Students must be able to correctly **state** the following theorems:
Intermediate Value Theorem (both forms), Extreme Value Theorem, Rolle's Theorem, Mean Value Theorem, the second derivative test.

Students must be able to prove the following theorems:

- Differentiating functions using the definition (limit of the differential quotient)
- Prove that $\dfrac{d}{dx}(\sin x) = \cos x$ and $\dfrac{d}{dx}(\cos x) = -\sin x$ and $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$
- If a function is differentiable at a number x, then it is continuous there.
- The product rule and quotient rule for derivatives.
- The Mean Value Theorem.

1. Prove that the function $f(x) = x^6 + x^4 - 1$ has at least one zero.
2. Find all values of c that satisfy the conclusion of the Mean Value Theorem.

   a) $f(x) = x + \dfrac{1}{x}$ on $[1, 6]$

   b) $f(x) = \sqrt{x}$ on $[4, 9]$

   c) $f(x) = \sin x$ on $\left[0, \dfrac{\pi}{2}\right]$

   d) $f(x) = \ln x$ on $[2, 8]$

   e) $f(x) = x^3 - 2x + 1$ on $[1, 4]$

   f) $f(x) = \dfrac{1}{x^2}$ on $[1, 3]$

3. Let $f(x) = \dfrac{1}{x}$. Show that there is no value of c in $(-1, 2)$ such that $f'(c) = \dfrac{f(2) - f(-1)}{2 - (-1)}$. Is this a contradiction to the Mean Value Theorem?
4. Find all relative and absolute maximums and minimums for each of the following functions.

   a) $f(x) = 3x^5 - 20x^3 - 6$ on $[-3, 3]$

   b) $f(x) = x^3 - 2x^2 - 15x + 8$ on $[-4, 5]$

   c) $f(x) = 2x^6 - 3x^4 + 1$ on $[-1, 1]$

   d) $f(x) = -3x^4 - 8x^3 + 48x^2 + 60$ on $[-5, 3]$

5. Find two nonnegative numbers whose sum is 9 and so that the product of one number and the square of the other number is the greatest possible.
6. An open rectangular box with square base is to be made from $60\,\text{ft}^2$ of material. What dimensions will result in a box with the largest possible volume?
7. A container in the shape of a right circular cylinder with no top has surface area $10\,\text{ft}^2$. What height h and base radius r will maximize the volume of the cylinder ?
8. We want to make a flower bed in the shape of a sector in a circle. If we want the area of the garden to be $100\,\text{ft}^2$, what dimensions would guarantee the least perimeter?
9. Compute each of the following limits.

   a) $\displaystyle\lim_{x\to\infty} \frac{3^{2x-1}}{2^{3x+1}}$

   b) $\displaystyle\lim_{x\to\infty} \frac{3^{2x+1}}{2^{4x-1}}$

   c) $\displaystyle\lim_{x\to\infty} \frac{2^{x-1} - 2^{-x}}{2^{x+1} + 2^{-x}}$

   d) $\displaystyle\lim_{x\to\infty} x\left(\frac{1}{2} - \frac{1}{2 + \frac{1}{x}}\right)$

e) $\lim\limits_{x\to\infty}\left(5^{x+1}-5^{x}\right)$

f) $\lim\limits_{x\to\infty}\dfrac{\cos^2 x}{x}$

g) $\lim\limits_{x\to\infty}\left(\sqrt{2x}-\sqrt{x}\right)$

h) $\lim\limits_{x\to\infty}\dfrac{3\cdot 2^{2x}-2^{x}+1}{-4^{x}+2^{x}-1}$

i) $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$

10. Compute each of the following limits. Assume that $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}=e$ and $\lim\limits_{x\to-\infty}\left(1+\dfrac{1}{x}\right)^{x}=e$.

a) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)$

b) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{3x}$

c) $\lim\limits_{x\to\infty}\left(1+\dfrac{4}{x}\right)^{x}$

d) $\lim\limits_{x\to\infty}\left(1-\dfrac{1}{x}\right)^{x}$

e) $\lim\limits_{x\to\infty}\left(1-\dfrac{2}{x+1}\right)^{x}$

f) $\lim\limits_{x\to\infty}\left(1-\dfrac{4}{x}\right)^{2x}$

11. Differentiate each of the following.

a) $f(x)=-4x^3\cos x+\dfrac{\sin x}{x}$

b) $f(x)=\sqrt{x^5}\left(\cos^2 x+\sin^2 x\right)$

c) $f(x)=(2\sin x-\cos x+1)\left(3x^2-x+1\right)$

d) $f(x)=\dfrac{1}{3}x^3\ln x-\dfrac{1}{9}x^3-3$

e) $f(x)=\dfrac{3x^2-8x+2}{\sin x}$

f) $f(\theta)=\tan\theta$

g) $f(x)=\dfrac{\ln x}{\sqrt{x}}$

12. Find an equation for the tangent line drawn to the graph of $f(x)=\cos x$ at $x=\dfrac{\pi}{3}$.

13. a) Find the values of a and b so that $f(x)=2x^3-ax^2+bx+4$ has a relative maximum at $x=-2$ and a relative minimum at $x=5$.

b) Find the value of a if $f(x)=\dfrac{(x-a)(x-2)}{x^2}$ has a relative minimum at $x=3$.

14. a) State and prove the quotient rule for differentiation.

b) State and prove the Mean Value Theorem. (You may assume Rolle's Theorem.)

Answers

1. Claim: the function $f(x) = x^6 + x^4 - 1$ has at least one zero.

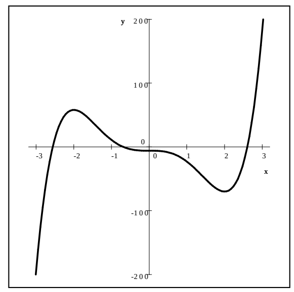
   Proof: Since f is a polynomial, it is continuous on $\mathbb{R}$. $f(0) = -1$ and $f(1) = 1$ and so by the Intermediate Value Theorem, f has a zero between 0 and 1.

2. a) $\sqrt{6}$ b) $\dfrac{25}{4}$ c) $\cos^{-1}\left(\dfrac{2}{\pi}\right) \approx 0.880\,69$ d) $\dfrac{6}{\ln 4} \approx 4.328\,085$ e) $\sqrt{7}$ f) $\sqrt[3]{\dfrac{9}{2}}$

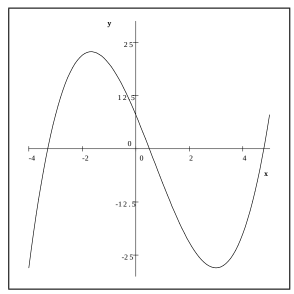
3. $\dfrac{f(2) - f(-1)}{2 - (-1)} = \dfrac{1}{2}$ and $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$ We solve $-\dfrac{1}{x^2} = \dfrac{1}{2}$ and find that this equation has no real solution.

   This is not a contradiction to the Mean Value Theorem because f is NOT differentiable on $(-1, 1)$ and NOT continuous on $[-1, 2]$.

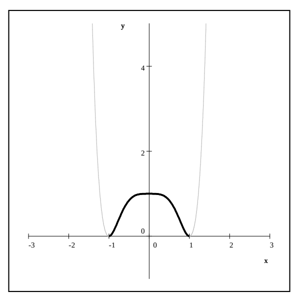
4. a) a) $f(x) = 3x^5 - 20x^3 - 6$ on $[-3, 3]$
   relative maximum: $(-2, 58)$
   absolute maximum: $(3, 183)$
   relative minimum: $(2, -70)$
   absolute minimum: $(-3, -195)$



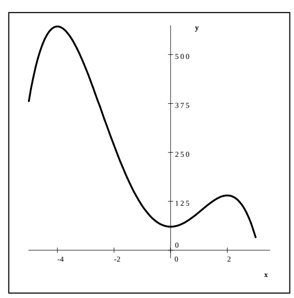
   b) $f(x) = x^3 - 2x^2 - 15x + 8$ on $[-4, 5]$
   relative maximum: $\left(-\dfrac{5}{3}, \dfrac{616}{27}\right)$
   absolute maximum: $\left(-\dfrac{5}{3}, \dfrac{616}{27}\right)$
   relative minimum: $(3, -28)$
   absolute minimum: $(3, -28)$ and $(-4, -28)$



   c) $f(x) = 2x^6 - 3x^4 + 1$ on $[-1, 1]$
   relative maximum: $(0, 1)$
   absolute maximum: $(0, 1)$
   relative minimum: none
   absolute minimum: $(-1, 0)$ and $(1, 0)$



   d) $f(x) = -3x^4 - 8x^3 + 48x^2 + 60$ on $[-5, 3]$
   relative maximum: $(-4, 572)$ and $(2, 140)$
   absolute maximum: $(-4, 572)$
   relative minimum: $(0, 60)$
   absolute minimum: $(3, 33)$



5. 3 and 6 with a product of 108

6. base: $2\sqrt{5}\,\text{ft}$ height: $\sqrt{5}\,\text{ft}$ $V_{\max} = 20\sqrt{5} = \text{ft}^3$

7. $r = \sqrt{\dfrac{10}{3\pi}}$ $h = r$

8. $r = 10\,\text{ft}$ and $\theta = 2$ radian

   Solution: $100 = \dfrac{1}{2}r^2\theta$ Solve for θ: $\theta = \dfrac{200}{r^2}$

   The perimeter is then $P = 2r + r\theta = 2r + r\left(\dfrac{200}{r^2}\right) = 2r + \dfrac{200}{r}$

   $P'(r) = 2 - \dfrac{200}{r^2}$ We solve for the zero of the derivative:

   $2 - \dfrac{200}{r^2} = 0 \implies r = \pm 10$ (the negative root is ruled out) and then $\theta = 2$ radian

9. a) ∞ b) 0 c) $\dfrac{1}{4}$ d) $\dfrac{1}{4}$ e) ∞ f) 0 g) ∞ h) -3 i) 2

10. a) 1 b) e^3 c) e^4 d) $\dfrac{1}{e}$ e) e^{-2} f) e^{-8}

11. a) $f'(x) = -12x^2\cos x + 4x^3\sin x + \dfrac{1}{x}\cos x - \dfrac{1}{x^2}\sin x$ b) $f'(x) = \dfrac{5}{2}x\sqrt{x}$

    c) $f'(x) = (6x-1)(2\sin x - \cos x + 1) + (2\cos x + \sin x)\left(3x^2 - x + 1\right)$

    d) $f'(x) = x^2\ln x$ e) $f'(x) = \dfrac{\sin x\,(6x-8) - \cos x\left(3x^2 - 8x + 2\right)}{\sin^2 x}$

    f) $f'(\theta) = \sec^2\theta$ g) $f'(x) = \dfrac{2 - \ln x}{2x^{3/2}}$

12. $-\dfrac{\sqrt{3}}{2}\left(x - \dfrac{\pi}{3}\right) = y - \dfrac{1}{2}$

13. a) $a = 9, b = -60$

    b) $f(x) = \dfrac{x^2 + x(-a-2) + 2a}{x^2} = 1 + \dfrac{-a-2}{x} + \dfrac{2a}{x^2}$

    Then $f'(x) = \dfrac{d}{dx}\left(1 + \dfrac{-a-2}{x} + \dfrac{2a}{x^2}\right) = \dfrac{1}{x^2}(a+2) - 4\dfrac{a}{x^3}$

    $f'(3) = 0 \implies \dfrac{1}{3^2}(a+2) - 4\dfrac{a}{3^3} = 0$ We solve for a and obtain 6

14. See handout