Please note that Quiz 5 will also cover material from Quiz 4 Review and Exam 1 Review. Study those topics as well.

1. Find the formula for the inverse for each of the following functions.

   a) $f(x) = 5 - \sqrt[3]{2x-1}$ b) $f(x) = e^{3x-1}$ c) $f(x) = \dfrac{3x-5}{x+6}$ d) $f(x) = \dfrac{2x-1}{3x-2}$

2. Solve each of the following inequalities.

   a) $x^2 + 4 > 6x$ b) $x^2 \le 6x$ c) $x^2 - 6x \le -11$ d) $4x - 1 \ge 4x^2$

3. Sketch the graph of each of the following functions.

   a) $f(x) = -(x+3)(x-1)^2$ b) $f(x) = (x+3)x^2(x-2)$ c) $f(x) = 9x - x^3$

4. Compute each of the following deriatives by taking the limit of the differential quotient.

   a) $f(x) = x^2$ c) $f(x) = x^3$ e) $f(x) = \dfrac{1}{x}$ g) $f(x) = \sin x$

   b) $f(x) = mx + b$ d) $f(x) = x^4$ f) $f(x) = \sqrt{x}$ h) $f(x) = \cos x$

5. Differentiate each of the following.

   a) $f(x) = 5x^4 - x^3 + 5x^2 - x - 1$ c) $f(x) = x^8 - 1$

   b) $f(x) = \sqrt{x^3} - \sqrt[5]{x^4}$ d) $f(x) = \sqrt{x} - \sqrt[3]{x} + \sqrt[4]{x}$

6. In each case, find an equation for the tangent line drawn to the graph of the function given, at the point given.

   a) $f(x) = x^3 - 2x^2 - x + 2$ at $x = 2$ c) $f(x) = 4x^3 - x^2 + x + 8$ at $x = 0$

   b) $f(x) = 2\sqrt{x} - 1$ at $x = 9$ d) $f(x) = -\dfrac{1}{x^2}$ at $x = 3$

7. Find the equation of all tangent lines drawn to the graph of $y = \dfrac{1}{x}$ with slope $-\dfrac{1}{9}$.

8. Assume that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ and compute each of the following limits.

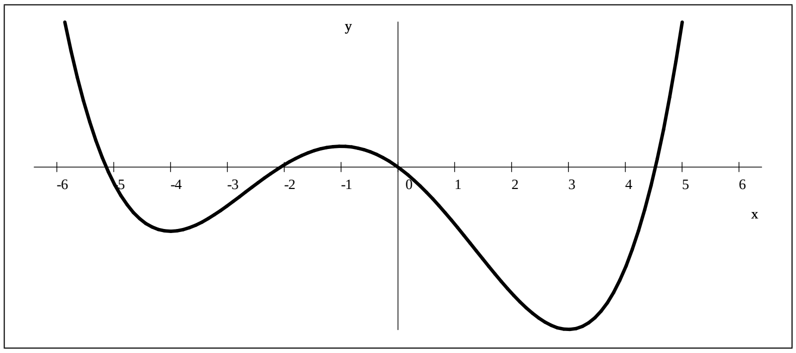
   a) $\lim\limits_{x\to 0} \dfrac{\sin x}{3x}$ b) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x}$ c) $\lim\limits_{x\to 0} \dfrac{\tan x}{x}$ d) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x}$ e) $\lim\limits_{x\to 0} \dfrac{x^2}{1-\cos x}$

9. In case of each of the following functions given, give a complete analysis of the function and then sketch the graph of the function.

   a) $f(x) = \log_3 x$ on $(0, \infty)$ b) $f(x) = \log_3 x$ on $[3, 81]$

10. The picture below shows the graph of f, where f is a function defined on domain $[-6, 6]$.

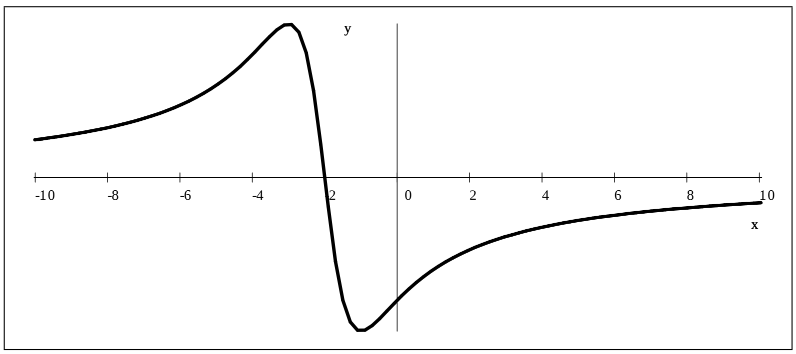
    a) On what intervals is f increasing? b) On what intervals is f decreasing?



11. The picture below shows the graph of f', the first derivative of f, where f is a function defined on domain $[-10, 10]$.

    a) On what intervals is f increasing? b) On what intervals is f decreasing?

    (Hint: you are looking at a velocity function and being asked about the location function.)



$y = f'(x)$

12. Suppose that the location function of an object is given $L(t) = t^3 + At^2 + Bt - 8$. Find the values of A and B if we know that $L(1) = -11$ and $v(2) = -7$.

13. a) Find an equation for the tangent line drawn to the graph of $f(x) = -2x^2 + 5x - 1$ at $x = -1$

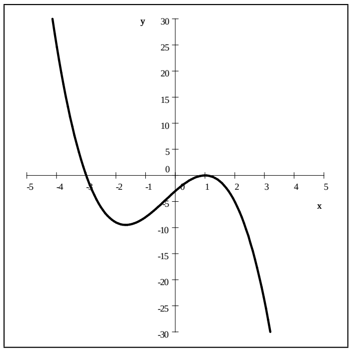
    b*) Find an equation for all tangent lines drawn to the graph of $y = -\frac{1}{2}x^2 + 3x + 5$ from the point $P(-1, 6)$. (Note: what makes this problem so diffferent from the previous one is the fact that P is NOT a point on the graph.)

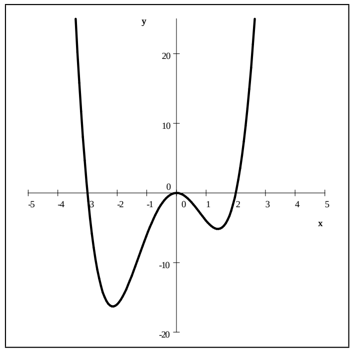
Answers

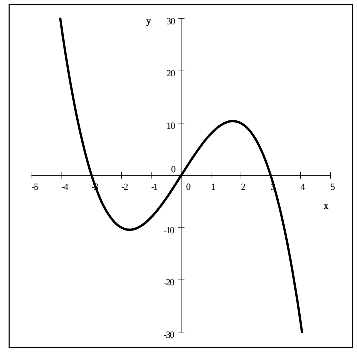
1. a) $f^{-1}(x) = \dfrac{1}{2}(5-x)^3 + \dfrac{1}{2}$ b) $f^{-1}(x) = \dfrac{1}{3}(\ln x + 1)$ d) $f^{-1}(x) = \dfrac{6x+5}{-x+3}$ d) $f^{-1}(x) = \dfrac{2x-1}{3x-2}$

2. a) $x < 3 - \sqrt{5}$ or $x > 3 + \sqrt{5}$ b) $0 \le x \le 6$ c) no solution d) $x = \dfrac{1}{2}$

3. a) $f(x) = -(x+3)(x-1)^2$ b) $f(x) = (x+3)x^2(x-2)$ c) $f(x) = 9x - x^3$







4. Compute each of the following deriatives by taking the limit of the differential quotient.

a) $f(x) = x^2$ - see handout Differentiation 1 - Proofs Claim 2A

b) $f(x) = mx + b$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[m(x+h)+b]-[mx+b]}{h} = \lim_{h\to 0}\frac{mx+mh+b-mx-b}{h} \\ &= \lim_{h\to 0}\frac{mh}{h} = \lim_{h\to 0} m = m \end{aligned}$$

c) $f(x) = x^3$ - see handout Differentiation 1 - Proofs Claim 2B

d) $f(x) = x^4$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^4-x^4}{h} = \lim_{h\to 0}\frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4-x^4}{h} \\ &= \lim_{h\to 0}\frac{4x^3h+6x^2h^2+4xh^3+h^4}{h} = \lim_{h\to 0}\frac{\cancel{h}\left(4x^3+6x^2h+4xh^2+h^3\right)}{\cancel{h}} \\ &= \lim_{h\to 0}\left(4x^3+6x^2h+4xh^2+h^3\right) = 4x^3 \end{aligned}$$

e) $f(x) = \dfrac{1}{x}$ - see handout Differentiation 1 - Proofs Claim 2C

f) $f(x) = \sqrt{x}$ - see handout Differentiation 1 - Proofs Claim 2D

g) $f(x) = \sin x$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\ &= \lim_{h\to 0}\left(\frac{\sin x\cos h-\sin x}{h}+\frac{\cos x\sin h}{h}\right) = \lim_{h\to 0}\sin x\cdot\frac{\cos h-1}{h}+\lim_{h\to 0}\cos x\cdot\frac{\sin h}{h} \\ &= \lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\frac{\cos h-1}{h}+\lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h} = \sin x\cdot 0+\cos x\cdot 1 = \cos x \end{aligned}$$

h) $f(x) = \cos x$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0} \frac{\cos x\cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h\to 0}\left(\frac{\cos x\cos h - \cos x}{h} - \frac{\sin x \sin h}{h}\right) = \lim_{h\to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \cdot \frac{\sin h}{h} \\ &= \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\sin h}{h} = \cos x \cdot 0 - \sin x \cdot 1 = -\sin x \end{aligned}$$

5. a) $f'(x) = 20x^3 - 3x^2 + 10x - 1$ b) $f'(x) = \frac{3}{2}x^{1/2} - \frac{4}{5}x^{-1/5} = \frac{3}{2}\sqrt{x} - \frac{4}{5\sqrt[5]{x}}$

   c) $f'(x) = 8x^7$ d) $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{3}x^{-2/3} + \frac{1}{4}x^{-3/4} = \frac{1}{2\sqrt{x}} - \frac{1}{3\sqrt[3]{x^2}} + \frac{1}{4\sqrt[3]{x^4}}$

6. a) $y = 3x - 6$ b) $\frac{1}{3}(x-9) = y - 5$ c) $y = x + 8$ d) $\frac{2}{27}(x-3) = y + \frac{1}{9}$

7. $y = -\frac{1}{9}x + \frac{2}{3}$ and $y = -\frac{1}{9}x - \frac{2}{3}$

8. Assume that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

   a) $\lim\limits_{x\to 0} \frac{\sin x}{3x} = \frac{1}{3} \lim\limits_{x\to 0} \frac{\sin x}{x} = \frac{1}{3} \cdot 1 = \frac{1}{3}$

   b) $\lim\limits_{x\to 0} \frac{\sin 3x}{x} = \lim\limits_{x\to 0}\left(\frac{\sin 3x}{x} \cdot \frac{3}{3}\right) = \lim\limits_{x\to 0}\left(\frac{\sin 3x}{3x} \cdot \frac{3}{1}\right) = \lim\limits_{x\to 0}\left(\frac{\sin 3x}{3x} \cdot 3\right) = 3 \lim\limits_{x\to 0} \frac{\sin 3x}{3x}$

   Let $y = 3x$. When x approaches zero, so does y.

   $3 \lim\limits_{x\to 0} \frac{\sin 3x}{3x} = 3 \lim\limits_{y\to 0} \frac{\sin y}{y} = 3 \cdot 1 = 3$

   c) $\lim\limits_{x\to 0} \frac{\tan x}{x} = \lim\limits_{x\to 0} \frac{\sin x}{x \cos x} = \lim\limits_{x\to 0} \frac{\sin x}{x} \cdot \lim\limits_{x\to 0} \frac{1}{\cos x} = 1 \cdot 1 = 1$

   d) $\lim\limits_{x\to 0} \frac{1-\cos x}{x} = \lim\limits_{x\to 0}\left(\frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}\right) = \lim\limits_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim\limits_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$

   $= \lim\limits_{x\to 0} \frac{\sin x}{x} \cdot \lim\limits_{x\to 0} \frac{\sin x}{1+\cos x} = 1 \cdot 0 = 0$

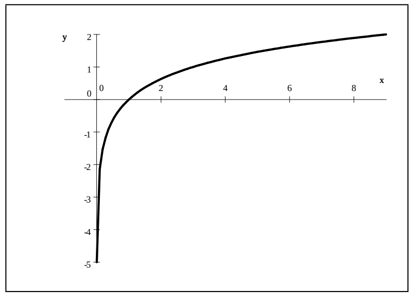
   e) $\lim\limits_{x\to 0} \frac{x^2}{1-\cos x} = \lim\limits_{x\to 0} \frac{x^2}{1-\cos x} \cdot \frac{1+\cos x}{1+\cos x} = \lim\limits_{x\to 0} \frac{x^2(1+\cos x)}{1-\cos^2 x} = \lim\limits_{x\to 0} \frac{x^2(1+\cos x)}{\sin^2 x}$

   $= \lim\limits_{x\to 0} \frac{x}{\sin x} \cdot \frac{x}{\sin x} \cdot (1+\cos x) = \lim\limits_{x\to 0} \frac{x}{\sin x} \cdot \lim\limits_{x\to 0} \frac{x}{\sin x} \cdot \lim\limits_{x\to 0} (1+\cos x) = 1 \cdot 1 \cdot 2 = 2$

9. In case of each of the following functions given, give a complete analysis of the function and then sketch the graph of the function.

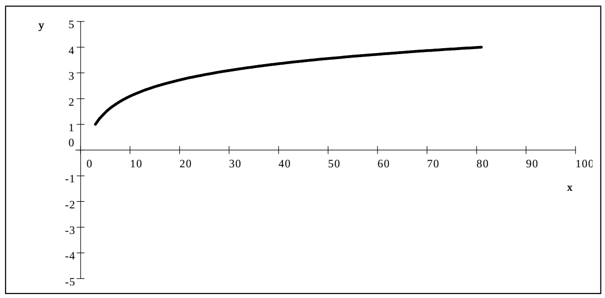
   a) $f(x) = \log_3 x$ on $(0, \infty)$

domain: $(0,\infty)$	$y-$intercept: none	maximum: none	one-to-one
range: $(-\infty,\infty)$	$x-$intercept: $(1,0)$	minimum: none	



b) $f(x) = \log_3 x$ on $[3, 81]$

domain: $[3, 81]$
range: $[1, 4]$
$y-$intercept: none
$x-$intercept: none
maximum: $(81, 4)$
minimum: $(3, 1)$
one-to-one



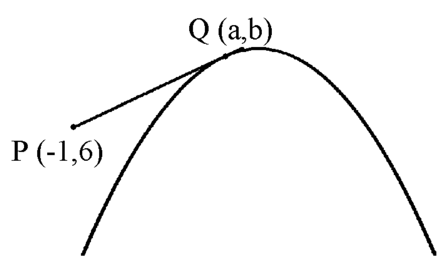
10. a) on $(-4, -1)$ and on $(3, 6)$ b) on $(-6, -4)$ and on $(-1, 3)$

11. a) on $(-10, -2)$ b) on $(-2, 10)$

12. $A = -5$ and $B = 1$

13. a) $y = 9x + 1$ b) $y = x + 7$ and $y = 7x + 13$

Solution for b): Let us denote the point of tangency by $Q(a, b)$. We express the slope between P and Q.



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{b - 6}{a - (-1)} = \frac{b - 6}{a + 1}$$

Since Q is a point on the parabole, we have $b = -\frac{1}{2}a^2 + 3a + 5$. So we now have

$$m = \frac{b - 6}{a + 1} = \frac{-\frac{1}{2}a^2 + 3a + 5 - 6}{a + 1} = \frac{-\frac{1}{2}a^2 + 3a - 1}{a + 1} = \frac{-a^2 + 6a - 2}{2a + 2}$$

The line is a tangent line to $f(x) = -\frac{1}{2}x^2 + 3x + 5$. The slope of the tangent line drawn to the graph of f at $Q(a,b)$ is the derivative of f, evaluated at $x = a$.

$$\begin{aligned} f(x) &= -\frac{1}{2}x^2 + 3x + 5 \\ f'(x) &= -x + 3 \\ f'(a) &= -a + 3 \quad \Longrightarrow \quad \text{this is the slope of line segment } PQ \end{aligned}$$

$$m = \frac{-a^2 + 6a - 2}{2a + 2} \quad \text{becomes} \quad -a + 3 = \frac{-a^2 + 6a - 2}{2a + 2}$$

We solve this for a.

$$\begin{aligned} -a + 3 &= \frac{-a^2 + 6a - 2}{2a + 2} \\ (-a + 3)(2a + 2) &= -a^2 + 6a - 2 \\ -2a^2 + 4a + 6 &= -a^2 + 6a - 2 \\ 0 &= a^2 + 2a - 8 \\ 0 &= (a + 4)(a - 2) \quad \Longrightarrow \quad a_1 = -4 \quad \text{and} \quad a_2 = 2 \end{aligned}$$

Recall that $b = -\frac{1}{2}a^2 + 3a + 5$.

$$b_1 = -\frac{1}{2}(-4)^2 + 3(-4) + 5 = -15 \text{ and } b_2 = -\frac{1}{2} \cdot 2^2 + 3 \cdot 2 + 5 = 9$$

So there are two tangent lines, and the points of tangencies are $Q_1(-4, -15)$ and $Q_2(2, 9)$. Recall that the slope of the tangent line is $m = -a + 3$. So $m_1 = 4 + 3 = 7$ and $m_2 = -2 + 3 = 1$. So the point slope form of the two tangent lines are

$$\begin{aligned} Q(-4, -15) \text{ and } m &= 7 \\ 7(x + 4) &= y + 15 \quad \Longrightarrow \quad y = 7x + 13 \end{aligned}$$

$$\begin{aligned} Q(2, 9) \text{ and } m &= 1 \\ 1(x - 2) &= y - 9 \quad \Longrightarrow \quad y = x + 7 \end{aligned}$$

So the tangent lines are $y = 7x + 13$ and $y = x + 7$.