1. Based on the picture depicting the graph of a function f, find each of the following limits.

a) $\displaystyle\lim_{x\to -3^-} f(x)$

b) $\displaystyle\lim_{x\to -3^+} f(x)$

c) $\displaystyle\lim_{x\to -3} f(x)$

d) $\displaystyle\lim_{x\to -2^-} f(x)$

e) $\displaystyle\lim_{x\to -2^+} f(x)$

f) $\displaystyle\lim_{x\to -2} f(x)$

g) $\displaystyle\lim_{x\to 0^-} f(x)$

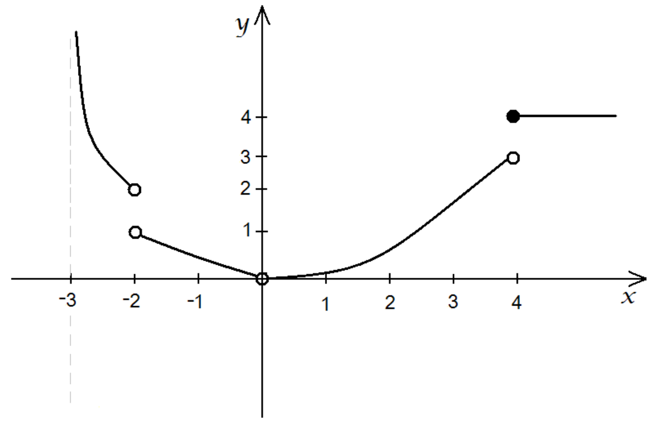
h) $\displaystyle\lim_{x\to 0^+} f(x)$

i) $\displaystyle\lim_{x\to 0} f(x)$

j) $\displaystyle\lim_{x\to 4^-} f(x)$

k) $\displaystyle\lim_{x\to 4^+} f(x)$

l) $\displaystyle\lim_{x\to 4} f(x)$



2. Find the equation for the inverse for each of the following functions given.

a) $f(x) = 3x - 1$

b) $g(x) = \log_2(5x - 1)$

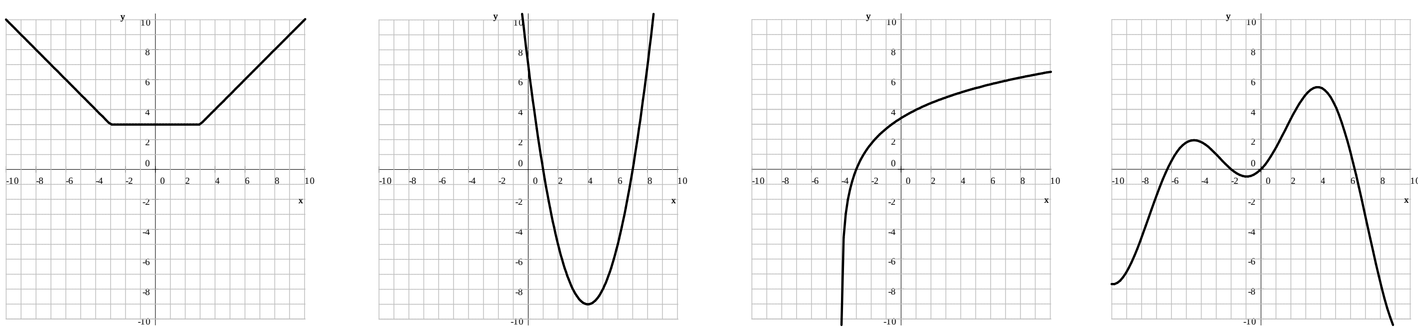
c) $h(x) = 1 + 3e^{2x-7}$

d) $p(x) = (x-2)^3$

e) $m(x) = \dfrac{x-1}{2x+7}$

3. In each case, the graph of a function is given. Sketch the graph of the inverse relation in the same coordinate system.

a) b) c) d)



4. Sketch the graph of and give a complete analysis for each of the following functions.

a) $f(x) = \sin x$ on $(-2\pi, 2\pi)$

b) $g(x) = 2 + \sqrt{4 - x^2}$

5. Compute each of the following limits.

a) $\displaystyle\lim_{x\to\infty} \frac{\cos x}{x}$

b) $\displaystyle\lim_{x\to 3\pi/4} \frac{\cos x}{\sin x}$

c) $\displaystyle\lim_{x\to\infty} \left(\sqrt{2x+1} - \sqrt{x+1}\right)$

d) $\displaystyle\lim_{x\to\infty} \left(\sqrt{x+1} - \sqrt{x-1}\right)$

e) $\displaystyle\lim_{x\to 3} \frac{x-3}{|x-3|}$

f) $\displaystyle\lim_{x\to 8} \frac{3 - \sqrt{x+1}}{8 - x}$

g) $\displaystyle\lim_{x\to\infty} \frac{2^x + 2^{-x}}{2^x - 2^{-x}}$

h) $\displaystyle\lim_{x\to -\infty} \frac{2^x + 2^{-x}}{2^x - 2^{-x}}$

i) $\displaystyle\lim_{x\to\infty} \frac{2^{2x-1} \cdot 3^{x+1}}{12^{x-1}}$

j) $\displaystyle\lim_{x\to -\infty} \tan^{-1} x$

k) $\displaystyle\lim_{x\to\infty} \tan^{-1} x$

6. a) Compute the average velocity of an object between $t_1 = 3\,\text{s}$ and $t_2 = 7\,\text{s}$ is its location function, measured in meters, is given by $L(t) = -t^2 + 6t - 1$

b) The location function of an object, measured in meters, is given by $L(t) = 2t^2 + At - 4$. Compute the value of A if we know that the average velocity of an object between $t_1 = 2\,\text{s}$ and $t_2 = 5\,\text{s}$ is $7\dfrac{\text{m}}{\text{s}}$.

7. Prove each of the following identities. (Assume the sum, difference, and double-angle formulas, especially if you have them memorized. What you don't have memorized, derive.)

a) $\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2=1+\sin x$ b) $1+\tan^2 x=\sec^2 x$ c) $\sin^2\frac{x}{2}=\frac{1-\cos x}{2}$

8. Recall that $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$. Use this fact to compute each of the following limits.

a) $\lim\limits_{x\to 0}\frac{\sin 5x}{x}$ c) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$ e*) $\lim\limits_{\theta\to\pi/4}\frac{\tan\theta-1}{\theta-\frac{\pi}{4}}$

b) $\lim\limits_{x\to 0}\frac{1-\cos x}{x}$ d) $\lim\limits_{x\to 0}\frac{\tan x}{x}$ f*) $\lim\limits_{x\to 0}\frac{\sqrt{1-\cos x}}{x}$

9. a) Find the perimeter of a 15−sided reguar polygon written into a circle with radius 10 m.

b) Find the perimeter of an n−sided regular polygon written into a circle with radius R. Use radians to measure angles.

c) Find the limit of the perimeter of an n−sided regular polygon written into a circle with radius R as n approaches infinity. Use radians to measure angles.

10. a) Find the area of a 15−sided regular polygon written into a circle with radius 10 m.

b) Find the area of an n−sided regular polygon written into a circle with radius R. Use radians to measure angles.

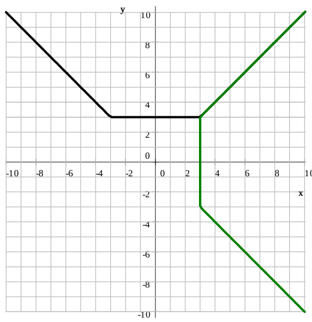
c) Find the limit of the area of an n−sided regular polygon written into a circle with radius R as n approaches infinity. Use radians to measure angles.

Answers

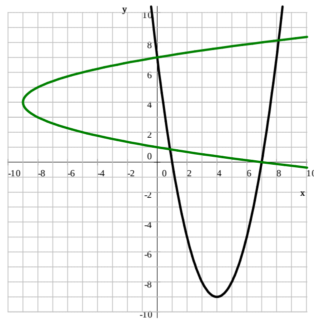
1. a) undefined b) ∞ c) undefined d) 2 e) 1 f) undefined g) 0 h) 0 i) 0
   j) 3 k) 4 l) undefined

2. a) $f^{-1}(x) = \frac{1}{3}x + \frac{1}{3}$ b) $g^{-1}(x) = \frac{1}{5}(2^x + 1)$ c) $h^{-1}(x) = \frac{1}{2}\ln\left(\frac{x-1}{3}\right) + \frac{7}{2}$ d) $p^{-1}(x) = \sqrt[3]{x} + 2$
   e) $m^{-1}(x) = \dfrac{7x+1}{-2x+1}$

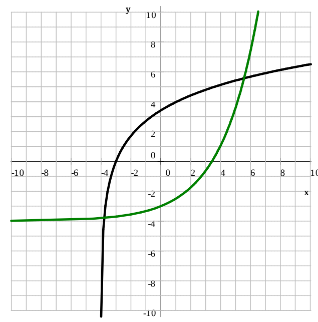
3. a)



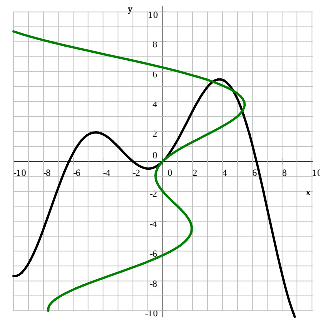
b)



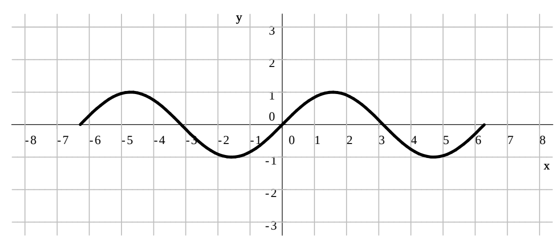
c)



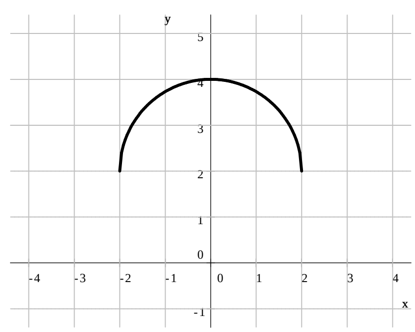
d)



4. a) $f(x) = \sin x$ on $(-2\pi, 2\pi)$
   domain: $(-2\pi, 2\pi)$
   range: $[-1, 1]$
   $y-$intercept: $(0,0)$
   $x-$intercept: $(-\pi, 0)$, $(0,0)$, $(\pi, 0)$
   maximum: $\left(\frac{\pi}{2}, 1\right)$, $\left(-\frac{3\pi}{2}, 1\right)$
   minimum: $\left(-\frac{\pi}{2}, -1\right)$, $\left(\frac{3\pi}{2}, -1\right)$
   one-to-one: no



   b) $g(x) = 2 + \sqrt{4 - x^2}$
   domain: $[-2, 2]$
   range: $[2, 4]$
   $y-$intercept: $(0, 4)$
   $x-$intercept: none
   maximum: $(0, 4)$
   minimum: $(-2, 2)$ and $(2, 2)$
   one-to-one: no



5. a) 0 b) -1 c) ∞ d) 0 e) undefined f) $\frac{1}{6}$ g) 1 h) -1 i) 18 j) $-\frac{\pi}{2}$ k) $\frac{\pi}{2}$

6. a) $-4\frac{\text{m}}{\text{s}}$ b) -7

7. a) $\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2 = 1 + \sin x$

$$\text{LHS} = \sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} = 1 + \sin x = \text{RHS}$$

b) $1+\tan^2 x = \sec^2 x$

$$\text{LHS} = 1+\tan^2 x = 1+\frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x}+\frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x = \text{RHS}$$

c) $\sin^2 \dfrac{x}{2} = \dfrac{1-\cos x}{2}$

We start with the double-angle formula for cosine, the one that involves $\sin^2 x$. Only, let us use a different letter - we will need x later.

$$\begin{aligned} \cos 2\alpha &= 1-2\sin^2\alpha \quad \text{solve for } \sin^2\alpha \\ 2\sin^2\alpha &= 1-\cos 2\alpha \\ \sin^2\alpha &= \frac{1-\cos 2\alpha}{2} \end{aligned}$$

Now we introduce a new variable: Let $x = 2\alpha$. Then clearly $\dfrac{x}{2} = \alpha$. Then our equation becomes

$$\sin^2 \frac{x}{2} = \frac{1-\cos x}{2}$$

8. a) $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{x} = 5$

$$\lim_{x\to 0}\frac{\sin 5x}{x} = \lim_{x\to 0}\frac{\sin 5x}{x}\cdot 1 = \lim_{x\to 0}\frac{\sin 5x}{x}\cdot\frac{5}{5} = \lim_{x\to 0}\frac{\sin 5x}{5x}\cdot 5 = 5\lim_{x\to 0}\frac{\sin 5x}{5x}$$

Let $y = 5x$. As x approaches zero, so does y. So the limit becomes

$$5\lim_{x\to 0}\frac{\sin 5x}{5x} = 5\lim_{y\to 0}\frac{\sin y}{y} = 5\cdot 1 = 5$$

b) $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x} = 0$

$$\begin{aligned} \lim_{x\to 0}\frac{1-\cos x}{x} &= \lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x} = \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)} = \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}\right) \\ &= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{1+\cos x} = 1\cdot 0 = 0 \end{aligned}$$

c) $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$

$$\begin{aligned} \lim_{x\to 0}\frac{1-\cos x}{x^2} &= \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x} = \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)} = \lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x}\right) \\ &= \lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\lim_{x\to 0}\frac{1}{1+\cos x} = \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2\cdot\lim_{x\to 0}\frac{1}{1+\cos x} = 1^2\cdot\frac{1}{2} = \frac{1}{2} \end{aligned}$$

d) $\displaystyle\lim_{x\to 0}\frac{\tan x}{x} = 1$

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{x} = \lim_{x\to 0}\frac{\sin x}{x\cos x} = \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right) = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x} = 1\cdot 1 = 1$$

e*) $\displaystyle\lim_{\theta\to\pi/4}\frac{\tan\theta-1}{\theta-\frac{\pi}{4}}=2$

This is clearly a $\frac{0}{0}$ type of an indeterminate. To simplify the denominator, we will introduce a new variable. Let $x=\theta-\frac{\pi}{4}$. As θ approaches $\frac{\pi}{4}$, x will approach zero. Also, solving $x=\theta-\frac{\pi}{4}$ for θ we get $\theta=x+\frac{\pi}{4}$. So our limit becomes

$$\lim_{\theta\to\pi/4}\frac{\tan\theta-1}{\theta-\frac{\pi}{4}}=\lim_{x\to 0}\frac{\tan\left(x+\frac{\pi}{4}\right)-1}{x}$$

Now the denominator is simple, but the numerator became more complex. We will expand $\tan\left(x+\frac{\pi}{4}\right)$ using the sum formula for tangent.

$$\begin{aligned}\lim_{x\to 0}\frac{\tan\left(x+\frac{\pi}{4}\right)-1}{x} &= \lim_{x\to 0}\frac{\dfrac{\tan x+\tan\frac{\pi}{4}}{1-\tan x\tan\frac{\pi}{4}}-1}{x}=\lim_{x\to 0}\frac{\dfrac{\tan x+1}{1-\tan x\cdot 1}-1}{x}=\lim_{x\to 0}\frac{\dfrac{\tan x+1}{1-\tan x}-\dfrac{1-\tan x}{1-\tan x}}{x}\\ &= \lim_{x\to 0}\frac{1}{x}\cdot\frac{\tan x+1-(1-\tan x)}{1-\tan x}=\lim_{x\to 0}\frac{1}{x}\cdot\frac{\tan x+1-1+\tan x}{1-\tan x}=\lim_{x\to 0}\frac{1}{x}\cdot\frac{2\tan x}{1-\tan x}\\ &= \lim_{x\to 0}\frac{\tan x}{x}\cdot\frac{2}{1-\tan x}=\lim_{x\to 0}\frac{\tan x}{x}\cdot\lim_{x\to 0}\frac{2}{1-\tan x}=1\cdot 2=2\end{aligned}$$

f*) $\displaystyle\lim_{x\to 0}\frac{\sqrt{1-\cos x}}{x}=\text{undefined}$

This is also a $\frac{0}{0}$ type of an indeterminate. Recall that $\sin^2\frac{x}{2}=\frac{1-\cos x}{2}$ and so $1-\cos x=2\sin^2\frac{x}{2}$.

$$\lim_{x\to 0}\frac{\sqrt{1-\cos x}}{x}=\lim_{x\to 0}\frac{\sqrt{2\sin^2\frac{x}{2}}}{x}=\lim_{x\to 0}\frac{\left|\sqrt{2}\sin\frac{x}{2}\right|}{x}$$

We will need to be a little bit careful because of the absolute value. If x is positive (recall it is also very close to zero) then so is $\sin\frac{x}{2}$. If x is negative, so is $\sin\frac{x}{2}$. We will separately evaluate the left-side and right-side limits. Let us introduce the new variable $y=\frac{x}{2}$:

$$\begin{aligned}\lim_{x\to 0^+}\frac{\sqrt{2}\left|\sin\frac{x}{2}\right|}{x} &= \lim_{x\to 0^+}\frac{\sqrt{2}\sin\frac{x}{2}}{x\cdot 1}=\sqrt{2}\lim_{x\to 0^+}\frac{\sin\frac{x}{2}}{x\cdot 1}=\sqrt{2}\lim_{x\to 0^+}\frac{\sin\frac{x}{2}}{x\cdot\frac{2}{2}}=\sqrt{2}\lim_{x\to 0^+}\frac{\sin\frac{x}{2}}{\frac{x}{2}\cdot 2}=\sqrt{2}\lim_{x\to 0^+}\frac{1}{2}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}}\\ &= \frac{\sqrt{2}}{2}\lim_{y\to 0^+}\frac{\sin y}{y}=\frac{\sqrt{2}}{2}\cdot 1=\frac{\sqrt{2}}{2}\end{aligned}$$

The other side goes similarly:

$$\begin{aligned}\lim_{x\to 0^-}\frac{\sqrt{2}\left|\sin\frac{x}{2}\right|}{x} &= \lim_{x\to 0^+}\frac{\sqrt{2}\left(-\sin\frac{x}{2}\right)}{x\cdot 1}=-\sqrt{2}\lim_{x\to 0^+}\frac{\sin\frac{x}{2}}{x\cdot 1}=-\sqrt{2}\lim_{x\to 0^+}\frac{\sin\frac{x}{2}}{x\cdot\frac{2}{2}}=-\sqrt{2}\lim_{x\to 0^+}\frac{\sin\frac{x}{2}}{\frac{x}{2}\cdot 2}=-\sqrt{2}\lim_{x\to 0^+}\frac{1}{2}\cdot\\ &= -\frac{\sqrt{2}}{2}\lim_{y\to 0^+}\frac{\sin y}{y}=-\frac{\sqrt{2}}{2}\cdot 1=-\frac{\sqrt{2}}{2}\end{aligned}$$

Since the right-hand side limit and the left-hand side limit are different, the two-sided limit is undefined.

9. a) $(15)\,(2)\,(10\,\mathrm{m})\left(\sin\dfrac{2\pi}{30}\right) = 300\left(\sin\dfrac{\pi}{15}\right)\ \mathrm{m} \approx 62.373\,507\,245\,327\,8\,\mathrm{m}$

b) $2nR\sin\dfrac{\pi}{n}$

c) $\displaystyle\lim_{n\to\infty}\left(2nR\sin\frac{\pi}{n}\right) = 2R\lim_{n\to\infty}\left(\frac{\sin\frac{\pi}{n}}{\frac{1}{n}}\right) = 2R\lim_{n\to\infty}\left(\frac{\pi}{\pi}\cdot\frac{\sin\frac{\pi}{n}}{\frac{1}{n}}\right) = 2\pi R\lim_{n\to\infty}\left(\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\right)$

Define $x = \dfrac{\pi}{n}$. As $n\to\infty$, clearly $x\to 0$. Thus

$$2\pi R\lim_{n\to\infty}\left(\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\right) = 2\pi R\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 2\pi R\cdot 1 = 2\pi R$$

10. a) $\dfrac{1}{2}\,(15)\,(10\,\mathrm{m})^2\sin\dfrac{2\pi}{15} = 305.052\,482\,306\,85\,\mathrm{m}^2$

b) $n\left(R\sin\left(\dfrac{\pi}{n}\right)\right)\left(R\cos\left(\dfrac{\pi}{n}\right)\right) = \dfrac{1}{2}nR^2\sin\dfrac{2\pi}{n}$

c) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{2}nR^2\sin\frac{2\pi}{n}\right) = \lim_{n\to\infty}\left(R^2\frac{\sin\frac{2\pi}{n}}{\frac{2}{n}}\right) = \lim_{n\to\infty}\left(\pi R^2\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right) = \pi R^2\lim_{n\to\infty}\left(\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right)$

Define $x = \dfrac{2\pi}{n}$. As $n\to\infty$, clearly $x\to 0$. Thus

$$\pi R^2\lim_{n\to\infty}\left(\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right) = \pi R^2\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = \pi R^2\cdot 1 = \pi R^2$$