1. Let A_1 and A_2 denote the area of two circles, C_1 and C_2, respectively. Find the ratio $\dfrac{A_1}{A_2}$ if we know that an arc subtended by a central angle of $45°$ in C_1 is as long as an arc subtended by a central angle of $30°$ in C_2.

2. Simplify each of the following expressions. Use exact values.

a) $\sec\left(\dfrac{11\pi}{6}\right)$ b) $\tan\left(\dfrac{2015\pi}{4}\right)$ c) $\cos\left(\dfrac{2\pi}{3}\right)$ d) $\sin\left(\dfrac{142\pi}{3}\right)$ e) $\dfrac{\tan 75° - 1}{\tan 75° + 1}$

3. Solve each of the following equations over the real numbers. Use exact values, and show all steps. Make sure to check your solution(s).

a) $-\cos 2x = \sin x$

b) $\sin 3x \cos 3x = \dfrac{\sqrt{3}}{4}$

c) $\sin x = \sin 2x$

d) $7\sin x + 1 = 6\cos^2 x$

e) $\sin x + 1 = 2\cos^2 x$

f) $\sin 3x = \cos 3x$

g) $\cos x - \sqrt{3}\sin x = -\sqrt{2}$

4. a) Solve $\sin x + 1 = 2\cos^2 x$

b) Solve $\sin 3x + 1 = 2\cos^2 3x$

c) List all solutions (in degrees) that fall between $0°$ and $360°$.

5. Graph each of the following functions.

a) $f(x) = -\dfrac{1}{2}\sin 2x + 1$ on $[-2\pi, 2\pi]$

b) $f(x) = -\dfrac{1}{2}\sin(2x - \pi) + 1$ on $[-2\pi, 2\pi]$

c) $f(x) = -3\cos\left(\dfrac{\pi x}{3}\right) - 2$ on $[-9, 9]$

d) $f(x) = -\sin\left(\dfrac{\pi x}{3} - \pi\right) + \dfrac{1}{2}$ on $[-3\pi, 3\pi]$

e) $f(x) = 3\sin(2x - \pi) + 1$ on $[-2\pi, 2\pi]$

f) $\tan x$ on $[-2\pi, 2\pi]$

g) $f(x) = \tan^{-1} x$

h) $f(x) = \sec x$

6. Find the exact value for each of the following expressions.

a) $\cos 22.5°$ b) $\cos 15° \cos 75°$ c) $\dfrac{\tan 65° - \tan 5°}{1 + (\tan 65°)\tan 5°}$ d) $\sin^{-1}\left(-\dfrac{1}{\sqrt{2}}\right)$ d) $\cos^{-1}\left(-\dfrac{1}{\sqrt{2}}\right)$

7. Prove each of the following identities.

a) $1 - \left(\cos\dfrac{x}{2} - \sin\dfrac{x}{2}\right)^2 = \sin x$

b) $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

c) $\sin 2x = \dfrac{1 - \tan^2\left(\dfrac{\pi}{4} - x\right)}{1 + \tan^2\left(\dfrac{\pi}{4} - x\right)}$

d) $\dfrac{\sin x + \sin 5x}{\cos x + \cos 5x} = \tan 3x$

e) $\sin 70° - \sin 50° = \sin 10°$

f) $\cos 2t = \dfrac{1 - \tan^2 t}{1 + \tan^2 t}$

g) $\cot\dfrac{x}{2} = \dfrac{\sin x}{1 - \cos x}$

8. Simplify each of the following expressions.

a) $\sin\left(\cos^{-1}\left(-\dfrac{3}{5}\right)\right)$

b) $\sin\left(\tan^{-1}(-2)\right)$

c) $\sin\left(2\cos^{-1}\left(\dfrac{1}{3}\right)\right)$

d) $\tan\left(2\tan^{-1}\left(\dfrac{3}{4}\right)\right)$

e) $\cos\left(2\tan^{-1}\left(\dfrac{1}{3}\right)\right)$

f) $\tan\left(\dfrac{1}{2}\cos^{-1}\left(-\dfrac{1}{2}\right)\right)$

g) $\sin\left(\dfrac{1}{2}\cos^{-1}\left(\dfrac{2}{3}\right)\right)$

h) $\tan\left(\tan^{-1}(2) + \tan^{-1}(3)\right)$

9. Simplify each of the following expressions.

a) $\sin\left(\cos^{-1}x\right)$ b) $\sin\left(\tan^{-1}x\right)$ c) $\sin\left(2\cos^{-1}x\right)$ d) $\tan\left(2\tan^{-1}x\right)$ e) $\cos\left(2\tan^{-1}x\right)$ f) $\tan\left(\frac{1}{2}\cos^{-1}x\right)$ g) $\sin\left(\frac{1}{2}\cos^{-1}x\right)$

10. Suppose that $\sin\alpha = -\frac{5}{13}$ and α is not in the fourth quadrant; $\cos\beta = \frac{7}{25}$ and β is not in the first quadrant. Find the exact value for each of the following.

a) $\tan(\alpha-\beta)$ b) $\cos(\alpha+\beta)$ c) $\cos 2\alpha$ d) $\tan\frac{\alpha}{2}$

11. Let x and y be angles such that $\sin x = -\frac{3}{5}$, $\cos y = -\frac{20}{29}$. In addition, we know that $180^\circ \le x \le 270^\circ$ and $90^\circ \le y \le 180^\circ$. Find the exact value of each of the following.

a) $\cos(x+y)$ b) $\sin(3x)$ c) $\tan(x-y)$

12. Express each of the following as a sum or difference.

a) $\sin 35^\circ \cos 25^\circ$ b) $\cos 25^\circ \cos 75^\circ$ c) $\cos 4x \cos 2x$

13. Express each of the following as a product.

a) $\sin 50^\circ + \sin 20^\circ$ b) $\sin 75^\circ - \sin 35^\circ$ c) $\cos 7x + \cos 3x$

14. Suppose that $\tan 2x = \frac{3}{4}$. Compute the exact value of a) $\cos 2x$ b) $\sin x$

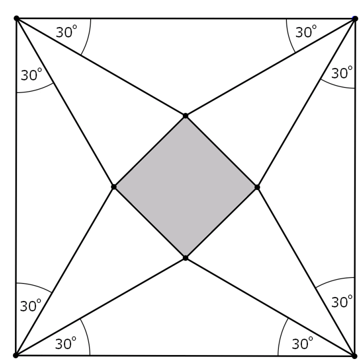
15. Find the exact value of $\tan\beta$ if β is the acute angle formed by $y = \frac{2}{3}x - 5$ and $y = -x + 1$.

16. Solve each of the following triangles.

a) $b = 248.6$, $c = 186.2$, and $\gamma = 43.1^\circ$ b) $\gamma = 42^\circ$, $a = 122\,\text{m}$, and $c = 70\,\text{m}$ c) $a = 5$, $b = 12$, and $c = 8$

17. Triangle ABC has sides of length 6, 7, and 8. Find the exact value of $\cos\alpha + \cos\beta + \cos\gamma$.

18. Consider the square with sides 1 meter shown on the picture. Find the exact value of the area of the shaded region.

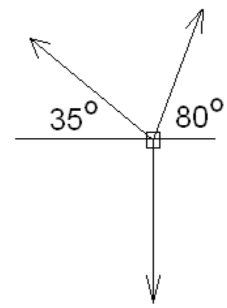


19. Consider the vectors $\underline{u} = 3\underline{i} + 4\underline{j}$ and $\underline{v} = 8\underline{i} - 15\underline{j}$. Find each of the following.

a) $-2\underline{u}$ b) $\|\underline{u}\|$ c) $\|\underline{v}\|$ d) $\underline{u}+\underline{v}$ e) $3\underline{u}-2\underline{v}$ f) $\underline{u}\cdot\underline{v}$

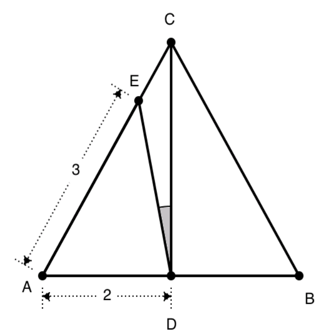
g) $(\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v})$ h) Find the angle formed by the vectors $\underline{u}$ and $\underline{v}$.

20. An object is held by ropes as shown on the picture. Find the forces in the ropes if the object weighs 100 N.



21. Find the smallest and largest value of the function $f(x) = 3\sin x - 8\cos x$.

22. Consider the regular triangle with sides 4 meter shown on the picture.



a) Find the exact value of the length of line segment CD.
b) Find the exact value of the length of line segment ED.
c) Find the exact value of $\cos\delta$ if δ is the shaded angle $\angle EDC$.

23. Find the exact value of $\tan x$ if we know that $\tan y = \dfrac{2}{7}$ and $\tan(x+y) = \dfrac{11}{12}$

24. Find the height of a tree if the angle of elevation of its top changes from 20° to 40° as the observer advances 75 ft towards its base.

25. The minute hand of a clock is 5 cm long. Find the speed of the top of the minute hand. Express your answer in meter per second.

26. Phoenix, AZ and Salt Lake City, UT have approximately the same longitude. The radius of the earth is approximately 3960 miles. The latitude of Phoenix is 33.5° and that of Salt Lake City is 40.7°. Find the distance to the nearest mile between the two cities.

27. A satellite can be seen over the same point on Earth above Phoenix, AZ. It is 400 miles above the surface. Find the speed of the satellite in miles per hour. (The latitude of Phoenix is 33.5° and the radius of the Earth is 3960 mi).

Answers

1. $\dfrac{4}{9}$ 2. a) $\dfrac{2\sqrt{3}}{3}$ b) -1 c) $-\dfrac{1}{2}$ d) $-\dfrac{\sqrt{3}}{2}$ e) $\dfrac{\sqrt{3}}{3}$

3. a) $\dfrac{\pi}{2} + 2k\pi,\ -\dfrac{\pi}{6} + 2k\pi,\ -\dfrac{5\pi}{6} + 2k\pi,\ k \in \mathbb{Z}$ b) $\dfrac{\pi}{9} + \dfrac{k\pi}{3}$ or $\dfrac{\pi}{18} + \dfrac{k\pi}{3}$ where $k \in \mathbb{Z}$

c) $k\pi,\quad \pm\dfrac{1}{3}\pi + 2k\pi$ where $k \in \mathbb{Z}$ d) $\dfrac{\pi}{6} + 2k\pi,\quad \dfrac{5\pi}{6} + 2k\pi$ where $k \in \mathbb{Z}$

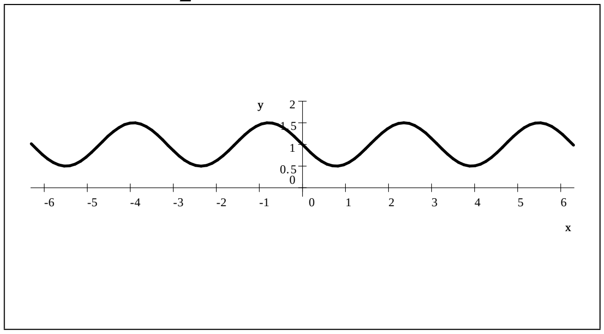
e) $-\frac{\pi}{2}+2k\pi,\quad \frac{\pi}{6}+2k\pi,\quad \frac{5\pi}{6}+2k\pi\quad k\in\mathbb{Z}$ f) $\frac{\pi}{12}+\frac{k\pi}{3},\quad k\in\mathbb{Z}$

g) $\frac{5\pi}{12}+2k\pi\qquad -\frac{13\pi}{12}+2k\pi\qquad k\in\mathbb{Z}$

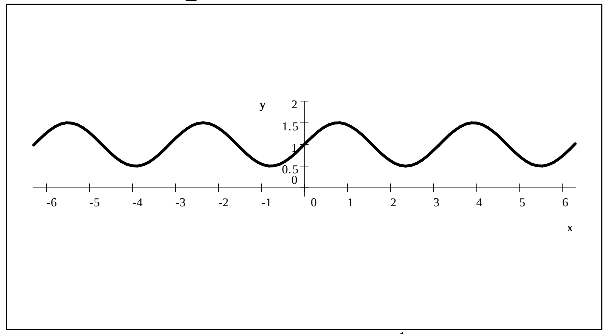
4. a) $-\frac{\pi}{2}+2k\pi,\quad \frac{\pi}{6}+2k\pi,\quad \frac{5\pi}{6}+2k\pi\quad k\in\mathbb{Z}$ b) $-\frac{\pi}{6}+\frac{2k\pi}{3},\quad \frac{\pi}{18}+\frac{2k\pi}{3},\quad \frac{5\pi}{18}+\frac{2k\pi}{3}\quad k\in\mathbb{Z}$

c) $10°,\ 50°,\ 90°,\ 130°,\ 170°,\ 210°,\ 250°,\ 290°,\ 330°$

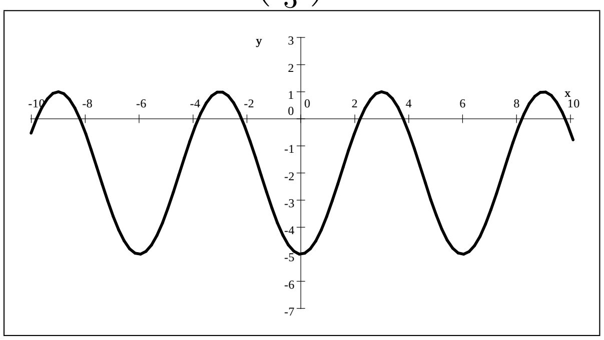
5. a) $f(x)=-\frac{1}{2}\sin 2x+1$ on $[-2\pi,2\pi]$



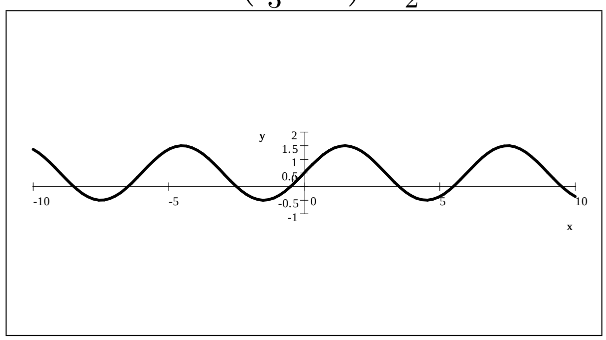
b) $f(x)=-\frac{1}{2}\sin(2x-\pi)+1$ on $[-2\pi,2\pi]$



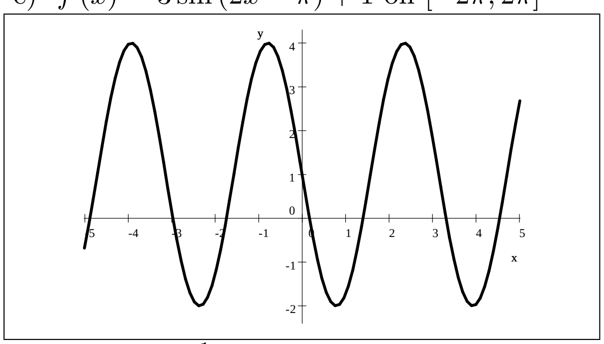
c) $f(x)=-3\cos\left(\frac{\pi x}{3}\right)-2$ on $[-9,9]$



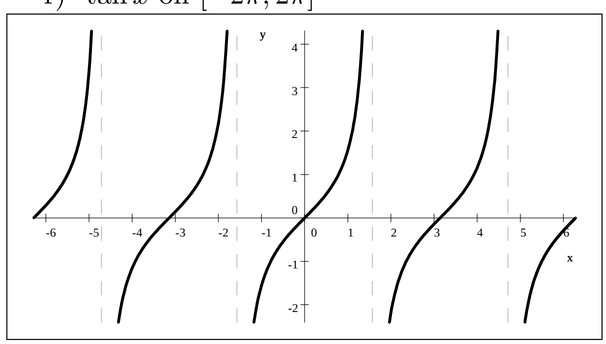
d) $f(x)=-\sin\left(\frac{\pi x}{3}-\pi\right)+\frac{1}{2}$ on $[-3\pi,3\pi]$



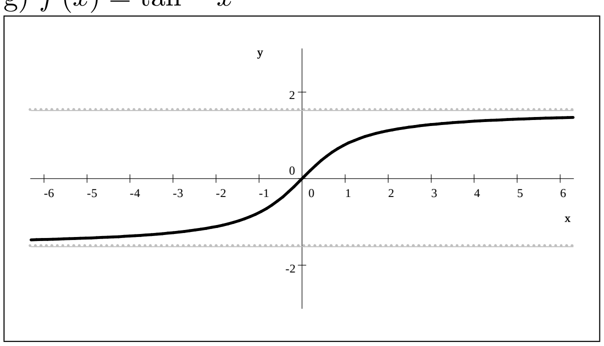
e) $f(x)=3\sin(2x-\pi)+1$ on $[-2\pi,2\pi]$



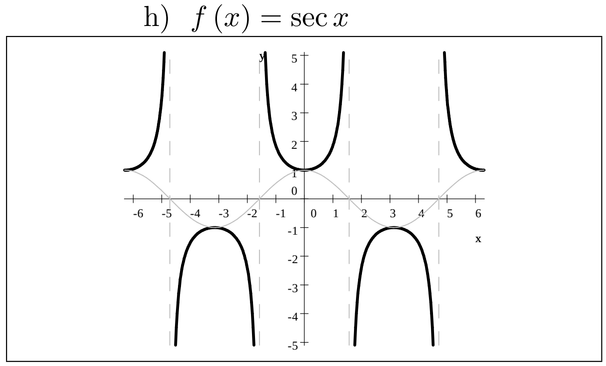
f) $\tan x$ on $[-2\pi,2\pi]$



g) $f(x)=\tan^{-1}x$



h) $f(x)=\sec x$



6. a) $\frac{1}{2}\sqrt{\sqrt{2}+2}$ b) $\frac{1}{4}$ c) $\sqrt{3}$ d) $-\frac{\pi}{4}$ e) $\frac{3\pi}{4}$

7. a) $1-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2=\sin x$

$$\text{LHS}=1-\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}\right)=1-(1-\sin x)=\text{RHS}$$

b) $\cos 4x=8\cos^4 x-8\cos^2 x+1$

$$\begin{aligned}\cos 4x &= \cos(2\cdot 2x)=2\cos^2(2x)-1=2\left(2\cos^2 x-1\right)^2-1=2\left(4\cos^4 x-4\cos^2 x+1\right)-1\\ &= 8\cos^4 x-8\cos^2 x+1\end{aligned}$$

c) $\sin 2x=\dfrac{1-\tan^2\left(\frac{\pi}{4}-x\right)}{1+\tan^2\left(\frac{\pi}{4}-x\right)}$

$$\text{RHS}=\frac{1-\tan^2\left(\frac{\pi}{4}-x\right)}{1+\tan^2\left(\frac{\pi}{4}-x\right)}=\frac{1-\left(\dfrac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right)^2}{1+\left(\dfrac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right)^2}=\frac{1-\left(\dfrac{1-\tan x}{1+\tan x}\right)^2}{1+\left(\dfrac{1-\tan x}{1+\tan x}\right)^2}=\frac{1-\dfrac{(1-\tan x)^2}{(1+\tan x)^2}}{1+\dfrac{(1-\tan x)^2}{(1+\tan x)^2}}$$

To clear the denominators, we multiply both numerator and denominator by $(1+\tan x)^2$

$$\begin{aligned}\text{RHS} &= \frac{1-\dfrac{(1-\tan x)^2}{(1+\tan x)^2}}{1+\dfrac{(1-\tan x)^2}{(1+\tan x)^2}}=\frac{1-\dfrac{(1-\tan x)^2}{(1+\tan x)^2}}{1+\dfrac{(1-\tan x)^2}{(1+\tan x)^2}}\cdot\frac{(1+\tan x)^2}{(1+\tan x)^2}=\frac{(1+\tan x)^2-(1-\tan x)^2}{(1+\tan x)^2+(1-\tan x)^2}\\ &= \frac{1+\tan^2 x+2\tan x-\left(1+\tan^2 x-2\tan x\right)}{1+\tan^2 x+2\tan x+\left(1+\tan^2 x-2\tan x\right)}=\frac{1+\tan^2 x+2\tan x-1-\tan^2 x+2\tan x}{1+\tan^2 x+2\tan x+1+\tan^2 x-2\tan x}\\ &= \frac{4\tan x}{2+2\tan^2 x}=\frac{2\tan x}{1+\dfrac{\sin^2 x}{\cos^2 x}}=\frac{2\tan x}{\dfrac{\cos^2 x}{\cos^2 x}+\dfrac{\sin^2 x}{\cos^2 x}}=\frac{2\tan x}{\dfrac{\sin^2 x+\cos^2 x}{\cos^2 x}}=\frac{2\tan x}{\dfrac{1}{\cos^2 x}}=2\tan x\cos^2 x\\ &= 2\frac{\sin x}{\cos x}\cos^2 x=2\sin x\cos x=\sin 2x=\text{LHS}\end{aligned}$$

d) $\dfrac{\sin x+\sin 5x}{\cos x+\cos 5x}=\tan 3x$

We re-write both numerator and denominator as products first.

$$\begin{aligned}\sin x &= \sin(3x-2x)=\sin 3x\cos 2x-\cos 3x\sin 2x\\ \sin 5x &= \sin(3x+2x)=\sin 3x\cos 2x+\cos 3x\sin 2x\end{aligned}$$

Thus $\sin x+\sin 5x=2\sin 3x\cos 2x$. Similarly,

$$\begin{aligned}\cos 5x &= \cos(3x+2x)=\cos 3x\cos 2x-\sin 3x\sin 2x\\ \cos x &= \cos(3x-2x)=\cos 3x\cos 2x+\sin 3x\sin 2x\end{aligned}$$

Thus $\cos x+\cos 5x=2\cos 3x\cos 2x$. Thus

$$\frac{\sin x+\sin 5x}{\cos x+\cos 5x}=\frac{2\sin 3x\cos 2x}{2\cos 3x\cos 2x}=\tan 3x$$

e) $\sin 70^\circ - \sin 50^\circ = \sin 10^\circ$

$$\begin{aligned} \text{LHS} &= \sin 70^\circ - \sin 50^\circ = \sin(60^\circ + 10^\circ) - \sin(60^\circ - 10^\circ) \\ &= \sin 60^\circ \cos 10^\circ + \cos 60^\circ \sin 10^\circ - (\sin 60^\circ \cos 10^\circ - \cos 60^\circ \sin 10^\circ) \\ &= \sin 60^\circ \cos 10^\circ + \cos 60^\circ \sin 10^\circ - \sin 60^\circ \cos 10^\circ + \cos 60^\circ \sin 10^\circ \\ &= 2\cos 60^\circ \sin 10^\circ = 2 \cdot \frac{1}{2} \sin 10^\circ = \sin 10^\circ = \text{RHS} \end{aligned}$$

f) $\cos 2t = \dfrac{1-\tan^2 t}{1+\tan^2 t}$

$$\text{RHS} = \frac{1-\tan^2 t}{1+\tan^2 t} = \frac{1 - \dfrac{\sin^2 t}{\cos^2 t}}{1 - \dfrac{\sin^2 t}{\cos^2 t}} = \frac{\cos^2 t - \sin^2 t}{\cos^2 t + \sin^2 t} = \frac{\cos 2t}{1} = \text{LHS}$$

g) $\cot \dfrac{x}{2} = \dfrac{\sin x}{1 - \cos x}$

Let $\beta = \dfrac{x}{2}$. Then $2\beta = x$ We will prove that $\cot \beta = \dfrac{\sin 2\beta}{1 - \cos 2\beta}$

$$\text{RHS} = \frac{\sin x}{1-\cos x} = \frac{\sin 2\beta}{1 - \cos 2\beta} = \frac{2\sin\beta\cos\beta}{1 - \left(1 - 2\sin^2\beta\right)} = \frac{2\sin\beta\cos\beta}{2\sin^2\beta} = \cot\beta = \cot\frac{x}{2} = \text{LHS}$$

8. a) $\frac{4}{5}$ b) $-\frac{2\sqrt{5}}{5}$ c) $\frac{4\sqrt{2}}{9}$ d) $\frac{24}{7}$ e) $\frac{4}{5}$ f) $\sqrt{3}$ g) $\frac{\sqrt{6}}{6}$ h) -1

9. a) $\sqrt{1-x^2}$ b) $\frac{x}{\sqrt{x^2+1}}$ c) $2x\sqrt{1-x^2}$ d) $\frac{2x}{1-x^2}$ e) $\frac{1-x^2}{x^2+1}$ f) $\frac{\sqrt{1-x^2}}{1+x}$ g) $\sqrt{\frac{1-x}{2}}$

10. a) $-\frac{323}{36}$ b) $-\frac{204}{325}$ c) $\frac{119}{169}$ d) -5 11. a) $\frac{143}{145}$ b) $-\frac{117}{125}$ c) $\frac{144}{17}$

12. a) $\frac{1}{2}(\sin 60^\circ + \sin 10^\circ)$ b) $\frac{1}{2}(\cos 50^\circ + \cos 100^\circ)$ c) $\frac{1}{2}(\cos 6x + \cos 2x)$

13. a) $2\sin 35^\circ \cos 15^\circ$ b) $2\cos 55^\circ \sin 20^\circ$ c) $2\cos 5x \cos 2x$ 14. a) $\pm\frac{4}{5}$ b) $\pm\frac{\sqrt{10}}{10}, \pm\frac{3\sqrt{10}}{10}$ 15. 5

16. a) $\beta_1 = 65.819^\circ$, $\alpha_1 = 71.081^\circ$, $a_1 = 257.790$ and $\beta_2 = 114.181^\circ$, $\alpha_2 = 22.719^\circ$ $a_2 = 105.247$

b) no solution c) $\alpha = 17.612^\circ$ $\beta = 133.433^\circ$ $\gamma = 28.955^\circ$ 17. $\frac{47}{32}$ 18. $\frac{2-\sqrt{3}}{3}$

19. a) $-6\underline{i} - 8\underline{j}$ b) 5 c) 17 d) $11\underline{i} - 11\underline{j}$ e) $-7\underline{i} + 42\underline{j}$ f) -36 g) -264 h) 115.0576°

20. 19.160 N, 90.38343 N 21. smallest: $-\sqrt{73}$ largest: $\sqrt{73}$ 22. a) $\sqrt{12}$ b) $\sqrt{7}$ c) $\frac{3\sqrt{21}}{14}$ 23. $\frac{1}{2}$

24. 48.209 ft 25. $8.73 \times 10^{-5}\,\frac{\text{m}}{\text{s}}$ 26. 498 miles 27. $951.83508\,\frac{\text{mi}}{\text{h}}$

Last revised: April 25, 2015