Students must be able to correctly **state** the following theorems:
Intermediate Value Theorem (both forms), Extreme Value Theorem, Rolle's Theorem, Mean Value Theorem, the second derivative test.
Students must be able to state the definitions of basic properties of functions (continuous, increasing, concave up, etc.)

Students must be able to prove the following theorems:

- Differentiating functions using the definition (limit of the differential quotient)
- Prove that $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ and $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- Prove that $\frac{d}{dx}(e^x) = e^x$ and that $\frac{d}{dx}(a^x) = a^x \ln a$
- If a function is differentiable at a number x, then it is continuous there.
- The product rule and quotient rule for derivatives.
- The Mean Value Theorem.

1. Compute the inverse for each of the following functions.

   a) $f(x) = (2x-1)^3$
   b) $f(x) = \log_2(5x-12)$
   c) $g(x) = e^{3x-1} + 5$
   d) $h(a) = \dfrac{3a-7}{5a+1}$
   e) $f(x) = \dfrac{2x-7}{3x-2}$

2. Differentiate each of the following.

   a) $f(x) = 3^{\sin x}$
   b) $f(x) = \left(3x^2 - 8x + 7\right)^{27}$
   c) $f(x) = \cos^2 x + \sin^2 x$
   d) $f(x) = \sin(10x)$
   e) $f(\theta) = \dfrac{1}{\cos^4 \theta}$
   f) $g(x) = \dfrac{1}{5x-8}$
   g*) $f(x) = \ln(\sec x + \tan x)$
   h) $f(y) = \log_2\left(\sqrt{y^4+1}\right)$
   i) $f(x) = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}$
   j) $h(\theta) = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta$
   k) $f(x) = \dfrac{x}{x^2+1}$
   l) $f(\alpha) = \dfrac{\tan \alpha}{1 + \tan^2 \alpha}$
   m) $f(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$
   n) $f(x) = (3x-10)^{20}(5x+1)^8$

3. Let g be a differentiable function with $g(2) = 4$ and $g'(2) = -3$. Compute the exact value of $f'(2)$ if f is defined as

   a) $f(x) = 2g(x) - 3$
   b) $f(x) = (g(x))^3$
   c) $f(x) = \ln(g(x))$
   d) $f(x) = \cos(g(x))$
   e) $f(x) = \dfrac{1}{(g(x))^3}$
   f) $f(x) = \dfrac{1}{g(x)}$

4. Find all relative maximums and minimums of $f(x) = (x-3)^4 (x-6)^7$.

5. Give a complete analysis for each of the following functions.

   a) $f(x) = (x-14)\left(x^2+5x-50\right)$ on $[-11, 15]$

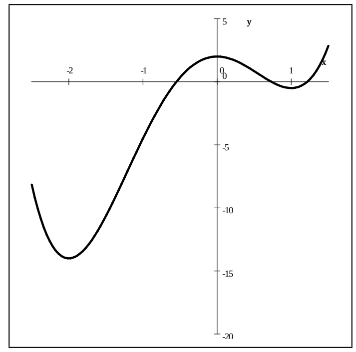
   b) $f(x) = e^{2x}(x-5)^2$ on $[-2, 6]$.

   c) $f(x) = x\sqrt{1-x^2}$

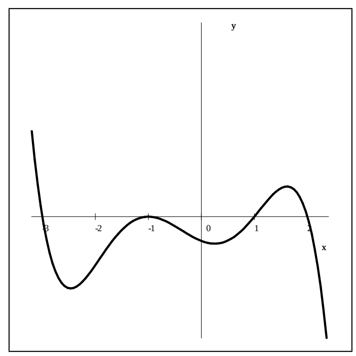
6. Sketch the graph of $f(x) = \sin^{-1} x$ and state its basic properties. (domain, range, increasing/decreasing)

7. a) Find an equation for the tangnet line drawn to $f(x) = \ln\left(x^2+1\right)$ at $x = 3$.

8. a) The graph on the picture shows f'. Where does f have points of inflection?



   b) The graph on the picture shows f''. Where does f have points of inflection?



9. Find the point P on the parabola $y = x^2$ that is closest to the point $Q(0, 5)$.

10. Compute each of the following indefinite integrals.

    a) $\int \sin\theta \, d\theta$

    b) $\int \left(\frac{3}{x} + \sqrt{x}\right) dx$

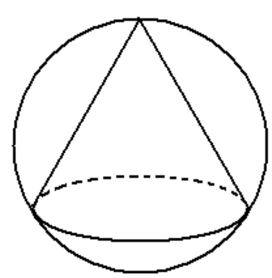
    c) $\int 2^x dx$

    d) $\int x^{100} dx$

    e) $\int (e^x - \sin x) \, dx$

    f) $\int \left(ax^2 + bx + c\right) dx$

11. a) Consider a spehere with radius R and all right cones we can write into it as shown on the picture below. Which one has the greatest volume?



    b) A company determines that the if n is the number of items produced, they can all be sold at a price of $p(n) = \sqrt{800 - 0.5n}$ for each item. What is the greatest revenue possible? How many items should the company produce?

c) What is the smallest possible value of the sum of the square of a positive number and the reciprocal of the same number?

d) A company determines that if the spend P on promoting a product and R on research to improve it, then they can sell approximately P^5R^3 many items of it. The company has a llocated a total of \$500 000 for promotion and research. How much should they spend on promotion and on research to maximize the number of items sold?

12. How many points of inflections does the function $f(x) = xe^{2x}$ have?

13. Prove that the function $f(x) = 4x^3 - 12x + 3x^4 + 8$ has at least one relative maximum or minimum.

14. Find all values of c that satisfy the conclusion of the Mean Value Theorem.

a) $f(x) = \ln(3x-1)$ on $[1,5]$ b) $f(x) = e^x$ on $[0,3]$ c) $f(x) = (2x-1)^3$ on $[-1,2]$

Answers

1. a) $f^{-1}(x) = \dfrac{\sqrt[3]{x}+1}{2}$ b) $f^{-1}(x) = \dfrac{2^x+12}{5}$ c) $g^{-1}(x) = \dfrac{\ln(x-5)+1}{3}$

d) $h^{-1}(a) = \dfrac{a+7}{-5a+3}$ e) $f^{-1}(x) = \dfrac{2x-7}{3x-2}$

2. a) $f'(x) = (\cos x)\cdot(\ln 3)\cdot 3^{\sin x}$ b) $f'(x) = 54\left(3x^2-8x+7\right)^{26}(3x-4)$ c) $f'(x) = 0$

d) $f'(x) = 10\cos 10x$ e) $f'(\theta) = \dfrac{4\sin\theta}{\cos^5\theta}$ f) $g'(x) = -\dfrac{5}{(5x-8)^2}$ g*) $f'(x) = \sec x$

h) $f'(y) = \dfrac{2y^3}{(\ln 2)(y^4+1)}$ i) $f'(x) = xe^{3x}$ j) $h'(\theta) = \dfrac{1}{2}\cos 2\theta + \dfrac{1}{2} = \cos^2\theta$

k) $f'(x) = \dfrac{-x^2+1}{(x^2+1)^2}$ l) $f'(\alpha) = \dfrac{-\tan^2\alpha+1}{\tan\alpha^2+1} = \cos 2\alpha$ m) $f'(x) = \dfrac{1}{2}\left(e^x - e^{-x}\right)$

n) $f'(x) = 40(5x+1)^7(3x-10)^{20} + 60(5x+1)^8(3x-10)^{19} = 20(21x-17)(5x+1)^7(3x-10)^{19}$

3. a) -6 b) -144 c) $-\dfrac{3}{4}$ d) $3\sin 4$ e) $\dfrac{9}{256}$ f) $\dfrac{3}{16}$

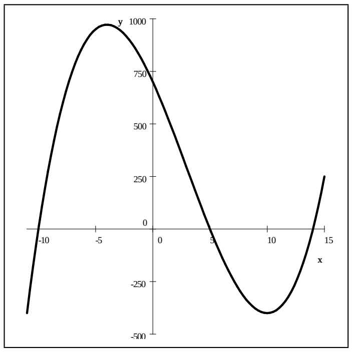
4. $f'(x) = 4(x-3)^3(x-6)^7 + 7(x-3)^4(x-6)^6 = (11x-45)(x-3)^3(x-6)^6$

relative maximum at $x = 3$ and relative minimum at $x = \dfrac{45}{11}$

5. a) $f(x) = (x-14)\left(x^2+5x-50\right) =$
$= (x+10)(x-5)(x-14)$
$= x^3 - 9x^2 - 120x + 700$
domain: $[-11, 15]$
$y-$intercept: $(0, 700)$
$x-$intercepts: $(-10,0)$, $(5,0)$, and $(14,0)$
$f'(x) = 3x^2 - 18x - 120 = 3(x+4)(x-10)$
relative minimum: at $x = -4$
relative maximum: at $x = 10$
$f(-11) = -400$
$f(-4) = 972$
$f(10) = -400$ and $f(15) = 250$
absolute maximum: $(-4, 972)$
absolute minimum: $(-11, -400)$ and $(10, -400)$
range: $[-400, 972]$
$f''(x) = 6x - 18 = 6(x-3)$
concave down on $(-11, 3)$
concave up on $(3, 15)$
point of inflection: at $x = 3$
continuous on $[-11, 15]$

not one-to-one

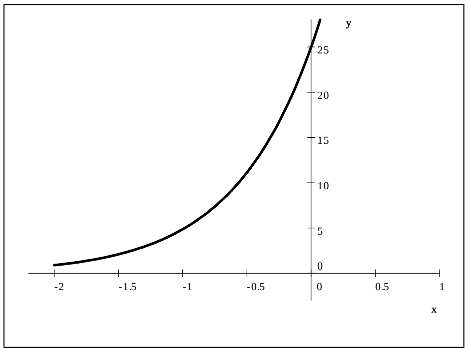
end-behavior: N/A



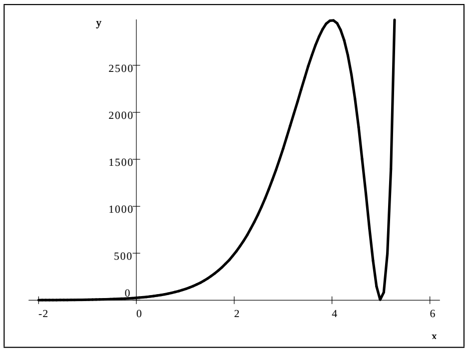
b) $f(x) = e^{2x}(x-5)^2$

domain: $[-2, 6]$
$y-$intercept: $(0, 25)$
$x-$intercept: $(5, 0)$
$f'(x) = 2e^{2x}(x^2 - 9x + 20) = 2e^{2x}(x-4)(x-5)$
relative minimum: at $x = 5$
relative maximum: at $x = 4$
$f(-2) = 49e^{-4} \approx 0.8975$
$f(6) = e^{12} \approx 162754.79$
$f(5) = 0$ and $f(4) = e^8 \approx 2980.96$
absolute maximum: $(6, e^{12})$
absolute minimum: $(5, 0)$
range: $[0, e^{12}]$
$f''(x) = 2e^{2x}(2x^2 - 16x + 31)$

concave up on $\left(-2, 4 - \frac{\sqrt{2}}{2}\right)$
concave down on $\left(4 - \frac{\sqrt{2}}{2}, 4 + \frac{\sqrt{2}}{2}\right)$
concave up on $\left(4 + \frac{\sqrt{2}}{2}, 6\right)$
points of inflection: at $x = 4 - \frac{\sqrt{2}}{2}$ and $x = 4 + \frac{\sqrt{2}}{2}$
continuous on $[-2, 6]$
not one-to-one
end-behavior: N/A



f on $[-2, 0]$



f on $[0, 6]$

c) $f(x) = x\sqrt{1-x^2}$

domain: $[-1, 1]$
$y-$intercept: $(0, 0)$
$x-$intercept: $(-1, 0)$, $(0, 0)$, $(1, 0)$
$f'(x) = \dfrac{-2x^2 + 1}{\sqrt{1-x^2}}$
relative minimum: at $x = -\frac{\sqrt{2}}{2}$

relative maximum: at $x = \frac{\sqrt{2}}{2}$
$f(-1) = 0$ $\quad f(1) = 0$
$f\left(-\frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$

$f\left(\frac{\sqrt{2}}{2}\right)=\frac{1}{2}$

absolute maximum: $\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right)$

absolute minimum: $\left(-\frac{\sqrt{2}}{2},-\frac{1}{2}\right)$

range: $\left[-\frac{1}{2},\frac{1}{2}\right]$

$f''(x)=\dfrac{2x^3-3x}{(1-x^2)\sqrt{1-x^2}}$

concave up on $\left(-1,-\frac{\sqrt{6}}{2}\right)$

concave down on $\left(-\frac{\sqrt{6}}{2},0\right)$

concave up on $\left(0,\frac{\sqrt{6}}{2}\right)$

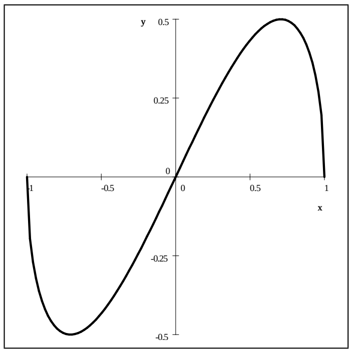
concave down on $\left(\frac{\sqrt{6}}{2},1\right)$

points of inflection: at $x=-\frac{\sqrt{6}}{2}$, 0, and $x=\frac{\sqrt{6}}{2}$

continuous on $[-1,1]$

not one-to-one

end-behavior: N/A



6. see handout

7. a) $y=\frac{3}{5}(x-3)+\ln 10$

8. a) at $x=-2,0,$ and 1 b) at $x=-3,1,$ and 2

9. There are two points: $\left(-\frac{3\sqrt{2}}{2},\frac{9}{2}\right)$ and $\left(\frac{3\sqrt{2}}{2},\frac{9}{2}\right)$

10. a) $-\cos\theta+C$ b) $3\ln|x|+\frac{2}{3}\sqrt{x^3}+C$ c) $\frac{2^x}{\ln 2}+C$ d) $\frac{x^{101}}{101}+C$ e) $e^x+\cos x+C$

    f) $\frac{1}{3}ax^3+\frac{1}{2}bx^2+cx+D$ (D is a constant)

11. a) $h=\frac{4}{3}R$ and $r=\frac{2\sqrt{2}}{3}R$ then $V=\frac{32}{81}\pi R^3$ b) \$1067 items for a total of \$17 418.59

    c) $\frac{3}{2}\sqrt[3]{2}$ d) \$312 500 on promotion and \$187 500 on research

12. One, at $x=-1$

13. Solution: $f'(x)=12x^2+12x^3-12=12\left(x^2+x^3-1\right)$

    Since $f'(0)=-12$ and $f'(1)=12$ and both f and f' are polynomials (so continuous on $\mathbb{R}$) f' will change sign through a zero soemwhere between $x=0$ and $x=1$. That's where f will have a relative minimum.

14. a) $\frac{4}{\ln 7}+\frac{1}{3}$ b) $\ln\frac{e^3-1}{3}=-\ln 3+\ln\left(e^3-1\right)$ c) $\frac{1\pm\sqrt{3}}{2}$