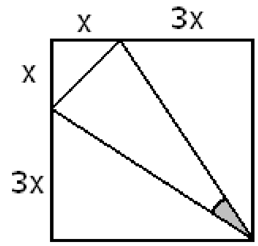
Please note that Quiz 7 may cover topics from Quizzes 1-6 and Exams 1 and 2, even if they do not appear here.

1. Given that θ is the angle shaded on the picture, compute the exact value of each of the following:
   a) $\cos\theta$
   b) $\sin\theta$
   c) $\tan\theta$



2. Solve each of the following equations.

   a) $\cos 2x = -\sin x$
   b) $\sin 2x = -\sin x$
   c) $\sin 3x = -\dfrac{1}{\sqrt{2}}$
   d) $\sin x = \cos x$
   e) $\sin 2x = \cos 2x$

3. Compute the exact value of $\dfrac{1+\tan 15°}{1-\tan 15°}$. (Hint: there is a really smart way of doing this that requires very little computation!)

4. Compute the exact value of each of the following.

   a) $\sin 15°$
   b) $\tan 75°$
   c) $\sin 20° \cos 40° + \cos 20° \sin 40°$
   d) $\dfrac{\tan 60° - \tan 30°}{1 + \tan 60° \tan 30°}$
   e) $\sin 10° \cos 130° - \sin 130° \cos 10°$

5. Suppose that α and β are angles with $\tan\alpha = -\dfrac{1}{2}$ and $\cos\beta = \dfrac{1}{3}$. We also know that neither α nor β belongs to the fourth quadrant. Compute the exact value of each of the following.

   a) $\sin\alpha$ and $\cos\alpha$
   b) $\sin\beta$ and $\tan\beta$
   c) $\sin 2\alpha$
   d) $\cos 2\alpha$
   e) $\sin 4\alpha$
   f) $\sin 3\alpha$
   g) $\tan 2\beta$
   h) $\sin(\alpha - \beta)$
   i) $\cos(\alpha - \beta)$
   j) $\tan 4\beta$

6. Find the exact value of $\tan y$ if we know that $\tan x = 2$ and $\tan(x+y) = -2$.

7. Find the exact value of $\tan x$ if we know that $\tan 2x = \dfrac{20}{21}$.

8. Find the exact value of $\cos 4x$ if we know that $\cos x = \dfrac{1}{3}$.

9. Find all solutions of the equation $\sin 4x = -1$ that fall between $0°$ and $360°$.

10. Solve each of the following triangles:

   a) $b = 16\,\text{ft}$, $\alpha = 38°$, and $\beta = 83°$.
   b) $a = 4\,\text{cm}$, $b = 7\,\text{cm}$, and $\alpha = 58°$.
   c) $a = 6\,\text{in}$, $b = 4\sqrt{3}\,\text{in}$, and $\alpha = 60°$.
   d) $a = 12\,\text{m}$, $c = 15\,\text{m}$, and $\alpha = 32°$
   e) $a = 4\,\text{m}$, $c = 7\,\text{m}$ and $\gamma = 70°$

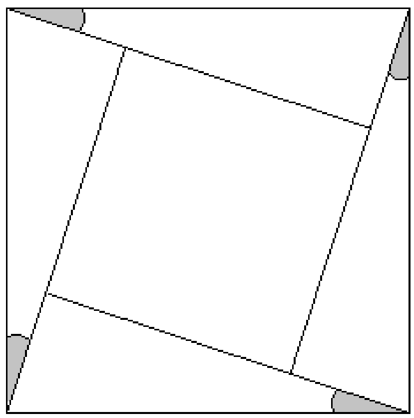
11. We know that in triangle ABC that $a = b = 20$. Find the value of c that guarantees that the area of the triangle is the greatest possible.

12. Consider the line $y = \frac{3}{4}x$. Find the equation for the line that bisects the angle that is formed between the line and the positive part of the $x-$axis.

13. Point D is on side AB of triangle ABC, with $\angle ACD = \angle BCD = 60°$, $AC = 5$, and $BC = 15$. Find the length of line segment CD.

14. (Enrichment) Prove that in any triangle ABC, the angle bisector of angle α splits the opposite side (namely a) into two parts whose ratio is the same as $\frac{b}{c}$. (Hint: try to use the Law of Sines.)

15. (Enrichment) The rectangles shown on the picture are squares. Given that the smaller square's area is exactly half of the area of the larger square, find the exact value of the shaded angle.



Answers

1. a) $\dfrac{24}{25}$ b) $\dfrac{7}{25}$ c) $\dfrac{7}{24}$

2. a) $-\dfrac{\pi}{6}+2k\pi,\quad -\dfrac{5\pi}{6}+2k\pi,\quad \dfrac{\pi}{2}+2k\pi$ where $k\in\mathbb{Z}$ b) $\pm\dfrac{2\pi}{3}+2k\pi,\quad k\pi,$ where $k\in\mathbb{Z}$

   c) $-\dfrac{\pi}{12}+\dfrac{2}{3}k\pi\quad -\dfrac{\pi}{4}+\dfrac{2}{3}k\pi$ where $k\in\mathbb{Z}$ d) $\dfrac{\pi}{4}+k\pi$ e) $\dfrac{\pi}{8}+\dfrac{k\pi}{2}$

3. $\sqrt{3}$

4. a) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ b) $2+\sqrt{3}$ c) $\dfrac{\sqrt{3}}{2}$ d) $\dfrac{\sqrt{3}}{3}$ e) $-\dfrac{\sqrt{3}}{2}$

5. a) $\sin\alpha=\dfrac{1}{\sqrt{5}}=\dfrac{\sqrt{5}}{5}$ and $\cos\alpha=-\dfrac{2}{\sqrt{5}}=-\dfrac{2\sqrt{5}}{5}$ b) $\sin\beta=\dfrac{\sqrt{8}}{3}=\dfrac{2\sqrt{2}}{3}$ and $\tan\beta=2\sqrt{2}$

   c) $\sin 2\alpha=-\dfrac{4}{5}$ d) $\cos 2\alpha=\dfrac{3}{5}$ e) $\sin 4\alpha=-\dfrac{24}{25}$ f) $\sin 3\alpha=\dfrac{11\sqrt{5}}{25}$ g) $\tan 2\beta=-\dfrac{4\sqrt{2}}{7}$

   h) $\sin(\alpha-\beta)=\dfrac{\sqrt{5}+4\sqrt{10}}{15}$ i) $\cos(\alpha-\beta)=\dfrac{\sqrt{10}}{30}$ j) $\tan 4\beta=-\dfrac{56\sqrt{2}}{17}$

6. $\dfrac{4}{3}$ 7. $\dfrac{2}{5}$ or $-\dfrac{5}{2}$ 8. $\dfrac{17}{81}$ 9. $67.5^\circ,\ 157.5^\circ,\ 247.5^\circ,\ 337.5^\circ$

10. a) $\gamma=59^\circ,\ a\approx 9.924\,56\,\text{ft},\ c\approx 13.817\,671\,64\,\text{ft}$ b) no solution

   c) $\beta=90^\circ,\ \gamma=30^\circ,\ c=2\sqrt{3}\,\text{in}$

   d) $\gamma_1\approx 41.483\,098^\circ$ $\beta_1\approx 106.516\,902^\circ$ and $b_1\approx 21.710\,535\,35\,\text{m}$

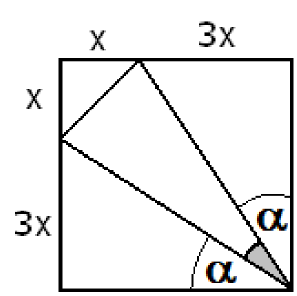
   $\gamma_2\approx 138.516\,902^\circ$ $\beta_2\approx 9.483\,098^\circ$ and $b_2\approx 3.730\,907\,55\,\text{m}$

   e) $\alpha\approx 32.477\,421^\circ$ $\beta\approx 77.522\,579^\circ$ $b\approx 7.273\,302\,4\,\text{m}$ 11. $20\sqrt{2}$ 12. $y=\dfrac{1}{3}x$

13. $\dfrac{15}{4}$ 14. see solutions 15. 15°

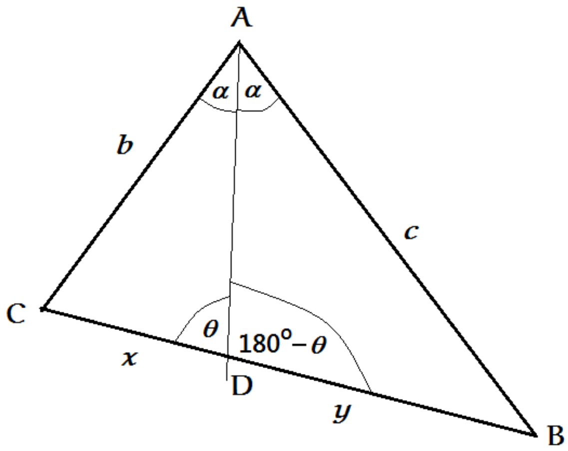
Solutions, Hints

1. Start with α and use the compound angle formulas.



13. Hint: Label the missing side as x and try to apply the area formula $A = \frac{1}{2}ab\sin\gamma$ in several ways for the same triangle to obtain an equation.

14. Consider the picture given. Line AD is the angle bisector for the angle at A. Let us apply the Law of Sines in triangle ADC.



$$\frac{\sin\alpha}{x} = \frac{\sin\theta}{b}$$

Let us also apply the Law of Sines in triangle ABD.

$$\frac{\sin\alpha}{y} = \frac{\sin(180° - \theta)}{c}$$

We also know that $\sin(180° - \theta) = \sin\theta$. So, we can re-write the second equation as

$$\frac{\sin\alpha}{y} = \frac{\sin\theta}{c}$$

We are almost done: let us express $\frac{\sin\alpha}{\sin\theta}$ in both equations:

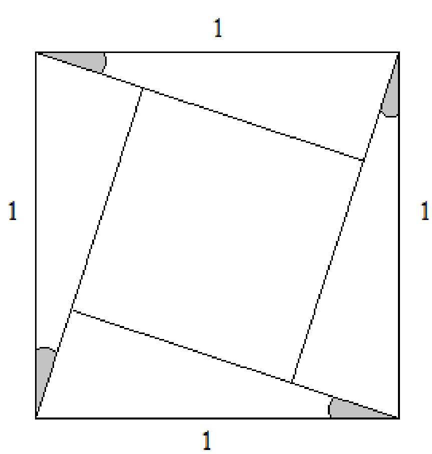
$$\frac{\sin\alpha}{\sin\theta} = \frac{x}{b} = \frac{y}{c}$$

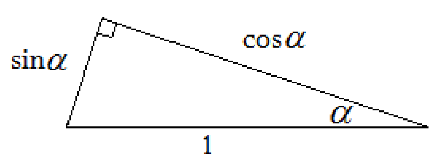
and so

$$\frac{x}{b} = \frac{y}{c} \quad \text{is the same as} \quad \frac{x}{y} = \frac{b}{c}$$

15. The rectangles shown on the picture are squares. Given that the smaller square's area is exactly half of the area of the larger square, find the exact value of the shaded angle.

Solution: Let us denote the shaded angle by α. Let us assume that the square has sides of length one (why are we allowed to do that?)





Then the shorter sides of the four identical right triangles are $\sin\alpha$ and $\cos\alpha$. Notice that the sides of the smaller square inside are the difference of $\cos\alpha$ and $\sin\alpha$. If the larger square has sides 1, then it has area 1.

The smaller square must have area $\frac{1}{2}$. So we have the following trigonometric equation:

$$\begin{aligned}(\cos\alpha-\sin\alpha)^2 &= \frac{1}{2}\\ \cos^2\alpha+\sin^2\alpha-2\sin\alpha\cos\alpha &= \frac{1}{2}\end{aligned}$$

We apply the Pythagorean identity: $\cos^2\alpha+\sin^2\alpha=1$ and the double-angle formula for sine: $2\sin\alpha\cos\alpha=\sin 2\alpha$. So our next line is

$$\begin{aligned}1-\sin 2\alpha &= \frac{1}{2}\\ -\sin 2\alpha &= -\frac{1}{2}\\ \sin 2\alpha &= \frac{1}{2}\end{aligned}$$

We solve for 2α:

$$2\alpha = 30^\circ + k\cdot 360^\circ \quad \text{or} \quad 2\alpha = 150^\circ + k\cdot 360^\circ \quad \text{where } k\in\mathbb{Z}$$

and divide by 2 :

$$\alpha = 15^\circ + k\cdot 180^\circ \quad \text{or} \quad \alpha = 75^\circ + k\cdot 180^\circ \quad \text{where } k\in\mathbb{Z}$$

Since α is an acute angle, only 15° and 75° are possible candidates for α. Looking at the picture, the solution is clearly the smaller angle, 15°.