1. Consider the function $f(x) = \dfrac{x}{x^2+1}$.

a) Find the domain of f.

b) Find the vertical asymptotes of the graph of f, discuss points of discontinuity, and state the intervals where f is continuous.

c) Compute eah of the following integrals.

i) $\displaystyle\int \frac{x}{x^2+1}dx$ ii) $\displaystyle\int_{-2}^{2} \frac{x}{x^2+1}dx$ iii) $\displaystyle\int_{0}^{\infty} \frac{x}{x^2+1}dx$

d) Find the limits $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. State horizontal asymptote(s).

e) Find both coordinates of all relative extrema.

f) Prove that the relative maximum found in part f) is indeed a maximum and not a minimum or point of inflection.

g) Prove that the relative maximum found in part f) is an absolute maximum as well.

h) Based on previous results, discuss absolute extrema, boundedness and state the range.

i) Find both coordinates of all points of inflection.

2. Let f be a function given by $f(x) = \ln\left(\dfrac{x}{x-1}\right)$

a) Find the domain of f.

b) Find the value of the derivative of f at $x=-1$.

c) Find the formula for the inverse of f.

3. A particle starts at $t=0$ and moves along the $x-$axis so that its position at any time $t \geq 0$ is given by $x(t) = (t-1)^3(2t-3)$.

a) Find the velocity function $v(t)$ of the particle.

b) For what values of t is the object moving to the left?

c) Find all values of t for which the object is moving and the acceleration is zero.

4. We are on the surface of the Moon. The gravitational acceleration there is $g = -1.6\ \frac{\text{m}}{\text{s}^2}$. A rock is thrown vertically upward, from an initial height of 19.2 m, with an initial velocity of $8\ \frac{\text{m}}{\text{s}}$.

a) Find the velocity function $v(t)$ of the object.

b) Find the location function $s(t)$ of the object.

c) Find the maximal height that the rock will reach.

d) How long until the rock hits the ground?

e) What is the velocity of the rock when it hits the ground?

5. A function f has derivative $f'(x) = -18(x+5)^3(x+4)^2(x+2)x^6(2-x)^5(4-x)^2$.

a) Graph f'.

b) Find all critical points of f and classify each of them as a maximum, minimum, or a point of inflection.

c) How many points of inflection does f have?

6. Reconstruct the degree 3 polynomial $P(x)$, given

   a) the points $(-2, 69)$, $(-1, 14)$, $(2, -7)$, and $(1, 6)$ on the graph.

   b) that $P(0) = -2$, $P'(0) = 3$, $P''(0) = 12$, and $P'''(0) = -240$.

7. Find the equation(s) of the tangent line(s) drawn to the graph of $2x^2 + y^2 = 5y - x$ at $x = -2$.

8. A company estimates that the total cost of producing q units is $C(q) = q^3 - 155q^2 + 6375q + 3000$

   a) What is the fixed cost?

   b) At what level of production will the total cost be minimized? What is the minimal cost?

   b) At what level of production will the profit be maximized, provided that we can sell every item we produce, for \$775?

9. Integrate

   a) $\displaystyle\int_1^{14} (2x-1)^{2/3}\, dx$ c) $\displaystyle\int_1^{\infty} (2x-1)^{2/3}\, dx$ e) $\displaystyle\int_3^{\infty} \frac{1}{(2x-1)^2} dx$

   b) $\displaystyle\int x2^x dx$ d) $\displaystyle\int xe^{-3x} dx$ f) $\displaystyle\int_1^{\infty} \frac{x^2}{\sqrt{x^3+2}} dx$

10. Find the average value of the function $f(x) = \ln x$ on the interval $[1, 10]$.

11. Find the area of the region determined by the graphs of $f(x) = x^2 - 16x - 24$ and $g(x) = -x^2 + 16$.

Answers

1. a) $\mathbb{R}$ b) there is no vertical assymptote and f is continuous everywhere.

   c) i) $\frac{1}{2}\ln\left(x^2+1\right) + C$ ii) 0 iii) ∞ d) $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to\infty} f(x) = 0$

   e) relative minimum: $\left(-1, -\frac{1}{2}\right)$; relative maximum: $\left(1, \frac{1}{2}\right)$

   f) The derivative. $f'(x) = \dfrac{-\left(x^2-1\right)}{\left(x^2+1\right)^2}$ changes sign from positive to negative at $x = 1$.

   g) When x is negative, then so is $f(x)$. When x is positive, $f(x)$ is increasing on $[0, 1]$ and decreasing on $[1, \infty)$.

   h) Since the relative extrema are both absolute as well, the range is $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and f is clearly bounded.

   i) $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$ and $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right)$

2. a) $(-\infty, 0) \cup (1, \infty)$ b) $-\dfrac{1}{2}$ c) $f^{-1}(x) = \dfrac{e^x}{e^x - 1}$

Solution for c: We solve for x in $y = \ln\left(\dfrac{x}{x-1}\right)$ and then swap x and y.

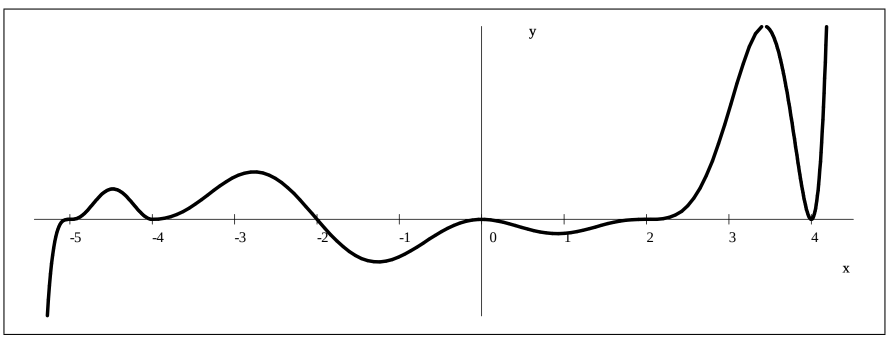
$$\begin{aligned} y &= \ln\left(\frac{x}{x-1}\right) \\ e^y &= \frac{x}{x-1} \\ e^y(x-1) &= x \\ e^y x - e^y &= x \qquad \text{add } e^y, \text{ subtract } x \\ e^y x - x &= e^y \\ x(e^y - 1) &= e^y \\ x &= \frac{e^y}{e^y - 1} \qquad \Longrightarrow \qquad y = \frac{e^x}{e^x - 1} \end{aligned}$$

3. a) $v(t) = (t-1)^2(8t - 11)$ b) $[0, 1) \cup \left(1, \dfrac{11}{8}\right)$ c) $\dfrac{5}{4}$

4. a) $v(t) = -1.6t + 8 = -1.6(t - 5)$ b) $s(t) = -0.8t^2 + 8.0t + 19.2 = -0.8(t+2)(t-12)$

c) $39.2\,\mathrm{m}$ d) $12\,\mathrm{s}$ e) $-11.2\dfrac{\mathrm{m}}{\mathrm{s}}$

5. a) b) 8



6. a) $P(x) = -5x^3 + 7x^2 + x + 3$ b) $P(x) = -40x^3 + 6x^2 + 3x - 2$

7. $y = -7x - 12$ and $y = 7x + 17$

8. a) \$3000 b) \$31 125 when $q = 75$ c) $q = 80$

Solution for c) Find the maximum of $(775q) - (q^3 - 155q^2 + 6375q + 3000)$

9. a) $\dfrac{363}{5}$ b) $2^x\left(\dfrac{x}{\ln 2} - \dfrac{1}{(\ln 2)^2}\right) + C$ c) ∞ d) $-\dfrac{1}{3}xe^{-3x} - \dfrac{1}{9}e^{-3x} + C$

e) $\dfrac{1}{10}$ f) ∞

Solution for b) We will integrate by parts. Let $g(x) = x$ where we will use the formula $\int f'g = fg - \int fg'$. Then clearly $f'(x) = 2^x$. To find $f(x)$, we integrate $\int 2^x dx = \dfrac{2^x}{\ln 2} + C$. Thus we have the following table

$$\begin{aligned} f(x) &= \frac{2^x}{\ln 2} \qquad && g(x) = x \\ f'(x) &= 2^x && g'(x) = 1 \end{aligned}$$

$$\begin{aligned}\int f'g &= gf - \int fg' \\ \int 2^x (x)\,dx &= x\frac{2^x}{\ln 2} - \int \frac{2^x}{\ln 2}dx \\ \int 2^x (x)\,dx &= x\frac{2^x}{\ln 2} - \frac{1}{\ln 2}\int 2^x dx = x\frac{2^x}{\ln 2} - \frac{1}{\ln 2}\frac{2^x}{\ln 2} + C\end{aligned}$$

Solution for d) We integrate by parts. We choose x for $g(x)$. Then $f'(x) = e^{-3x}$ and to find x, we have to integrate. We use substitution. Let $u = -3x \implies du = -3dx \quad dx = \dfrac{du}{-3}$

$$\int e^{-3x}dx = \int e^u \frac{du}{-3} = -\frac{1}{3}\int e^u du = -\frac{1}{3}e^u + C = -\frac{1}{3}e^{-3x} + C$$

Thus $f(x) = -\dfrac{1}{3}e^{-3x}$

$$\begin{aligned} f(x) &= -\frac{1}{3}e^{-3x} & g(x) &= x \\ f'(x) &= e^{-3x} & g'(x) &= 1\end{aligned}$$

$$\begin{aligned}\int f'g &= gf - \int fg' \\ \int e^{-3x}(x)\,dx &= \left(-\frac{1}{3}e^{-3x}\right)x - \int -\frac{1}{3}e^{-3x}dx = -\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}dx \\ &= -\frac{1}{3}xe^{-3x} + \frac{1}{3}\left(-\frac{1}{3}e^{-3x}\right) + C = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C\end{aligned}$$

Solution for e) Let us first solve the indefinite integral. We substitute $u = 2x - 1$

$$\begin{aligned} u &= 2x - 1 \\ du &= 2dx \implies dx = \frac{du}{2}\end{aligned}$$

$$\begin{aligned}\int \frac{1}{(2x-1)^2}dx &= \int \frac{1}{u^2}\frac{du}{2} = \frac{1}{2}\int \frac{1}{u^2}du = \frac{1}{2}\int u^{-2}du = \frac{1}{2}\frac{u^{-1}}{-1} + C \\ &= -\frac{1}{2u} + C = -\frac{1}{2(2x-1)} + C\end{aligned}$$

Now the improper integral is a limit of definite integrals:

$$\begin{aligned}\int_3^\infty \frac{1}{(2x-1)^2}dx &= \lim_{N\to\infty}\int_3^N \frac{1}{(2x-1)^2}dx = \lim_{N\to\infty} -\frac{1}{2(2x-1)}\Big|_3^N \\ &= -\frac{1}{2}\lim_{N\to\infty}\left(\frac{1}{2N-1} - \frac{1}{2(3)-1}\right) = -\frac{1}{2}\left(-\frac{1}{5}\right) = \frac{1}{10}\end{aligned}$$

Solution for f) Let us first solve the indefinite integral. We substitute $u = x^3 + 2$

$$\begin{aligned} u &= x^3 + 2 \\ du &= 3x^2dx \implies dx = \frac{du}{3x^2}\end{aligned}$$

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^3+2}}dx &= \int \frac{x^2}{\sqrt{u}}\frac{du}{3x^2} = \frac{1}{3}\int \frac{1}{\sqrt{u}}du = \frac{1}{3}\int u^{-1/2}du = \frac{1}{3}\frac{u^{1/2}}{\frac{1}{2}} + C \\ &= \frac{2}{3}\sqrt{u} + C = \frac{2}{3}\sqrt{x^3+2} + C\end{aligned}$$

Now the improper integral is a limit of definite integrals:

$$\begin{aligned}\int_1^\infty \frac{x^2}{\sqrt{x^3+2}}dx &= \lim_{N\to\infty}\int_1^N \frac{x^2}{\sqrt{x^3+2}}dx = \lim_{N\to\infty}\frac{2}{3}\sqrt{x^3+2}\Big|_1^N \\ &= \frac{2}{3}\lim_{N\to\infty}\left(\sqrt{N^3+2}-\sqrt{1^3+2}\right) = \infty\end{aligned}$$

10. $\frac{10}{9}\ln 10 - 1$

11. 576