Review Problems

1. Simplify each of the following.

a) $\log_{25}\left(\frac{1}{125}\right)$

b) $3^{\log_9 5}$

c) $\left(\sqrt{6+2\sqrt{5}}-\sqrt{6-2\sqrt{5}}\right)^2$

d) $\cos\left(\frac{13\pi}{6}\right)$

e) $\csc\left(\frac{13\pi}{4}\right)$

f) $\log_6 18+\log_6 2$

g) $\log_{10} 2+\log_{10} 50-1$

h) $\ln\left(5e^2\right)$

i) $\dfrac{\csc 30^\circ+\csc 60^\circ+\csc 90^\circ}{\sec 0^\circ+\sec 30^\circ+\sec 60^\circ}$

j) $\sin 143^\circ\cos 82^\circ+\cos 143^\circ\sin 82^\circ$

k) $\sin 120^\circ\cos 150^\circ-3\tan 330^\circ-\tan 495^\circ$

l) $\cos\left(-\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{3}\right)-\tan(3\pi)$

m) $\cos\left(-\frac{13\pi}{3}\right)+\sin\frac{13\pi}{4}-\tan\frac{9\pi}{4}+\cos\left(\frac{13\pi}{3}\right)$

n) $\sin 20^\circ\cos(-70^\circ)+\cos 380^\circ\cos(-20^\circ)$

o) $\tan 75^\circ$

p) $\cos 68^\circ\sin 8^\circ-\sin 68^\circ\cos 8^\circ$

q) $\cos 48^\circ\cos 3^\circ+\sin 48^\circ\sin 3^\circ$

r) $\cos(-105^\circ)$

s) $\tan 22.5^\circ$

t) $\dfrac{\tan\frac{2\pi}{15}+\tan\frac{\pi}{5}}{1-\left(\tan\frac{2\pi}{15}\right)\left(\tan\frac{\pi}{5}\right)}$

2. Solve each of the following inequalities.

a) $\dfrac{2x+3}{5x-1}\geq\dfrac{3}{8}$ b) $\dfrac{x+8}{2x-7}>1$

3. Solve the equation $3x^2-x-1=0$ by completing the square. Use exact values Check your solution.

4. Find the domain of each of the following expressions.

a) $\log_5\left(-x^2+2x+20\right)$

b) $\dfrac{1}{\sqrt{2x-6}-6}$

c) $\dfrac{1}{\log_2(x-5)+\log_2(x+5)}$

d) $\sqrt{(x+2)x(x-5)}$

e) $\log_7\left(x^2+3\right)$

f) $\ln\left(\dfrac{x-5}{x+6}\right)$

g) $\ln(x-5)-\ln(x+6)$

h) $\dfrac{\ln(x+1)}{\ln(x-1)}$

i) $\cot x$

j) $\dfrac{\cos x+1}{2\sin x-1}$

5. Solve each of the following equations.

a) $\log_3(x-7)(2x+7)=4$

b) $\log_3(x-7)+\log_3(2x+7)=4$

c) $\log_2(2x-3)-\log_2(x+1)=-3$

d) $\log_2(3-x)+\log_2(-x-4)=3$

e) $\log_3(x-5)-\log_3(2x-11)=-1$

f) $1-\sin x=2\cos^2 x$

g) $\cos x=\cos 2x$

h) $\tan^2 x=3$

i) $\sin^2 x=\sin x$

j) $3\cos x+2\cos^2 x=2$

k) $\sin x=\sin 2x$

l) $\cos 2x+5\cos x=-3$

m) $2+3\sin x=\cos 2x$

n) $\sin 2x=2\cos x$

o) $\sin x+\cos x=0$

p) $2^{x+1}\cdot 5=3^{x-1}$

q) $5^{2x-1}=2^{x+3}$

r) $9^{x+1}-28\cdot 3^x=-3$

s) $5\cdot 4^x+8\cdot 10^x=4\cdot 25^x$

6. Suppose that $\log_2 3=x$. Express each of the following in terms of x.

a) $\log_2 6$

b) $\log_2 12$

c) $\log_2 18$

d) $\log_2 72$

e) $\log_2\left(\frac{3}{2}\right)$

f) $\log_3 2$

g) $\log_2\left(\frac{4}{27}\right)$

h) $\log_6 12$

7. Suppose that $\log_2 3 = a$ and $\log_2 5 = b$. Express each of the following in terms of a.and b.

   a) $\log_2 15$ b) $\log_2 30$ c) $\log_2 240$ d) $\log_3 15$ e) $\log_6 30$

8. a) Suppose that $a = \log_5 4$ and $b = \log_2 6$. Express $\log_3 5$ in terms of a and b.

   b) Suppose that $\log_3 6 = x$ and $\log_5 9 = y$. Express $\log_{18} 100$ in terms of x and y.

9. a) The vertices of a triangle are $A(-4,1)$, $B(5,1)$, and $C(-4,7)$. Compute the exact value of the area of the triangle.
   b) The vertices of a triangle are $A(\log_2 x, \log_2 y)$, $B(\log_2 4x, \log_2 y)$, and $C(\log_2 x, \log_2 8y)$. Compute the exact value of the area of the triangle.

10. Find the smallest value of $a^2 + b^2$ if $a + 3b = 50$

11. Find the smallest possible value of $2a^2 + 3b^2$ if $a + b = 20$.

12. a) Prove that the area of any triangle can be computed as $A = \dfrac{1}{2}ab\sin\gamma$.

    b) Find the value of x so that the area of the triangle with sides 12 cm, 12 cm and x, is the greatest?

13. The first eight elements in an arithmetic sequence add up to 604. The next eight elements add up to 156. Find the first element and common difference in the sequence.

14. Let (a_n) be a non-constant arithmetic sequence with $s_{10} = \dfrac{s_{20}}{4}$. Find all natural numbers k so that $a_1 a_k = a_4^2$.

15. Graph each of the following functions.

    a) $f(x) = \dfrac{2}{x+1} + 3$ b) $f(x) = 2^{x-3} - 3$ c) $f(x) = |x+3| - 2$ d) $f(x) = (x-1)^3 - 1$

16. We placed \$3000 into a bank account with an annual compound interest rate of 4%, compounded annually. How long until the account reaches

    a) \$10 000 b) \$20 000 c) \$30 000 d) \$40 000 e) \$50 000 f) \$60 000

17. Simplify each of the following.

    a) $\sin(x + 180°)$ b) $\cos(180° - x)$ c) $\tan(90° - x)$

18. Prove each of the following identities.

    a) $\cot 2x = \dfrac{\cot^2 x - 1}{2\cot x}$ c) $\cos 3x = 4\cos^3 x - 3\cos x$

    b) $4\sin^4 x = 1 - 2\cos 2x + \cos^2 2x$ d) $\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$

19. Suppose that $\sin\alpha = \dfrac{3}{5}$ and α is not in the first quadrant; $\cos\beta = -\dfrac{7}{25}$ and β is not in the third quadrant. Find the exact value for each of the following.

    a) $\sin 2\beta$ c) $\sin 2\alpha$ e) $\sin 3\alpha$ g) $\tan(\alpha + \beta)$

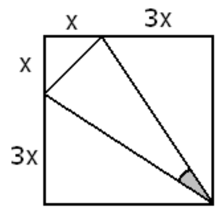
    b) $\cos 2\beta$ d) $\cos 2\alpha$ f) $\cos(\alpha - \beta)$ h) $\tan(\alpha - \beta)$

20. Find the exact value of $\tan\alpha$ if α is the acute angle formed by the lines $2x + 3y = -12$ and $2x - 7y = 1$.

21. Find the exact value of $\tan x$ if $\tan y = \dfrac{2}{3}$ and $\tan(x + y) = 4$.

22. Find the equation of the circle that passes through the points $A(0,2)$, $B(-2,-2)$, and $C(-8,-4)$.

23. Compute the exact value of the cosine of the angle shaded on the picture below.



24. Two circles, of radii 10 cm and 14 cm are placed so that their centers are 20 cm apart. Compute tbe exact value of the sine of the angle formed by the common tangent lines drawn to the circles.

Answers

1.) a) $-\frac{3}{2}$ b) $\sqrt{5}$ c) 4 d) $\frac{\sqrt{3}}{2}$ e) $-\sqrt{2}$ f) 2 g) 1 h) $2+\ln 5$ i) 1 j) $-\frac{1}{\sqrt{2}}$

k) $\sqrt{3}+\frac{1}{4}$ l) $\sqrt{3}$ m) $-\frac{\sqrt{2}}{2}$ n) 1 o) $\sqrt{3}+2$ p) $-\frac{\sqrt{3}}{2}$ q) $\frac{1}{\sqrt{2}}$ r) $\frac{\sqrt{2}-\sqrt{6}}{4}$

s) $\sqrt{2}-1$ t) $\sqrt{3}$ 2.) a) $(-\infty,-27]\cup\left(\frac{1}{5},\infty\right)$ b) $\left(\frac{7}{2},15\right)$ 3.) $\frac{1+\sqrt{13}}{6},\frac{1-\sqrt{13}}{6}$

4.) a) $1-\sqrt{21}<x<1+\sqrt{21}$ b) $x\geq 3,\quad x\neq 21$ c) $x>5,\quad x\neq\sqrt{26}$ d) $-2\leq x\leq 0$ or $x\geq 5$

e) $\mathbb{R}$ f) $x<-6$ or $x>5$ g) $x>5$ h) $x>1$ and $x\neq 2$ i) $x\neq k\pi \quad k\in\mathbb{Z}$

j) $x\neq\frac{\pi}{6}+2k\pi,\ x\neq\frac{5\pi}{6}+2k\pi$

5.) a) $-\frac{13}{2},10$ b) 10 c) $\frac{5}{3}$ d) -5 e) no solution f) $\frac{\pi}{2}+2k\pi,\ -\frac{\pi}{6}+2k\pi,\ -\frac{5\pi}{6}+2k\pi$ where $k\in\mathbb{Z}$

g) $2k\pi,\pm\frac{2\pi}{3}+2k\pi$ where $k\in\mathbb{Z}$ h) $\pm\frac{\pi}{3}+k\pi$ where $k\in\mathbb{Z}$ i) $\frac{\pi}{2}+2k\pi,\ k\pi$ where $k\in\mathbb{Z}$

j) $\pm\frac{\pi}{3}+2k\pi$ where $k\in\mathbb{Z}$ k) $k\pi,\pm\frac{\pi}{3}+2k\pi$ where $k\in\mathbb{Z}$ l) $\pm\frac{2\pi}{3}+2k\pi$ where $k\in\mathbb{Z}$

m) $x=-\frac{\pi}{2}+2k_1\pi,\ -\frac{\pi}{6}+2k_2\pi,\ -\frac{5\pi}{6}+2k_3\pi$ where $k_1,k_2,k_3\in\mathbb{Z}$ n) $\frac{\pi}{2}+k\pi$ where $k\in\mathbb{Z}$

o) $-\frac{\pi}{4}+k\pi$ where $k\in\mathbb{Z}$ p) $\log_{3/2}30=\frac{\ln 30}{\ln 3-\ln 2}$ q) $\log_{25/2}40=\frac{\ln 40}{\ln 25-\ln 2}$ r) $1,-2$ s) 1

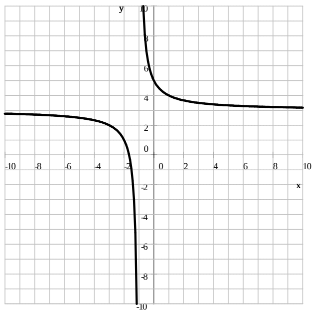
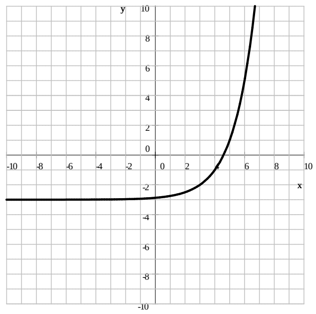
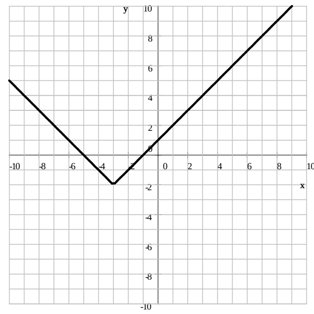
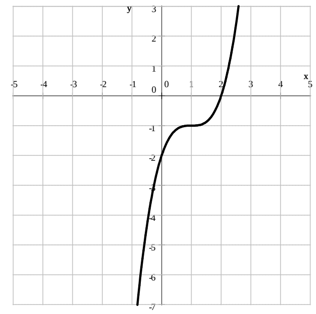
6.) a) $x+1$ b) $x+2$ c) $2x+1$ d) $2x+3$ e) $x-1$ f) $\frac{1}{x}$ g) $2-3x$ h) $\frac{x+2}{x+1}$

7.) a) $a+b$ b) $a+b+1$ c) $a+b+4$ d) $\frac{a+b}{a}$ e) $\frac{a+b+1}{a+1}$

8.) a) $\dfrac{2}{a(b-1)}$ b) $\dfrac{2x-2+\dfrac{4}{y}}{x+1}$ 9.) a) 27 unit^2 b) 3 unit^2 10.) 250 11.) 480

12.) a) $A=\dfrac{1}{2}ah$ and $h=b\sin\gamma$ b) $12\sqrt{2}\,\text{cm}$ 13.) $a=100, d=-7$ 14.) 25

15.) a) $f(x)=\dfrac{2}{x+1}+3$ b) $f(x)=2^{x-3}-3$ c) $f(x)=|x+3|-2$ d) $f(x)=(x-1)^3-1$

16.) a) $\dfrac{\ln 10-\ln 3}{\ln 1.04}=30.6974$ during the 31st year b) $\dfrac{\ln 20-\ln 3}{\ln 1.04}\approx 48.37036$ during the 49th year

c) $\dfrac{\ln 10}{\ln 1.04}\approx 58.7084$ during the 59th year d) $\dfrac{\ln 40-\ln 3}{\ln 1.04}\approx 66.043347$ during the 67th year

e) $\dfrac{\ln 50-\ln 3}{\ln 1.04}\approx 71.7328$ during the 72nd year f) $\dfrac{\ln 20}{\ln 1.04}=76.3814$ during the 77th year

17.) a) $-\sin x$ b) $-\cos x$ c) $\cot x$

18.) a) $\cot 2x=\dfrac{\cot^2 x-1}{2\cot x}$

$$\begin{aligned}\text{RHS} &= \frac{\cot^2 x-1}{2\cot x}=\frac{\dfrac{\cos^2 x}{\sin^2 x}-1}{2\dfrac{\cos x}{\sin x}}=\frac{\dfrac{\cos^2 x}{\sin^2 x}-\dfrac{\sin^2 x}{\sin^2 x}}{2\dfrac{\cos x}{\sin x}}=\frac{\dfrac{\cos^2 x-\sin^2 x}{\sin^2 x}}{\dfrac{2\cos x}{\sin x}}=\frac{\cos^2 x-\sin^2 x}{\sin^2 x}\cdot\frac{\sin x}{2\cos x}\\ &= \frac{\cos^2 x-\sin^2 x}{2\sin x\cos x}=\frac{\cos 2x}{\sin 2x}=\cot 2x=\text{LHS}\end{aligned}$$

c) $\cos 3x=4\cos^3 x-3\cos x$ b) $4\sin^4 x=1-2\cos 2x+\cos^2 2x$

$$\begin{aligned}\text{LHS} &= 4\sin^4 x=4\left(\sin^2 x\right)^2=\left(2\sin^2 x\right)^2=\left(2\sin^2 x-1+1\right)^2=(-\cos 2x+1)^2=\\ &= \cos^2 2x-2\cos 2x+1=\text{RHS}\end{aligned}$$

c) $\cos 3x=4\cos^3 x-3\cos x$

$$\begin{aligned}\text{LHS} &= \cos 3x=\cos(x+2x)=\cos x\cos 2x-\sin x\sin 2x=\cos x\left(2\cos^2 x-1\right)-\sin x(2\sin x\cos x)=\\ &= 2\cos^3 x-\cos x-2\sin^2 x\cos x=2\cos^3 x-\cos x-2\left(1-\cos^2 x\right)\cos x=\\ &= 2\cos^3 x-\cos x-2\cos x\left(1-\cos^2 x\right)=2\cos^3 x-\cos x-2\cos x+2\cos^3 x=4\cos^3 x-3\cos x=\text{RHS}\end{aligned}$$

d) $\sin\left(\frac{\pi}{2}-x\right)=\cos x$

$$\text{LHS}=\sin\left(\frac{\pi}{2}\right)\cos x-\cos\left(\frac{\pi}{2}\right)\sin x=1\cos x-0\sin x=\cos x=\text{RHS}$$

19.) a) $-\frac{336}{625}$ b) $-\frac{527}{625}$ c) $-\frac{24}{25}$ d) $\frac{7}{25}$ e) $\frac{117}{125}$ f) $\frac{4}{5}$ g) $\frac{117}{44}$ h) $\frac{3}{4}$ 20.) $\frac{20}{17}$ 21.) $\frac{10}{11}$

22.) $(x+7)^2+(y-3)^2=50$ 23.) $\frac{24}{25}$ 24.) $\frac{4\sqrt{6}}{25}$

Solution for 8:

a) Given that $a=\log_5 4$ and $b=\log_2 6$. Express $\log_3 5$ in terms of a and b.
Solution: In the statement $a=\log_5 4$ we switch to base 2:

$a=\log_5 4=\frac{\log_2 4}{\log_2 5}=\frac{2}{\log_2 5}$ Thus $a=\frac{2}{\log_2 5}$ we solve for $\log_2 5=\frac{2}{a}$

$b=\log_2 6=\log_2(2\cdot 3)=\log_2 3+\log_2 2=\log_2 3+1$ Thus $b=\log_2 3+1$ We solve for $\log_2 3=b-1$

Now we're ready: $\log_3 5=\frac{\log_2 5}{\log_2 3}=\frac{\frac{2}{a}}{b-1}=\frac{2}{a(b-1)}$

b) Given that $\log_3 6=x$ and $\log_5 9=y$. Express $\log_{18} 100$ in terms of x and y.
Solution: We will switch to base 3.

$x=\log_3 6=\log_3(2\cdot 3)=\log_3 3+\log_3 2=1+\log_3 2$ Thus $x=1+\log_3 2$, we solve for $\log_3 2$ and get $\log_3 2=x-1$

$y=\log_5 9=\frac{\log_3 9}{\log_3 5}=\frac{2}{\log_3 5}$ Thus $y=\frac{2}{\log_3 5}$ We solve for $\log_3 5$ and get $\log_3 5=\frac{2}{y}$

We are now ready to solve the problem:

$$\begin{aligned}\log_{18} 100 &= \frac{\log_3 100}{\log_3 18}=\frac{\log_3\left(2^2\cdot 5^2\right)}{\log_3\left(2\cdot 3^2\right)}=\frac{\log_3\left(2^2\right)+\log_3\left(5^2\right)}{\log_3(2)+\log_3\left(3^2\right)}=\frac{2\log_3 2+2\log_3 5}{\log_3 2+2}\\ &= \frac{2(x-1)+2\frac{2}{y}}{x-1+2}=\frac{2x-2+\frac{4}{y}}{x+1}\cdot\frac{y}{y}=\frac{2xy-2y+4}{xy+y}\end{aligned}$$

Answer: $\frac{2x-2+\frac{4}{y}}{x+1}=\frac{2xy-2y+4}{xy+y}=\frac{2(xy-y+2)}{y(x+1)}$ All three forms are acceptable as final answer.

Solution for 13.

The problem becomes routine when we realize that what is given is that
$s_8=604$ and $s_{16}=604+156=760$.

Solution for 14.

Let a and d denote the first element and common difference. Then

$$\begin{aligned}
s_{10} &= \frac{s_{20}}{4} \\
\frac{2a+9d}{2}(10) &= \frac{1}{4}\frac{2a+19d}{2}(20) && \text{divide by 10} \\
\frac{2a+9d}{2} &= \frac{1}{4}\frac{2a+19d}{2}(2) && \text{simplify} \\
\frac{2a+9d}{2} &= \frac{1}{4}(2a+19d) && \text{multiply by 4} \\
2(2a+9d) &= 2a+19d \\
4a+18d &= 2a+19d && \text{subtract } 2a \\
2a+18d &= 19d && \text{subtract } 18d \\
2a &= d
\end{aligned}$$

$$\begin{aligned}
a_1 a_k &= a_4^2 \\
a(a+(k-1)d) &= (a+3d)^2 && \text{substitute } d=2a \\
a(a+(k-1)(2a)) &= (a+3(2a))^2 && \text{distribute } 2a \\
a(a+2ak-2a) &= (a+6a)^2 && \text{combine like terms} \\
a(2ak-a) &= (7a)^2 && \text{distribute } a \\
2a^2k-a^2 &= 49a^2 && \text{subtract } 49a^2 \\
2a^2k-50a^2 &= 0 && \text{factor} \\
2a^2(k-25) &= 0 \\
a &= 0 \text{ or } k=25
\end{aligned}$$

If $a=0$, then $d=2a=0$ as well, but the problem stated that the sequence is not constant. Thus $k=25$.