Exam 2 will cover the following topics. (All topics covered by Quizzes 1-8 and Exam 1). These topics include functions and their graphs, inverse functions, exponents and logarithms, limits, differentiation by using the definition, the sum rule, the constant multiplier rule, the generalized power rule and the product rule, derivative of logarithmic functions, and applications of the derivative: optimization problems, tangent lines, antiderivatives, motion problems, extrema, and concavity behavior of functions

Review Problems

1. We placed \$100 into a bank account with an annual compound interest rate of 8%. Rounded to the nearest penny, how much money do we have in the account after 20 years if the bank compounds

   a) annually b) semi-annually c) monthly d) daily e) continuously?

2. We placed \$5000 into a bank account with an annual compound interest rate of 9%, compounded annually. How long until the account reaches \$8000?

3. We placed some money into a bank account with an annual compound interest rate of 10%, compounded continuously. How much is this amount if after five years, the account contains \$20000?

4. Simplify each of the following expressions.

   a) $\log_9\left(\dfrac{1}{\sqrt{27}}\right)$ b) $\log_{27}\left(\dfrac{1}{9^a}\right)$

5. Find each of the following limits.

   a) $\displaystyle\lim_{x\to 1+}\frac{x^2-1}{x^2-3x+2}$ c) $\displaystyle\lim_{x\to\infty}\frac{2x^2+3x+1}{3x^2-5x+2}$ e) $\displaystyle\lim_{x\to 3}\frac{x^2-3x}{x^2-9}$

   b) $\displaystyle\lim_{x\to 2+}\frac{x^2-1}{x^2-3x+2}$ d) $\displaystyle\lim_{x\to\infty}\frac{-x^3+2x+1}{x-3}$ f) $\displaystyle\lim_{x\to 3+}\frac{x^2+3x}{x^2-9}$

6. Find the derivative for each of the following functions.

   a) $f(x)=\dfrac{5x^3-1}{x^3}$ c) $f(x)=\sqrt[7]{x}-x^7+\sqrt{7x}-e^7$ e) $f(x)=\left(x^3-5x^2+1\right)\ln x$

   b) $f(x)=17x-24\sqrt{x}$ d) $f(x)=\log_2 x$

7. Compute the following indefinite integrals.

   a) $\displaystyle\int 12x^2-2x+1\,dx$ c) $\displaystyle\int x^5-2ax-a^2\,dx$

   b) $\displaystyle\int x^2+x+1+\frac{1}{x}+\frac{1}{x^2}\,dx$ d) $\displaystyle\int x^5-2ax-a^2\,da$

8. Consider $f(x)=x^3-7x+6$. Find the equation of the tangent line drawn to the graph of $f(x)=x^3-7x+6$ at $x=-1$.

9. Write an equation of all tangent lines drawn to the graph of $f(x)=\dfrac{1}{2}x^2-3x+1$ from the point $(-1,4)$.

10. Find an equation of the tangent line drawn to the graph of $f(x)=x^5-4x^2+1$ at $x=2$.

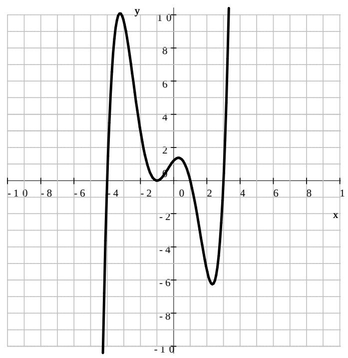
11. We know the following things about a function f. $f'(x)=20x^3-3$ and $f(-1)=16$. Find f.

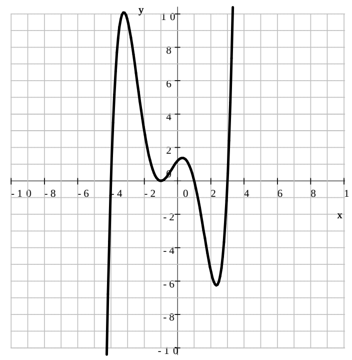
12. Find an equation of $F(x)$ if $F'(x)=10x^4-6x^2+4x-5$ and $F(-1)=8$.

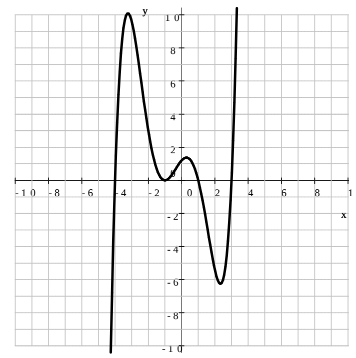
13. Give a complete analysis for the function $f(x)=(138-6x)\left(x^2-7x-8\right)$ on the interval $[-5,24]$.

14. The pictures below show the graph of a function $f(x)$. For each case, graph $g(x)$ in the same coordinate system, where $g(x)$ is given by

a) $g(x) = \dfrac{1}{f(x)}$ b) $g(x) = f'(x)$ c) $g(x)$ where $g'(x) = f(x)$







15. A closed box with a square base is to have a volume of 250 cubic meters. The material for the top and bottom of the box costs \$ 2 per square meter, and the material for the sides costs \$ 1 per square meter. Can the box be constructed for less than \$ 300?

16. A manufacturer estimates that when q units of a particular product are produced each month, the total cost will be $C(q) = 5q + 17000$ dollars, and all q units can be sold at a price of $p(q) = 65 - \dfrac{q}{100}$ dollars per unit.

a) Find the fixed cost.

b) Determine the level of production that results in a maximum profit.

c) What is the maximum profit?

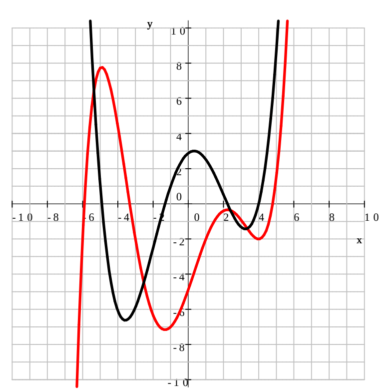
17. We shoot a small object upward, from the top of a tower. The acceleration function of the object is $a(t) = -10$. (Location is measured in meters, velocity in $\dfrac{\mathrm{m}}{\mathrm{s}}$, acceleration in $\dfrac{\mathrm{m}}{\mathrm{s}^2}$. Positive direction is upward.)

a) Given that $v(0) = 160$, find $v(t)$, the velocity function of the object.

b) Given that $h(0) = 525$, find $h(t)$, the location function of the object.

c) Find the maximum height that the object reaches.

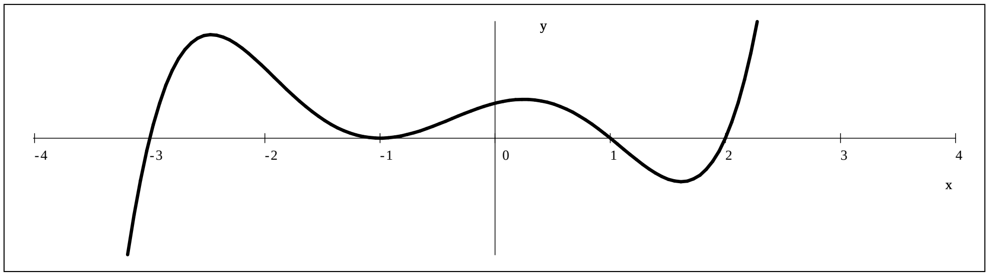
18. The graph below shows a function f and its first derivative, f'. Which is which?



19. Plot the graph of $g(x) = \dfrac{-(x+2)(x+1)^2(x-2)}{(x+1)x^2(x-1)(x-2)}$

20. Consider $f(x) = \dfrac{x+4}{3x-5}$. Find the equation for the inverse of f, $f^{-1}(x)$.

21. The graph below shows f', the first derivative of a function f.



a) Find all values of x for which the function f has a local maximum at x.

b) Find all values of x for which the function f has a local minimum at x.

c) How many points of inflection does f have?

d) Sketch the graph of f.

22. Suppose that f is a function with derivative

$$f'(x) = \left(16 - x^2\right)(1-x)^2 x \left(9 - x^2\right)(x+2)(2-x)(x+3)$$

a) Graph f'.

b) Find all x for which f has a relative maximum.

c) Find all x for which f has a relative minimum.

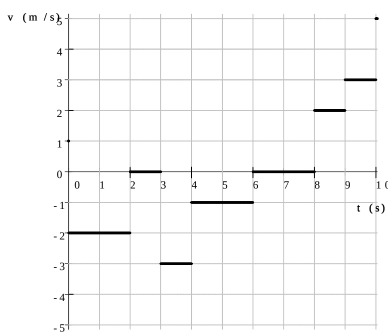
d) How many points of inflection does f have?

e) Plot the graph of f in the same coordinate system with f'.

f) Is it possible that f does not have any $x-$intercept?

g) Plot the graph of f'' in the same coordinate system with f'.

23. An object is moving along a vertical line. The picture below shows the velocity of an object (measured in $\frac{\text{m}}{\text{s}}$) as a function of time (measured in seconds)



a) Assume that $h(0) = 5$. Graph the location function, $h(t)$.

b) Assume that $h(0) = 2$. Graph the location function, $h(t)$.

Review Problems - Answers

1. a) $\$100 \cdot 1.08^{20} \simeq \$466.10$ b) $\$100\left(1+\dfrac{0.08}{2}\right)^{20(2)} \simeq \$480.10$ c) $\$100\left(1+\dfrac{0.08}{12}\right)^{20(12)} \simeq \$492.68$

   d) $\$100\left(1+\dfrac{0.08}{360}\right)^{20(360)} \simeq \$495.22$ e) $\$100 \cdot e^{0.08(20)} \simeq \$495.30$

2. in the sixth year
   Solution: Solve $5000\,(1.09)^n = 8000$ for n.

$$\begin{aligned} 5000 \cdot 1.09^n &= 8000 && \text{divide by } 5000 \\ 1.09^n &= 1.6 && \text{take } \ln \text{ of both sides} \\ \ln 1.09^n &= \ln 1.6 \\ n \ln 1.09 &= \ln 1.6 \\ n &= \frac{\ln 1.6}{\ln 1.09} \cong 5.453\,889\,46 \end{aligned}$$

3. \$12130. 61
   Solution: Solve $xe^{0.1(5)} = 20000$ for x.

$$\begin{aligned} xe^{0.1(5)} &= 20000 && \text{divide by } e^{0.5} \\ x &= \frac{20000}{e^{0.5}} \cong 12130.\,613\,2 \end{aligned}$$

4. a) $-\dfrac{3}{4}$ b) $\dfrac{-2a}{3}$
   Solution: a)

$$\log_9\left(\frac{1}{\sqrt{27}}\right) = \frac{\log_3\left(\frac{1}{\sqrt{27}}\right)}{\log_3 9} = \frac{\log_3\left(27^{-1/2}\right)}{\log_3 9} = \frac{-\frac{1}{2}\log_3 27}{2} = \frac{-\frac{3}{2}}{2} = -\frac{3}{4}$$

   b) Since $\dfrac{1}{9^a} = 9^{-a} = \left(3^2\right)^{-a} = 3^{-2a}$

$$\log_{27}\left(\frac{1}{9^a}\right) = \frac{\log_3\left(\frac{1}{9^a}\right)}{\log_3 27} = \frac{\log_3\left(3^{-2a}\right)}{3} = \frac{-2a}{3}$$

5. a) -2 b) ∞ c) $\dfrac{2}{3}$ d) $-\infty$ e) $\dfrac{1}{2}$ f) ∞

6. a) $f'(x) = \dfrac{3}{x^4}$ b) $f'(x) = 17 - \dfrac{12}{\sqrt{x}}$ c) $f'(x) = \dfrac{1}{7}x^{-6/7} - 7x^6 + \sqrt{7}\left(\dfrac{1}{2\sqrt{x}}\right)$

   d) $f'(x) = \dfrac{1}{x\ln 2}$ e) $f'(x) = \left(3x^2 - 10x\right)\ln x + \dfrac{x^3 - 5x^2 + 1}{x}$

7. a) $4x^3 - x^2 + x + C$ b) $\dfrac{x^3}{3} + \dfrac{x^2}{2} + x + \ln|x| - \dfrac{1}{x} + C$ c) $\dfrac{x^6}{6} - ax^2 - a^2x + C$ d) $x^5a - xa^2 - \dfrac{a^3}{3} + C$

8. $y = -4x + 8$
Solution: $f(-1) = (-1)^3 - 7(-1) + 6 = -1 + 7 + 6 = 12$, thus the tangent line has to pass through $(-1, 12)$. Compute the first derivative of f : $f'(x) = 3x^2 - 7$. The derivative at -1 takes the value $f'(-1) = 3(-1)^2 - 7 = -4$, thus the tangent line there has slope -4. Thus the tangent line is a straight line with slope -4, passing through $(-1, 12)$. The equation is then $y - 12 = -4(x + 1)$.

9. $y = -5x - 1$ and $y = -3x + 1$

10. $y = 64x - 111$
Solution: $f(2) = 17$, thus the tangent line has to pass through $(2, 17)$. Compute the first derivative of f : $f'(x) = 5x^4 - 8x$. The derivative at 2 takes the value $f'(2) = 64$ thus the tangent line there has slope 64. The equation is $y - 17 = 64(x - 2)$.

11. $f(x) = 5x^4 - 3x + 8$
Solution: We first find the antidervative of f. Since the derivative of x^4 is $4x^3$, the antiderivative of $f'(x) = 20x^3 - 3$ are the family $5x^4 - 3x + C$ where C can be any number. We find the value of C using the fact that $f(-1) = 16$.

$$\begin{aligned} 16 &= 5(-1)^4 - 3(-1) + C \\ 16 &= 8 + C \\ 8 &= C \end{aligned}$$

12. $F(x) = 2x^2 - 5x - 2x^3 + 2x^5 + 1$

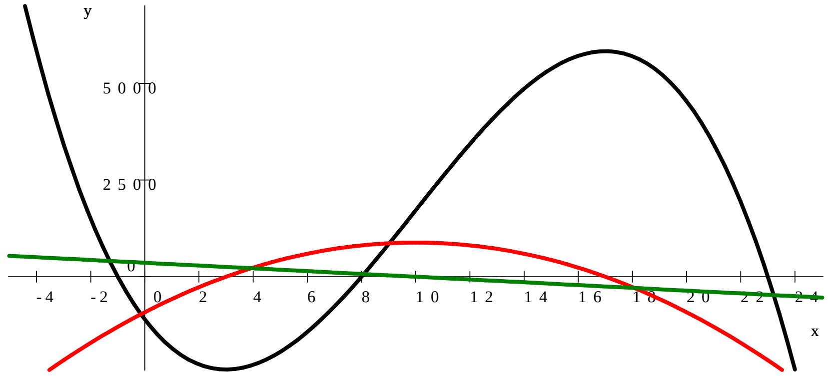
13. Complete analysis of the function $f(x) = (138 - 6x)(x^2 - 7x - 8)$ on domain $[-5, 24]$.
Solution:
domain: $[-5, 24]$
We multiply out f, and also obtain its fully factored form.

$$\begin{aligned} f(x) &= -6(x + 1)(x - 8)(x - 23) \\ f(x) &= -6x^3 + 180x^2 - 918x - 1104 \end{aligned}$$

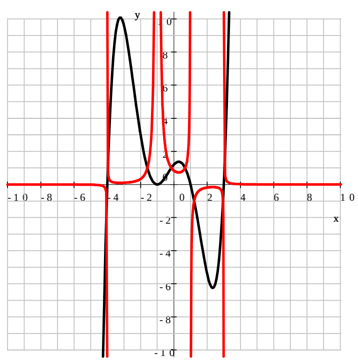
The factored form gives us the $x-$intercepts: $(-1, 0)$, $(8, 0)$, and $(23, 0)$
The polynomial form gives us the $y-$intercept: $(0, -1104)$
end-behavior: N/A since the function is restricted.
We now use the polynomial form of f to differentiate f. We factor the derivative, which is quadratic.

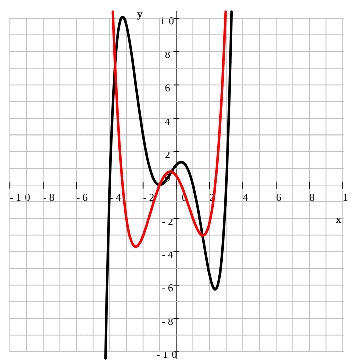
$$\begin{aligned} f'(x) &= -18x^2 + 360x - 918 \\ &= -18(x - 3)(x - 17) \end{aligned}$$

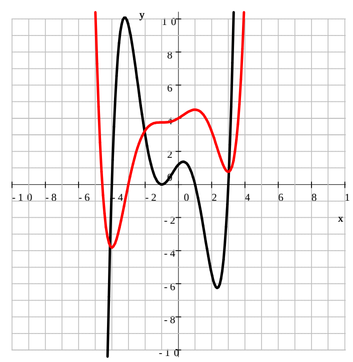
This is clearly an upside-down parabola, with two zeroes at 3 and 17. Since f' changes sign from negative to positive at $x = 3$, f has a relative minimum there. Since f' changes sign from positive to negative at $x = 17$, f has a relative minimum there. Thus f is increasing on $[3, 17]$; decreasing on $[-5, 3]$ and $[17, 24]$.
It is clear from f' (but we can also use $f''(x) = -36x + 360 = -36(x - 10)$) that f' is increasing on $(-\infty, 10)$ and decreases on $(10, \infty)$, thus f will turn at $x = 10$ from concave up to concave down. Thus f is concave up on $[-5, 10]$ and concave down on $[10, 24]$ and there is a point of inflection at $x = 10$.
We compute $f(-5)$, $f(3)$, $f(10)$, $f(17)$ and $f(24)$
relative minimum: $(3, -2400)$
absolute minimum: $(3, -2400)$ and $(24, -2400)$
relative maximum: $(17, 5832)$
absolute maximum: $(-5, 8736)$ $\Longrightarrow$ Now we know the range: $[-2400, 5832]$
point of inflection: $(10, 1716)$
continuous, bounded (both above and below)



14. a) $g(x) = \dfrac{1}{f(x)}$ b) $g(x) = f'(x)$ c) $g(x)$ where $g'(x) = f(x)$







15. no, a 5 by 5 by 10 box costs exactly \$300

Solution: Let x denote the side of the square base, and h denote the height of the box. Then $V = hx^2$ gives us

$$hx^2 = 250 \quad \Longrightarrow \quad h = \frac{250}{x^2}$$

We now set up the cost function, $C(x)$. The top and bottom each cost \$2 per square meter, and have area x^2. The four sides each have area $xh = x\left(\dfrac{250}{x^2}\right) = \dfrac{250}{x}$ and cost \$1 per square meter. Thus

$$C(x) = 2 \cdot 2 \cdot x^2 + 4 \cdot 1 \cdot \frac{250}{x} = 4x^2 + \frac{1000}{x} = 4x^2 + 1000x^{-1}$$

We are looking for the minimum of $C(x)$

$$\begin{aligned} C'(x) &= 8x + 1000(-1)x^{-2} = 8x - \frac{1000}{x^2} = \frac{8x^3 - 1000}{x^2} = \frac{8\left(x^3 - 125\right)}{x^2} \\ &= \frac{8(x-5)\left(x^2 + 5x + 25\right)}{x^2} \end{aligned}$$

The last form shows that C' has only one zero, at $x = 5$. Since both x^2 and $x^2 + 5x + 25$ are positive for all values of x, this expression will change from negative to positive at $x = 5$, indicating a minimum. Thus, the lowest possible cost will be associated with $x = 5$. The actual cost is then

$$C(5) = 4 \cdot 5^2 + \frac{1000}{5} = 300$$

Thus, we can not construct this box for less than \$300.

16. a) \$17 000 b) 3000 unit per month c) \$73 000
Solution:

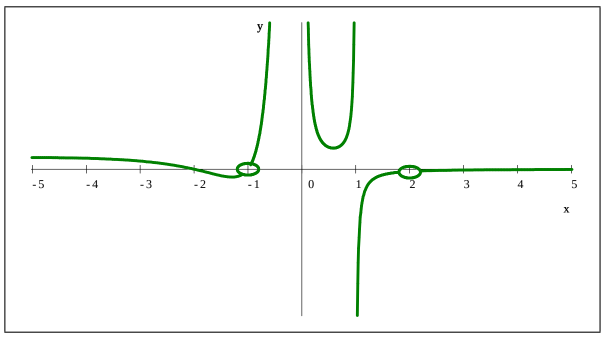
$$\begin{aligned} P(q) &= R(q) - C(q) = q \cdot p(q) - C(q) = q\left(65 - \frac{q}{100}\right) - (5q + 17\,000) \\ &= 65q - \frac{1}{100}q^2 - 5q - 17\,000 = -\frac{1}{100}q^2 + 60q - 17\,000 \\ &= -\frac{1}{100}\left(q^2 - 6000q + 1700\,000\right) = -\frac{1}{100}\left((q-3000)^2 - 3000^2 + 1700\,000\right) \\ &= -\frac{1}{100}\left((q-3000)^2 - 7300\,000\right) = -\frac{1}{100}(q-3000)^2 + 73\,000 \end{aligned}$$

Since we have an upside down parabola, the extrema is indeed a maximum.

17. a) $v(t) = -10t + 160$ b) $h(t) = -5t^2 + 160t + 525$ c) $h_{\max} = 1805$ m

18. the red is f, the black is f'

19. $g(x) = \dfrac{-(x+2)(x+1)^2(x-2)}{(x+1)x^2(x-1)(x-2)}$



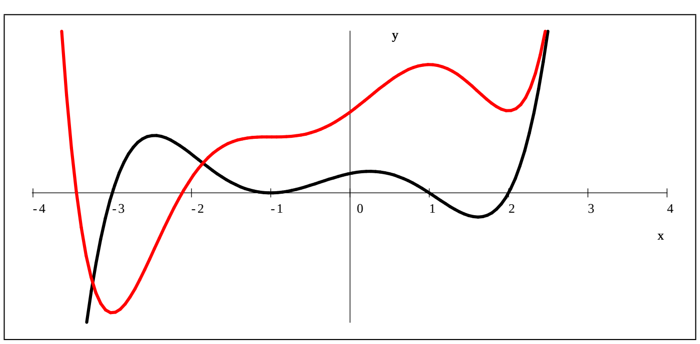
20. $f^{-1}(x) = \dfrac{5x+4}{3x-1}$
Solution: First we drop the function notation and write y instead of $f(x)$. Then we solve for x.

$$\begin{aligned} y &= \frac{x+4}{3x-5} \quad \text{multiply by } 3x-5 \\ y(3x-5) &= x+4 \\ 3xy - 5y &= x+4 \end{aligned}$$

$$3xy - x = 5y + 4$$
$$x(3y-1) = 5y+4$$
$$x = \frac{5y+4}{3y-1}$$

Thus the inverse function of $f(x)$ is $f^{-1}(x) = \dfrac{5x+4}{3x-1}$

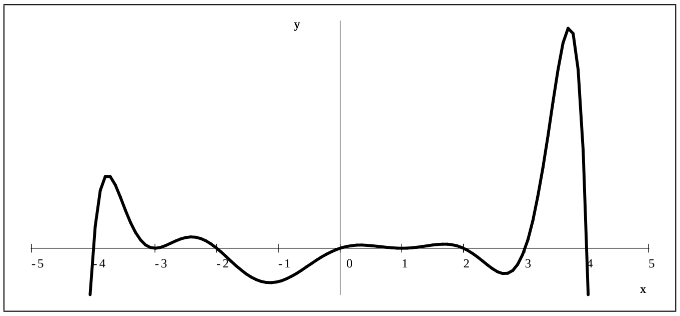
21. a) 1 b) $-3, 2$ c) 4 d)

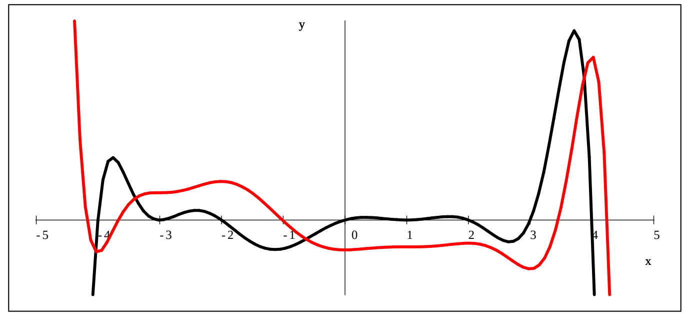


22. Suppose that f is a function with derivative

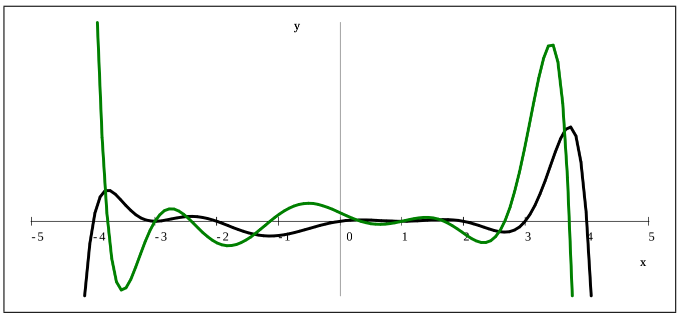
$$f'(x) = \left(16 - x^2\right)(1-x)^2 x \left(9 - x^2\right)(x+2)(2-x)(x+3)$$

a) b) $-2, 2, 4$ c) $-4, 0, 3$ d) 9 e) f) no





g)



23. a) red graph b) green graph