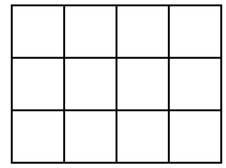
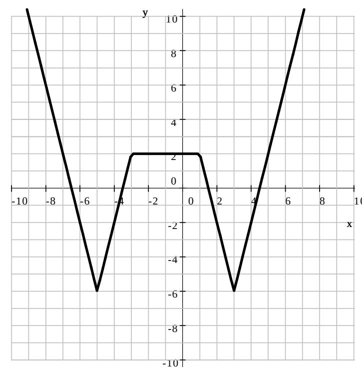
Please note that Quiz 8 will also cover topics covered on Quizzes 1-7 and Exam 1. Please review those topics as well, even if they do not appear on this document.

1. Use systematic listing to answer the following questions.
   a) How many rectangles are there on the picture?
   b) How many squares are there on the picture?



2. How many diagonals are there in a regular polygon with 20 sides?

3. Find the set of all values of x for which $P(x, y)$ is on the graph shown below and
   a) $y > 2$
   b) $y \geq 2$
   c) $y < -2$



4. Completely factor each of the following over the real numbers.
   a) $x^2 - 4y^2 + 3m^2x^2 - 12m^2y^2$
   b) $2x^5 - 16x^2$
   c) $x^2y^2 - y^2 - 4x^2 + 4$
   d) $y^4 - x^3 + x^3y^4 - 1$
   e) $4x^2 - 4x + 1 - y^6$

5. Simplify each of the following.
   a) $\left(\sqrt{10} - 3\right)^2 \left(\sqrt{10} + 3\right)^2$
   b) $\left(\sqrt[4]{x}\right)^{20}$
   c) $\sqrt[4]{x^{20}}$
   d) $\dfrac{1}{\sqrt{7} - 2} - \dfrac{1}{\sqrt{7} + 2}$
   e) $\left(-\dfrac{1}{a^{-3}b}\right)^{-2}$
   f) $\left(-\dfrac{1}{a^{-2}b}\right)^{-3}$

6. a) Find all real numbers with the following property: the number is 2 greater than its own reciprocal.
   b) Prove that the number(s) you found has that property.

7. Find an equation for each of the following lines.
   a) the line that passes through $P(-12, 7)$ and is parallel to $5x - 3y = 15$
   b) the line that passes through $P(-12, 7)$ and is perpendicular to $5x - 3y = 15$
   c) the line passing through $A(-5, 2)$ and $B(3, -4)$

8. An arch is in the shape of a semicircle. At a point along the base 4 feet from an end of the arch, the height of the arch is 12 feet. Find the maximum height of the arch.

9. Prove that there is no triangle whose sides are in the ratio of $1 : 2 : 3$.

10. Solve each of the following inequalities.
    a) $x^2 + 4 > 6x$ b) $x^2 \leq 6x$ c) $x^2 - 6x \leq -11$ d) $4x - 1 \geq 4x^2$

11. Graph each of the following.
    a) $y = -2x^2 + 12x - 10$ b) $(x - 2)^2 + (y + 1)^2 = 9$ c) $x = y^2$

12. Consider the circle $(x-1)^2+(y+2)^2=25$.
   a) Find all points on the circle with $x-$coordinate -9
   b) Find all points on the circle with $x-$coordinate -4
   c) Find all points on the circle with $x-$coordinate -2

13. Suppose that a and b are real numbers such that the sum of a and three times b is 12. Find each of the following.
   a) the greatest value of ab b) the smallest value of a^2+b^2 c) the greatest value of $5b^2-a^2$

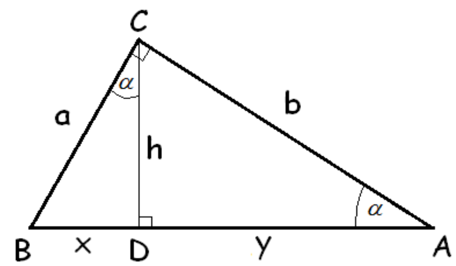
14. The shortest side of a right triangle is $20\,\text{cm}$. The difference between the other two sides is $8\,\text{cm}$. Find the sides of the triangle.

15. Prove that for every positive number x, the following is true: $x+\dfrac{1}{x}\geq 2$.

16. Consider the triangle ABC where $A(-4,4)$, $B(1,7)$, and $C(6,-1)$. Compute each of the following.
   a) the length of line segment AC b) the midpoint of line segment AB

17. Consider the triangle ABC where $A(-4,4)$, $B(1,7)$, and $C(6,-1)$.
   a) Find an equation for the line that connects A and C.
   b) Find an equation for the altitude drawn to side AC.

18. Consider the right triangle shown on the picture. Prove each of the following.
   a) $h=\sqrt{xy}$ b) $b=\sqrt{y(x+y)}$



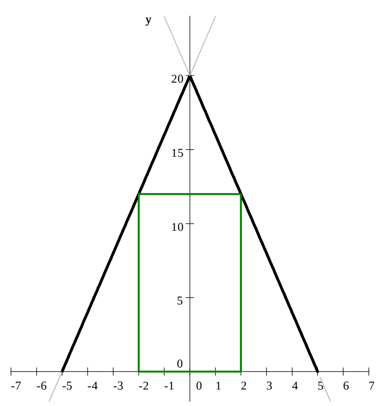
19. Consider the circle $2y-3x+x^2+y^2=x+45$. Find an equation for the tangnet line drawn to this circle at the point $P(-5,-2)$.

20. A person is standing 12 feet away from a street light that is 30 feet tall. If she is 5 feet tall, how long is her shadow?

21. Suppose that m and n are real number such that m is 5 greater than twice n. Find each of the following.
   a) the smallest value of n^2+m^2 b) the smallest value of nm c) the greatest value of n^2-m^2

22. We want to build a fence around a rectangular garden. One side is bordered by a river so no fence is needed there. Given that we have 500 feet of fencing, what is the greatest area that can be surrounded by the 500 feet of fencing? What dimensions will give us this maximum area?

23. Suppose that x is a real number with $-5<x<5$ and that rectangle $ABCD$ is positioned such that $A(x,0)$ and $B(-x,0)$, and C is on the line $y=4x+20$, and D is on the line $y=-4x+20$. What is the greatest possible value for the area of such a rectangle?



24*. A lattice point is a point whose both coordinates are integers. Can you find an equation of a line with
   a) no lattice points?
   b) exactly one lattice point?
   c) exactly two lattice points?

Answers

1. a) 60 b) 20   2. 170

3. a) $(-\infty,-7)\cup(5,\infty)$
   b) $(-\infty,-7]\cup[-3,1]\cup[5,\infty)$
   c) $(-6,-4)\cup(2,4)$

4. a) $(x+2y)(x-2y)\left(3m^2+1\right)$
   b) $2x^2(x-2)\left(x^2+2x+4\right)$
   c) $(x+1)(x-1)(y+2)(y-2)$
   d) $(x+1)\left(x^2-x+1\right)\left(y^2+1\right)(y+1)(y-1)$
   e) $\left(2x-1+y^3\right)\left(2x-1-y^3\right)$

5. a) 1 b) x^5 c) $\left|x^5\right|$ d) $\frac{4}{3}$ e) $\frac{b^2}{a^6}$ f) $-\frac{b^3}{a^6}$

6. a) $1-\sqrt{2}$ and $1+\sqrt{2}$
   b) If $x=1-\sqrt{2}$, then its reciprocal is

$$\begin{aligned}\frac{1}{1-\sqrt{2}} &= \frac{1}{1-\sqrt{2}}\cdot\frac{1+\sqrt{2}}{1+\sqrt{2}}\\ &= \frac{1+\sqrt{2}}{1-2}=\frac{1+\sqrt{2}}{-1}\\ &= -1-\sqrt{2}\end{aligned}$$

If we add 2 to this number, we get $-1-\sqrt{2}+2=1-\sqrt{2}$, and so x is indeed 2 greater than its reciprocal. And if $x=1+\sqrt{2}$, then its reciprocal is

$$\begin{aligned}\frac{1}{1+\sqrt{2}} &= \frac{1}{\sqrt{2}+1}\cdot\frac{\sqrt{2}-1}{\sqrt{2}-1}\\ &= \frac{\sqrt{2}-1}{1}=\sqrt{2}-1\end{aligned}$$

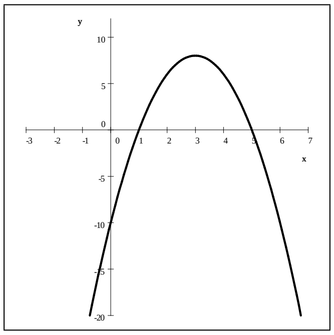
which is indeed 2 less than $\sqrt{2}+1$.

7. a) $y-7=\frac{5}{3}(x+12)$ or $y=\frac{5}{3}x+27$
   b) $-\frac{3}{5}(x+12)=y-7$ or $y=-\frac{3}{5}x-\frac{1}{5}$
   c) $y-2=-\frac{3}{4}(x+5)$ or $y+4=-\frac{3}{4}(x-3)$
   or $y=-\frac{3}{4}x-\frac{7}{4}$ 8. 20 feet

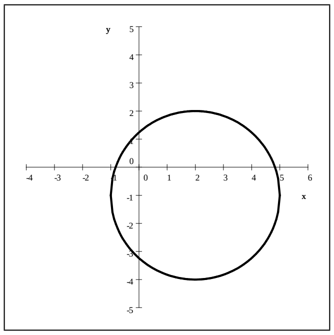
9. Such a triangle would have sides a, $2a$, and $3a$ long for some positive number a. But then the sides would fail the triangle inequality as $a+2a\not>3a$.

10. a) $x<3-\sqrt{5}$ or $x>3+\sqrt{5}$ b) $0\le x\le 6$
    c) no solution   d) $x=\frac{1}{2}$

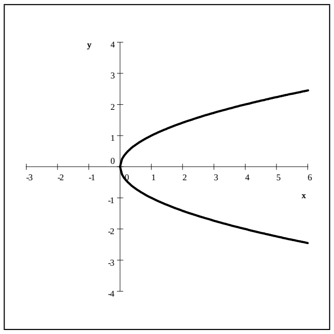
11. a) $y=-2x^2+12x-10$



b) $(x-2)^2+(y+1)^2=9$



c) $x=y^2$



12. a) there is no such point   b) $(-4,-2)$
    c) $(-2,2)$ and $(-2,-6)$

13. a) 12 (when $b=2$ and $a=6$)
    b) $\frac{72}{5}$ (when $b=\frac{18}{5}$ and $a=\frac{6}{5}$)
    c) 180 (when $b=9$ and $a=-15$)

14. 20 cm, 21 cm, and 29 cm

15. Claim: If $x > 0$, then $x + \frac{1}{x} \geq 2$.

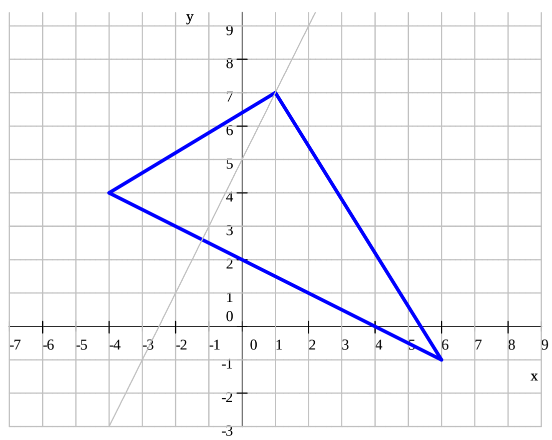
Proof: Let x be any positive number.
Then $(x-1)^2 \geq 0$ since it is a square.

$$\begin{array}{rcll} (x-1)^2 & \geq & 0 & \\ x^2 - 2x + 1 & \geq & 0 & \text{add } 2x \\ x^2 + 1 & \geq & 2x & \text{divide by } x\text{; recall that } x > 0 \\ x + \frac{1}{x} & \geq & 2 & \end{array}$$

16. a) $5\sqrt{5}$ unit b) $\left(-\frac{3}{2}, \frac{11}{2}\right)$

17. a) $-\frac{1}{2}(x+4) = y - 4$ or $y = -\frac{1}{2}x + 2$

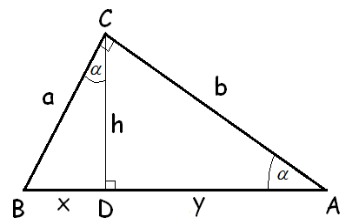
b) $y - 7 = 2(x-1)$ or $y = 2x + 5$



18. a) Claim: $h = \sqrt{xy}$
Proof: Triangles BDC and CDA are similar. Thus

$$\frac{\text{side opposite } \alpha}{\text{side opposite } \beta} = \frac{x}{h} = \frac{h}{y}$$

$$\frac{x}{h} = \frac{h}{y} \iff h^2 = xy \iff h = \sqrt{xy}$$



b) Claim: $b = \sqrt{y(x+y)}$
Proof: Triangles ACB and ADC are similar. Thus

$$\frac{\text{side opposite } \beta}{\text{hypotenuse}} = \frac{y}{b} = \frac{b}{x+y}$$

$$\frac{y}{b} = \frac{b}{x+y} \iff b^2 = y(x+y) \iff b = \sqrt{y(x+y)}$$

19. $y + 2 = -7(x+5)$ 20. 2.4 feet

21. a) 5, when $n = -2$ and $m = 1$

b) $-\frac{25}{8}$, when $n = -\frac{5}{4}$ and $m = \frac{5}{2}$

c) $\frac{25}{3}$, when $n = -\frac{10}{3}$ and $m = -\frac{5}{3}$

22. $31\,250\text{ ft}^2$, with a garden 125 ft by 250 ft 23. 50 unit^2

Last revised: September 28, 2017