Quiz 11 will cover the following topics: all topics from Quiz 1-10, Exams 1, 2, and integration by substitution (5.2), integration by parts (6.1) and improper integrals (6.3)

Review Problems

Compute each of the following integrals.

1. $\displaystyle\int xe^x \, dx$

2. $\displaystyle\int xe^{-4x} \, dx$

3. $\displaystyle\int \ln x \, dx$

4. $\displaystyle\int x\left(5x^2-2\right)^9 \, dx$

5. $\displaystyle\int_0^1 x\left(5x^2-2\right)^9 \, dx$

6. $\displaystyle\int \frac{1}{(3x-4)^5} \, dx$

7. $\displaystyle\int_2^3 \frac{1}{(3x-4)^5} \, dx$

8. $\displaystyle\int \frac{5x^2}{x^3-1} \, dx$

9. $\displaystyle\int_2^{10} \frac{5x^2}{x^3-1} \, dx$

10. $\displaystyle\int \frac{1}{4x-1} \, dx$

11. $\displaystyle\int_0^1 \frac{1}{4x-1} \, dx$

12. $\displaystyle\int \frac{1}{5}e^{(1/5)x} \, dx$

13. $\displaystyle\int_0^1 \frac{1}{5}e^{(1/5)x} \, dx$

14. $\displaystyle\int \left(5x^3-15x^2\right)\sqrt[5]{x^4-4x^3+1} \, dx$

15. $\displaystyle\int_0^4 \left(5x^3-15x^2\right)\sqrt[5]{x^4-4x^3+1} \, dx$

16. Consider the integral $\displaystyle\int \frac{x^2}{(x^3-8)^4} \, dx$. Compute each of the following.

a) $\displaystyle\int \frac{x^2}{(x^3-8)^4} \, dx$

b) $\displaystyle\int_0^1 \frac{x^2}{(x^3-8)^4} \, dx$

c) $\displaystyle\int_3^N \frac{x^2}{(x^3-8)^4} \, dx$

d) the improper integral $\displaystyle\int_3^\infty \frac{x^2}{(x^3-8)^4} \, dx$ as a limit, namely $\displaystyle\lim_{N\to\infty}\int_3^N \frac{x^2}{(x^3-8)^4} \, dx$

e) the improper integral $\displaystyle\int_0^2 \frac{x^2}{(x^3-8)^4} \, dx$ as a limit, namely $\displaystyle\lim_{a\to 2-}\int_0^a \frac{x^2}{(x^3-8)^4} \, dx$

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Review Problems - Answers

1.) $xe^x - e^x + C$ 2.) $-\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x} + C$ 3.) $x\ln x - x + C$ 4.) $\frac{(5x^2-2)^{10}}{100} + C$

5.) $\frac{2321}{4}$ 6.) $-\frac{1}{12(3x-4)^4} + C$ 7.) $\frac{203}{40\,000}$ 8.) $\frac{5}{3}\ln\left|x^3-1\right| + C$

9.) $\frac{5}{3}\ln 999 - \frac{5}{3}\ln 7$ 10.) $\frac{1}{4}\ln\left|4x-1\right| + C$ 11.) undefined 12.) $e^{(1/5)x} + C$

13.) $e^{1/5} - 1$ 14.) $\frac{25}{24}\left(x^4 - 4x^3 + 1\right)^{6/5} + C$ 15.) 0

16.) a) $\frac{-1}{9(x^3-8)^3} + C$ b) $\frac{169}{1580\,544}$ c) $\frac{1}{61\,731} - \frac{1}{9(N^3-8)^3}$ d) $\frac{1}{61\,731}$ e) ∞

Review Problems - Solutions

Compute each of the following integrals.

1.) $\int xe^x\,dx = \quad xe^x - e^x + C$

Solution: We will integrate this by parts, using the formula

$$\int f'g = fg - \int fg'$$

Let $g(x) = x$ and $f'(x) = e^x$ Then we obtain g' and f by differentiation and integration.

$f(x) = e^x$	$g(x) = x$
$f'(x) = e^x$	$g'(x) = 1$

$$\int f'g = fg - \int fg' \quad \text{becomes}$$

$$\int xe^x\,dx = xe^x - \int e^x dx = xe^x - e^x + C$$

We should check our result by differentiating the answer. Indeed,

$$\left(xe^x - e^x + C\right)' = e^x + xe^x - e^x = xe^x$$

2.) $\int xe^{-4x}\,dx = \quad -\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x} + C$

Solution: Let $g(x) = x$ and $f'(x) = e^{-4x}$ Then we obtain g' and f by differentiation and integration. To compute $f(x)$, we will use substitution. Let $u = -4x$ then $du = -4dx$ and so $dx = \frac{du}{-4}$.

$$f(x) = \int e^{-4x}\,dx = \int e^u\,\frac{du}{-4} = -\frac{1}{4}\int e^u\,du = -\frac{1}{4}e^u + C = -\frac{1}{4}e^{-4x} + C$$

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We will choose $C=0$ and so $f(x)=-\frac{1}{4}e^{-4x}$.

$f(x)=-\frac{1}{4}e^{-4x}$	$g(x)=x$
$f'(x)=e^{-4x}$	$g'(x)=1$

$$\begin{aligned}\int f'g &= fg-\int fg' \quad \text{becomes}\\ \int xe^{-4x}\,dx &= -\frac{1}{4}xe^{-4x}-\int -\frac{1}{4}e^{-4x}\,dx=-\frac{1}{4}xe^{-4x}+\frac{1}{4}\int e^{-4x}\,dx=-\frac{1}{4}xe^{-4x}+\frac{1}{4}\left(-\frac{1}{4}e^{-4x}\right)+C\\ &= -\frac{1}{4}xe^{-4x}-\frac{1}{16}e^{-4x}+C\end{aligned}$$

We check our result by differentiating the answer.

$$\left(-\frac{1}{4}xe^{-4x}-\frac{1}{16}e^{-4x}+C\right)'=$$

$$\begin{aligned}&= -\frac{1}{4}\left(xe^{-4x}\right)'-\frac{1}{16}\left(e^{-4x}\right)'=-\frac{1}{4}\left(e^{-4x}+x\left(-4e^{-4x}\right)\right)-\frac{1}{16}\left(-4e^{-4x}\right)\\ &= -\frac{1}{4}e^{-4x}+xe^{-4x}+\frac{1}{4}e^{-4x}=xe^{-4x}\end{aligned}$$

3.) $\int \ln x\,dx=$ $x\ln x-x+C$

Solution: Let $g(x)=\ln x$ and $f'(x)=1$ Then we obtain g' and f by differentiation and integration.

$f(x)=x$	$g(x)=\ln x$
$f'(x)=1$	$g'(x)=\frac{1}{x}$

$$\begin{aligned}\int f'g &= fg-\int fg' \quad \text{becomes}\\ \int \ln x\,dx &= x\ln x-\int x\cdot\frac{1}{x}\,dx=x\ln x-\int 1\,dx=x\ln x-x+C\end{aligned}$$

We check our result by differentiating the answer.

$$(x\ln x-x+C)'=\ln x+x\cdot\frac{1}{x}-1=\ln x$$

11.) $\int_0^1 \frac{1}{4x-1}\,dx=$ undefined

Solution: since $\frac{1}{4x-1}$ is not continuous at $x=\frac{1}{4}$, this is an improper integral.

$$\int_0^1 \frac{1}{4x-1}\,dx=\int_0^{1/4}\frac{1}{4x-1}\,dx+\int_{1/4}^1\frac{1}{4x-1}\,dx=\lim_{a\to 1/4-}\int_0^a\frac{1}{4x-1}\,dx+\lim_{b\to 1/4+}\int_b^1\frac{1}{4x-1}\,dx$$

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We use substitution to compute the indefinite integral, $\int \frac{1}{4x-1}\,dx = \frac{1}{4}\ln|4x-1| + C$. We now separately compute the improper integrals.

$$\begin{aligned}\int_0^{1/4} \frac{1}{4x-1}\,dx &= \lim_{a\to 1/4-} \int_0^a \frac{1}{4x-1}\,dx = \lim_{a\to 1/4-} \left(\frac{1}{4}\ln|4x-1|\Big|_0^a\right) \\ &= \lim_{a\to 1/4-} \left(\frac{1}{4}\ln|4a-1| - \frac{1}{4}\ln|4(0)-1|\right) = \frac{1}{4}\lim_{a\to 1/4-}(\ln|4a-1| - \ln 1) \\ &= \frac{1}{4}\lim_{a\to 1/4-}(\ln|4a-1|) = -\infty\end{aligned}$$

$$\begin{aligned}\int_{1/4}^{1} \frac{1}{4x-1}\,dx &= \lim_{b\to 1/4+} \int_b^1 \frac{1}{4x-1}\,dx = \lim_{b\to 1/4+} \left(\frac{1}{4}\ln|4x-1|\Big|_b^1\right) \\ &= \lim_{b\to 1/4+} \left(\frac{1}{4}\ln|4(1)-1| - \frac{1}{4}\ln|4b-1|\right) = \frac{1}{4}\lim_{b\to 1/4-}(\ln 3 - \ln|4b-1|) = \infty\end{aligned}$$

This sum $-\infty+\infty$ is an indeterminate, and so the answer is undefined.

14.) $\int (5x^3-15x^2)\sqrt[5]{x^4-4x^3+1}\,dx = \frac{25}{24}(x^4-4x^3+1)^{6/5} + C$

Solution: Let $u = x^4-4x^3+1$ Then $du = (4x^3-12x^2)\,dx$ and so $dx = \frac{du}{4x^3-12x^2} = \frac{du}{4x^2(x-3)}$

$$\int (5x^3-15x^2)\sqrt[5]{x^4-4x^3+1}\,dx =$$

$$\begin{aligned} &= \int (5x^3-15x^2)\sqrt[5]{u}\,\frac{du}{4x^2(x-3)} = \int 5x^2(x-3)\,u^{1/5}\,\frac{du}{4x^2(x-3)} \\ &= \int \frac{5}{4}u^{1/5}\,du = \frac{5}{4}\int u^{1/5}\,du = \frac{5}{4}\cdot\frac{u^{6/5}}{\frac{6}{5}} + C = \frac{5}{4}\cdot\frac{5}{6}u^{6/5} + C \\ &= \frac{25}{24}u^{6/5} + C = \frac{25}{24}(x^4-4x^3+1)^{6/5} + C\end{aligned}$$

We check via differentiation:

$$\left(\frac{25}{24}(x^4-4x^3+1)^{6/5} + C\right)' =$$

$$\begin{aligned} &= \frac{25}{24}\cdot\frac{6}{5}(x^4-4x^3+1)^{1/5}(4x^3-12x^2) = \frac{5}{4}(x^4-4x^3+1)^{1/5}(4)(x^3-3x^2) \\ &= \frac{5}{4}(4)(x^4-4x^3+1)^{1/5}(x^3-3x^2) = 5(x^4-4x^3+1)^{1/5}(x^3-3x^2) \\ &= (5x^3-15x^2)\sqrt[5]{x^4-4x^3+1}\end{aligned}$$

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16.) c) $\displaystyle\int_3^N \frac{x^2}{(x^3-8)^4}\,dx = \frac{1}{61\,731} - \frac{1}{9\,(N^3-8)^3}$

Solution:

$$\int_3^N \frac{x^2}{(x^3-8)^4}\,dx = \left.\frac{-1}{9\,(x^3-8)^3}\right|_3^N = \frac{-1}{9\,(N^3-8)^3} - \frac{-1}{9\,(3^3-8)^3}$$
$$= \frac{-1}{9\,(N^3-8)^3} + \frac{1}{61\,731} = \frac{1}{61\,731} - \frac{1}{9\,(N^3-8)^3}$$

d) $\displaystyle\int_3^\infty \frac{x^2}{(x^3-8)^4}\,dx = \lim_{N\to\infty}\int_2^N \frac{x^2}{(x^3-8)^4}\,dx = \frac{1}{61\,731}$

Solution:

$$\lim_{N\to\infty}\int_2^N \frac{x^2}{(x^3-8)^4}\,dx = \lim_{N\to\infty}\left(\frac{1}{61\,731} - \frac{1}{9\,(N^3-8)^3}\right) = \frac{1}{61\,731}$$

e) $\displaystyle\int_0^2 \frac{x^2}{(x^3-8)^4}\,dx = \lim_{a\to 2-}\int_0^a \frac{x^2}{(x^3-8)^4}\,dx = \infty$

Solution:

$$\lim_{a\to 2-}\int_0^a \frac{x^2}{(x^3-8)^4}\,dx = \lim_{a\to 2-}\left(\frac{-1}{9\,(a^3-8)^3} - \frac{-1}{9\,(0^3-8)^3}\right) = \lim_{a\to 2-}\left(\frac{1}{9\,(-8)^3} - \frac{1}{9\,(a^3-8)^3}\right) = \infty$$

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