfrac{2}{3}$ and $\tan(x+y)=4$.

13. Supose that x is an angle with $\cot x = -3$. Compute the exact value of each of the following.

a) $\cos x$ b) $\sin 2x$ c) $\cos 2x$

14. Compute the exact value of the tangent of the angle shaded on the picture.



15. Two circles, of radii 10 cm and 14 cm are placed so that their centers are 20 cm apart. Compute the exact value of the sine of the angle formed by the common tangent lines drawn to the circles.

16. Graph each of the following functions on $[-2\pi, 2\pi]$ and state domain, range, and basic properties for each function.

a) $f(x) = \sin x$ b) $f(x) = \cos x$ c) $f(x) = \tan x$

17. Find the value of x so that the area of the triangle with sides 12 cm, 12 cm and x, is the greatest?

18. We draw tangent lines to a circle from a point P outside of the circle. The line segment between P and a point of tangency is 3 units long. The line segment connecting P and the center of the circle intersects the circle in point Q. Line segment PQ is $\sqrt{3}$ units long. Compute the angle formed between the two tangent lines.

19. Solve each of the following triangles.

a) $\alpha = 42°$, $\beta = 55°$, $a = 7$ cm

b) $a = 10$ cm, $b = 4$ cm, $\gamma = 35°$

c) $a = 8$ cm, $b = 11$ cm, $\alpha = 35°$

d) $a = 5$ m, $b = 9$ m, $c = 7$ m

e) $a = 10$ m, $c = 7$ m, $\alpha = 28°$

f) $b = 12$ m, $c = 5$ m, $\gamma = 48°$

20. Suppose that α, β, and γ are angles in a triangle. Prove that

$$\frac{\cos\alpha}{a} + \frac{\cos\beta}{b} + \frac{\cos\gamma}{c} = \frac{a^2+b^2+c^2}{2abc}$$

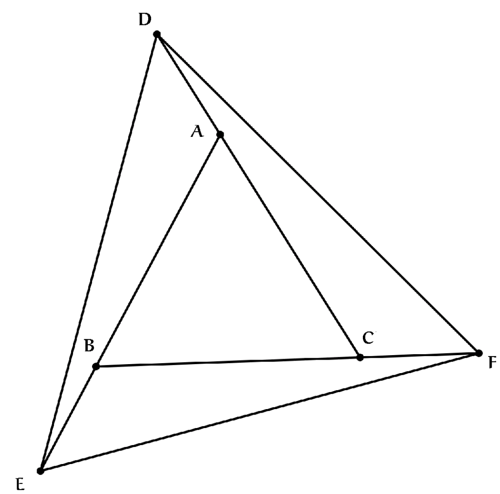
21. A water storage tank has the shape of a cylinder with radius 10 feet. It is mounted so that the circular cross sections are vertical. If the depth of the water is 15 feet, what percentage of the total capacity is used?

22. A pyramid has a square base. Find the angle that is formed between a triangular face and the base given that all edges are 20 meters long.

23. Find $\cos\alpha + \cos\beta + \cos\gamma$ in a triangle with sides 10 in, 8 in, and 5 in.

24. The latitude of New York City is 40.71°N. A satellite above New York City appears stationary in the sky. Find the speed of the satellite if it is at a height of 700 miles above the surface of Earth. Assume that the Earth is a sphere with radius 3960 miles long.

25. Suppose that ABC is an equilateral triangle with sides 1 unit long. We extend all three sides by a length of x as shown on the picture below. Find the exact value of x if we know that the equilateral triangle DEF has an area exactly three times as the area of triangle ABC.



26. Suppose that ABC is an equilateral triangle with sides 1 unit long.

a) Find the radius of the circumscribed circle. (That is a circle that contains vertices A, B, and C.)

b) Find the radius of the inscribed circle. (That is a circle inside the triangle, where sides a, b, and c are tangent lines to the circle.)

Answers

1.) a) $\frac{\sqrt{3}}{2}$ b) $-\sqrt{2}$ c) $\frac{\sqrt{2}-\sqrt{6}}{4}$ d) $\sqrt{3}+2$ e) $\sqrt{2}-1$ f) 1 g) 1 h) $\frac{\sqrt{2}}{2}$ i) $\sqrt{3}$

j) $-\frac{\sqrt{2}}{2}$ k) $\sqrt{3}+\frac{1}{4}$ l) 1 m) $-\frac{\sqrt{3}}{2}$ n) $\sqrt{3}$ o) $-\frac{\sqrt{2}}{2}$

2.) a) $-\frac{\pi}{4}+k\pi$ where $k\in\mathbb{Z}$ b) $\frac{\pi}{2}+2k\pi,\ -\frac{\pi}{6}+2k\pi,\ -\frac{5\pi}{6}+2k\pi$ where $k\in\mathbb{Z}$

c) $2k\pi, \pm\frac{2\pi}{3}+2k\pi$ where $k\in\mathbb{Z}$ d) $\pm\frac{\pi}{3}+k\pi$ where $k\in\mathbb{Z}$ e) $\frac{\pi}{2}+2k\pi,\ k\pi$ where $k\in\mathbb{Z}$

f) $\pm\frac{\pi}{3}+2k\pi$ where $k\in\mathbb{Z}$ g) $k\pi, \pm\frac{\pi}{3}+2k\pi$ where $k\in\mathbb{Z}$ h) $\pm\frac{2\pi}{3}+2k\pi$ where $k\in\mathbb{Z}$

i) $x=-\frac{\pi}{2}+2k_1\pi,\ -\frac{\pi}{6}+2k_2\pi,\ -\frac{5\pi}{6}+2k_3\pi$ where $k_1,k_2,k_3\in\mathbb{Z}$

j) $x=-\frac{\pi}{12}+k_1\frac{\pi}{3},\ -\frac{\pi}{36}+k_2\frac{\pi}{3},\ -\frac{5\pi}{36}+k_3\frac{\pi}{3}$ where $k_1,k_2,k_3\in\mathbb{Z}$ k) $\frac{\pi}{2}+k\pi$ where $k\in\mathbb{Z}$

l) $\frac{\pi}{9}+\frac{2}{3}k\pi$ $\frac{2\pi}{9}+\frac{2}{3}k\pi$ where $k\in\mathbb{Z}$ m) $k\pi$ $\pm\frac{2\pi}{3}=2k\pi$ where $k\in\mathbb{Z}$

n) $x=-\frac{\pi}{8}+\frac{k\pi}{2}$ where $k\in\mathbb{Z}$ 3.) $A=2nR^2\sin\left(\frac{180°}{n}\right)\cos\left(\frac{180°}{n}\right)$ 4.) $-\frac{\sqrt{3}}{2}$

5.) a) $36°+k\cdot72°$ where $k\in\mathbb{Z}$ $\pm12°+k\cdot72°$ where $k\in\mathbb{Z}$

b) 12°, 36°, 60°, 84° 108°, 132°, 156°, 180°, 204°, 228°, 252°, 276°, 300°, 324°, 348° 6.) $y=\frac{2}{3}x$

7.) a) $-\sin x$ b) $-\cos x$ c) $\cot x$ 8.) $64.010\,766\,4°$

9.) a) $\cot 2x=\frac{\cot^2 x-1}{2\cot x}$

$$\begin{aligned}\text{RHS} &= \frac{\cot^2 x-1}{2\cot x}=\frac{\frac{\cos^2 x}{\sin^2 x}-1}{2\frac{\cos x}{\sin x}}=\frac{\frac{\cos^2 x}{\sin^2 x}-\frac{\sin^2 x}{\sin^2 x}}{2\frac{\cos x}{\sin x}}=\frac{\frac{\cos^2 x-\sin^2 x}{\sin^2 x}}{\frac{2\cos x}{\sin x}}=\frac{\cos^2 x-\sin^2 x}{\sin^2 x}\cdot\frac{\sin x}{2\cos x}\\ &= \frac{\cos^2 x-\sin^2 x}{2\sin x\cos x}=\frac{\cos 2x}{\sin 2x}=\cot 2x=\text{LHS}\end{aligned}$$

b) $\cos 3x=4\cos^3 x-3\cos x$ b) $4\sin^4 x=1-2\cos 2x+\cos^2 2x$

$$\begin{aligned}\text{LHS} &= 4\sin^4 x=4\left(\sin^2 x\right)^2=\left(2\sin^2 x\right)^2=\left(2\sin^2 x-1+1\right)^2=\left(-\cos 2x+1\right)^2=\\ &= \cos^2 2x-2\cos 2x+1=\text{RHS}\end{aligned}$$

c) $\cos 3x=4\cos^3 x-3\cos x$

$$\begin{aligned}\text{LHS} &= \cos 3x=\cos\left(x+2x\right)=\cos x\cos 2x-\sin x\sin 2x=\cos x\left(2\cos^2 x-1\right)-\sin x\left(2\sin x\cos x\right)=\\ &= 2\cos^3 x-\cos x-2\sin^2 x\cos x=2\cos^3 x-\cos x-2\left(1-\cos^2 x\right)\cos x=\\ &= 2\cos^3 x-\cos x-2\cos x\left(1-\cos^2 x\right)=2\cos^3 x-\cos x-2\cos x+2\cos^3 x=4\cos^3 x-3\cos x=\text{RHS}\end{aligned}$$

d) $\sin\left(\frac{\pi}{2}-x\right)=\cos x$

$$\text{LHS}=\sin\left(\frac{\pi}{2}\right)\cos x-\cos\left(\frac{\pi}{2}\right)\sin x=1\cos x-0\sin x=\cos x=\text{RHS}$$

e) $\dfrac{\tan(45^\circ - x)}{\tan(45^\circ + x)} = \dfrac{1-\sin 2x}{1+\sin 2x}$

$$\begin{aligned}\text{LHS} &= \frac{\tan(45^\circ - x)}{\tan(45^\circ + x)} = \frac{\dfrac{\tan 45^\circ - \tan x}{1+\tan 45^\circ \tan x}}{\dfrac{\tan 45^\circ + \tan x}{1-\tan 45^\circ \tan x}} = \frac{\dfrac{1-\tan x}{1+1\cdot\tan x}}{\dfrac{1+\tan x}{1-1\cdot\tan x}} = \frac{1-\tan x}{1+\tan x}\cdot\frac{1-\tan x}{1+\tan x} = \left(\frac{1-\dfrac{\sin x}{\cos x}}{1+\dfrac{\sin x}{\cos x}}\right)^2 \\ &= \frac{\left(1-\dfrac{\sin x}{\cos x}\right)^2}{\left(1+\dfrac{\sin x}{\cos x}\right)^2} = \frac{\left(1-\dfrac{\sin x}{\cos x}\right)^2}{\left(1+\dfrac{\sin x}{\cos x}\right)^2}\cdot\frac{\cos^2 x}{\cos^2 x} = \frac{\left(\left(1-\dfrac{\sin x}{\cos x}\right)\cos x\right)^2}{\left(\left(1+\dfrac{\sin x}{\cos x}\right)\cos x\right)^2} = \frac{(\cos x - \sin x)^2}{(\cos x + \sin x)^2} \\ &= \frac{\cos^2 x + \sin^2 x - 2\sin x\cos x}{\cos^2 x + \sin^2 x + 2\sin x \cos x} = \frac{1-\sin 2x}{1+\sin 2x} = \text{RHS}\end{aligned}$$

f) $\sin(30^\circ + \beta) + \sin(30^\circ - \beta) = \cos\beta$

$$\begin{aligned}\text{LHS} &= \sin(30^\circ + \beta) + \sin(30^\circ - \beta) = \sin 30^\circ \cos\beta + \cos 30^\circ \sin\beta + \sin 30^\circ\cos\beta - \cos 30^\circ \sin\beta \\ &= 2\sin 30^\circ \cos\beta = 2\cdot\frac{1}{2}\cos\beta = \cos\beta = \text{RHS}\end{aligned}$$

g) $\dfrac{\cot x - \tan x}{\cot x + \tan x} = \cos 2x$

$$\text{LHS} = \frac{\cot x - \tan x}{\cot x + \tan x} = \frac{\dfrac{\cos x}{\sin x} - \dfrac{\sin x}{\cos x}}{\dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x}} = \frac{\left(\dfrac{\cos x}{\sin x} - \dfrac{\sin x}{\cos x}\right)\sin x\cos x}{\left(\dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x}\right)\sin x \cos x} = \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} = \frac{\cos 2x}{1} = \text{RHS}$$

h) $\dfrac{\cos(\alpha-\beta)}{\sin\alpha\sin\beta} = \cot\alpha\cot\beta + 1$

$$\text{LHS} = \frac{\cos(\alpha-\beta)}{\sin\alpha\sin\beta} = \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{\sin\alpha\sin\beta} = \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} + \frac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta} = \cot\alpha\cot\beta + 1 = \text{RHS}$$

i) $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$

$$\begin{aligned}\text{LHS} &= \frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x\cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y} = \frac{\dfrac{\sin x\cos y + \cos x\sin y}{\cos x\cos y}}{\dfrac{\sin x\cos y - \cos x \sin y}{\cos x \cos y}} = \frac{\dfrac{\sin x\cos y}{\cos x\cos y} + \dfrac{\cos x \sin y}{\cos x\cos y}}{\dfrac{\sin x\cos y}{\cos x\cos y} - \dfrac{\cos x\sin y}{\cos x \cos y}} \\ &= \frac{\tan x + \tan y}{\tan x - \tan y} = \text{RHS}\end{aligned}$$

j) $\tan^2 x = \dfrac{1-\cos 2x}{1+\cos 2x}$

$$\text{RHS} = \frac{1-\cos 2x}{1+\cos 2x} = \frac{1-\left(1-2\sin^2 x\right)}{1+(2\cos^2 x - 1)} = \frac{1-1+2\sin^2 x}{1+2\cos^2 x - 1} = \frac{2\sin^2 x}{2\cos^2 x} = \tan^2 x = \text{LHS}$$

10.) a) $-\dfrac{336}{625}$ b) $-\dfrac{527}{625}$ c) $-\dfrac{24}{25}$ d) $\dfrac{7}{25}$ e) $\dfrac{117}{125}$ f) $\dfrac{4}{5}$ g) $\dfrac{117}{44}$ h) $\dfrac{3}{4}$ 11.) $\dfrac{20}{17}$ 12.) $\dfrac{10}{11}$

13.) a) $\pm\dfrac{3\sqrt{10}}{10}$ b) $-\dfrac{3}{5}$ c) $\dfrac{4}{5}$ 14.) $\dfrac{12}{35}$ 15.) $\dfrac{4\sqrt{6}}{25}$

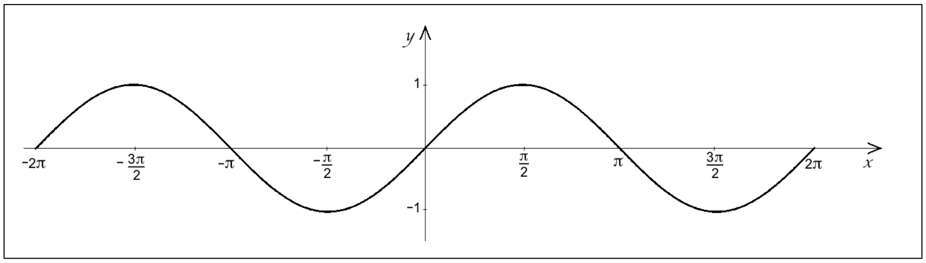
16.) a) $f(x) = \sin x$

domain: $\mathbb{R}$

range: $[-1, 1]$

periodic with period 2π: for all x, $\sin(x + 2\pi) = \sin x$

odd function: for all x, $\sin(-x) = -\sin x$



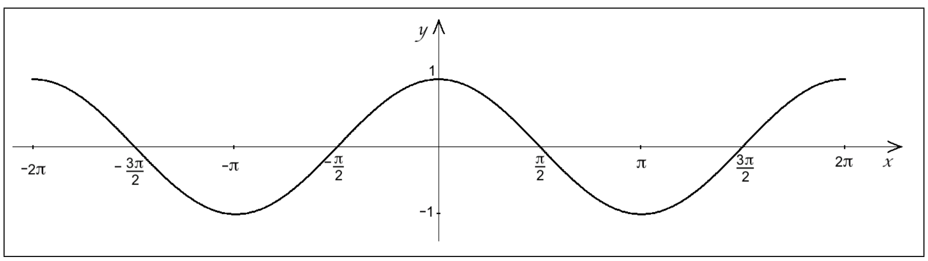
b) $f(x) = \cos x$

domain: $\mathbb{R}$

range: $[-1, 1]$

periodic with period 2π: for all x, $\cos(x + 2\pi) = \cos x$

even function: for all x, $\cos(-x) = \cos x$



c) $f(x) = \tan x$

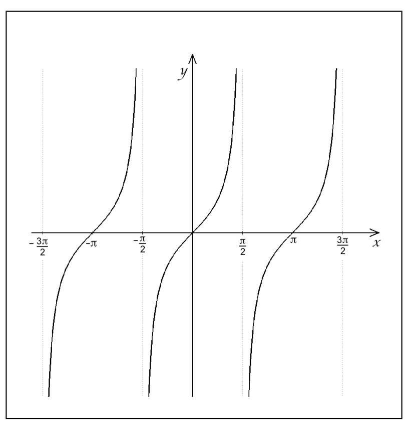
domain: $x \neq \dfrac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$

range: $\mathbb{R}$

periodic with period π: for all x, $\tan(x + \pi) = \tan x$

vertical asymptotes at $x = \dfrac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$

odd function: for all x, $\tan(-x) = -\tan x$



17.) $12\sqrt{2}\,\text{cm}$ 18.) $60°$

19.) a) $b \approx 8.569\,42\,\text{cm}$ $\gamma = 83°$ $c \approx 10.383\,358\,64\,\text{cm}$ b) $c \approx 7.1041\,\text{cm}$, $\alpha \approx 126.1582°$, $\beta \approx 18.8418°$

c) $c_1 \approx 13.9292\,\text{cm}$ $\beta_1 \approx 52.0612°$, $\alpha_1 \approx 92.9388°$ $c_2 \approx 4.0921\,\text{cm}$, $\beta_2 \approx 127.9388°$, $\alpha_2 \approx 17.0612°$

d) $\alpha = 33.5573°$, $\beta = 95.7392°$, $\gamma = 50.7035°$ e) $b \approx 15.6252\,\text{m}$, $\gamma = 19.1856°$, $\beta = 132.8144°$

f) no solution

20.) We state all three forms of the Law of Cosines and add the three lines.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\alpha \\ b^2 &= a^2 + c^2 - 2ac\cos\beta \\ c^2 &= a^2 + b^2 - 2ab\cos\gamma \end{aligned}$$

We add the three equations:

$$\begin{aligned} a^2 + b^2 + c^2 &= 2a^2 + 2b^2 + 2c^2 - 2bc\cos\alpha - 2ac\cos\beta - 2ab\cos\gamma \\ 2bc\cos\alpha + 2ac\cos\beta + 2ab\cos\gamma &= a^2 + b^2 + c^2 \end{aligned}$$

divide both sides by $2abc$

$$\begin{aligned} \frac{2bc\cos\alpha + 2ac\cos\beta + 2ab\cos\gamma}{2abc} &= \frac{a^2+b^2+c^2}{2abc} \\ \frac{\cos\alpha}{a} + \frac{\cos\beta}{b} + \frac{\cos\gamma}{c} &= \frac{a^2+b^2+c^2}{2abc} \end{aligned}$$

21.) $\dfrac{5\sqrt{75} + \dfrac{100\pi}{2\pi}\left(2\pi - 2\cos^{-1}\left(\dfrac{1}{2}\right)\right)}{100\pi} \approx 80.4\,498\,89\%$ 22.) $\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right) \approx 54.\,735\,61^\circ$

23.) $\dfrac{139}{160} + \dfrac{61}{100} + \left(-\dfrac{11}{80}\right) = \dfrac{1073}{800}$ 24.) $924.\,773\,766\,\dfrac{\text{mi}}{\text{h}}$ 25.) $\dfrac{1}{6}\sqrt{33} - \dfrac{1}{2}$ 26.) a) $\dfrac{\sqrt{3}}{3}$ b) $\dfrac{\sqrt{3}}{6}$

Last revised: March 28, 2015