Students must be able to correctly state the following theorems: Intermediate Value Theorem (both forms), Extreme Value Theorem, Rolle's Theorem, Mean Value Theorem, the second derivative test.

Students must be able to prove the following theorems:

- Differentiating functions using the definition (limit of the differential quotient)
- Prove that $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ and $\frac{d}{dx}(\ln x) = \frac{1}{x}$, $\frac{d}{dx}(e^x) = e^x$, and $\frac{d}{dx}(a^x) = a^x \cdot \ln a$
- If a function is differentiable at a number x, then it is continuous there.
- The product rule and quotient rule for derivatives.
- Rolle's Theorem.
- The Mean Value Theorem.

Review Problems

1. State and prove the mean value theorem.
2. The derivative of a function is given by $f'(x) = -2(x+3)(x+1)^2(x-2)^3(x-4)$.
   a) Sketch the graph of f'.
   b) Sketch the graph of f in the same coordinate system with f'.
3. Compute the inverse for each of the following functions.
   a) $f(x) = 3^{\sqrt{x}-1}$ b) $f(x) = \ln\left(\frac{1}{3}x - 1\right)$ c) $f(x) = 5\sqrt[3]{2x-1}$ d) $f(x) = \frac{x-3}{5x+8}$
4. Differentiate each of the following.
   a) $f(x) = e^{\sin^2 x} e^{\cos^2 x}$
   b) $f(x) = \ln(\tan x)$
   c) $f(x) = \tan \pi x$
   d) $f(x) = \ln(\sec x)$
   e) $x^3 y - x^2 y^3 = x - y$
   f) $\ln x - \ln y = \sin xy$
   g) $f(x) = \cos^{-1}(x^2)$
   h) $f(x) = \ln(\tan x)$
   i) $f(x) = \sin^2 x$
   j) $x^3 + y^3 = \ln(x+y)$
   k) $\sin x + \cos y = \tan xy$
   l) $(x^2 + y^2)^4 = 5x^3 y^4$
   m) $\sin(xy) = \arctan x + \arctan y$
5. a) Find an equation of all tangent lines drawn to the graph of $2x^2 + y^2 = 5y - x$ at $x = -2$.
   b) Find an equation of the tangent line drawn to the graph of $x^2 - y^3 + 2xy = y^2 - 5x - 12$ to the point $(-2, 1)$.
6. Let g be a differentiable function with $g(5) = 10$ and $g'(5) = -2$. Compute the exact value of $f'(5)$ if f is defined as
   a) $f(x) = 3g(x) + 1$
   b) $f(x) = (g(x))^3$
   c) $f(x) = \ln(g(x))$
   d) $f(x) = \cos(g(x))$
   e) $f(x) = \frac{1}{(g(x))^3}$
7. Find all relative maximums and minimums of $f(x) = (3x-1)^4(6-x)^7$.
8. Describe the concavity behavior of the graph of each of the following functions.

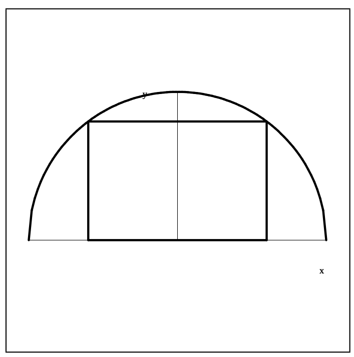
a) $f(x) = -x^3 + 30x^2 - 153x - 184$
b) $f(x) = x^3 - 6x^2 + 8x - 20$
c) $f(x) = x^4 - 10x^3 + 3x - 10$
d) $f(x) = xe^{2x}$
e) $f(x) = x^2 e^{2x}$
f) $f(x) = x\sqrt{1-x^2}$

9. Compute each of the following indefinite integrals.

a) $\int \sec^2\theta \, d\theta$
b) $\int \frac{2a-1}{a+3}\, da$
c) $\int \frac{x-1}{2x+1} dx$
d) $\int 2^{5x-1} dx$
e) $\int (\sin \pi x)\, dx$
f) $\int \frac{1}{5-x} dx$
g) $\int \frac{1}{1+x^2}\, dx$
h) $\int \frac{x^2}{1+x^2}\, dx$
i) $\int \frac{1}{\sqrt{1-x^2}}\, dx$

10. We would like to design a book. Each page should contain $60\,\text{cm}^2$ of text. The upper margin needs to be 3 cm wide and all other margins need to be 2 cm wide. What dimensions for the book would minimize the amount of paper we need to use to produce this book?

11. We write rectangles into a semicircle with radius 1 as shown on the picture. What dimensions will guarantee the maximal area of the rectangle?



12. Prove that if $f(x) = \sin^{-1} x$, then $f'(x) = \frac{1}{\sqrt{1-x^2}}$.

13. A company finds that if they spend x dollars on research and y dollars on marketing, then they can make a profit of $\sqrt{x}y^2$ dollars. How much should the company spend on research and marketing if they can spend a total of 200 000 dollars on these things?

14. The base radius and height of a cylinder are constantly changing but the volume of the cylinder is kept at a constant 600π in^3.

a) At a time t_1 the base radius is $r(t_1) = 10$ in and its rate of change is $r'(t_1) = 0.2\frac{\text{in}}{\text{s}}$. Compute the rate of change of the height of the cylinder $h(t)$ at time t_1.

b) At a time t_2 the height is $h(t_2) = 12$ in and its rate of change is $r'(t_2) = -0.5\frac{\text{in}}{\text{s}}$. Compute the rate of change of the radius of the cylinder $r(t)$ at time t_2.

15. The base radius and height of a cylinder are constantly changing but the volume of the cylinder is kept at a constant 600π in^3. The radius of the cylinder is changing at a constant rate of $5\frac{\text{in}}{\text{s}}$.

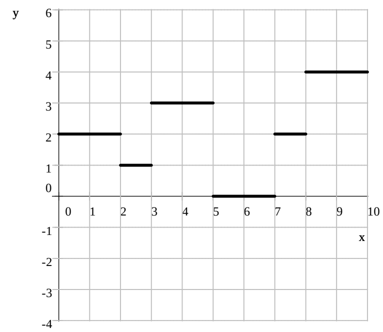
a) Find the rate of change of the height when the radius is 2 in.

b) Find the rate of change of the height when the radius is 100 in.

16. The base radius and height of a cylinder are constantly changing but the volume of the cylinder is kept at a constant 600π in^3. The height of the cylinder is changing at a constant rate of $0.8\frac{\text{in}}{\text{s}}$.
    a) Find the rate of change of the base radius when the height is 3 in.
    b) Find the rate of change of the base radius when the height is 20 in.

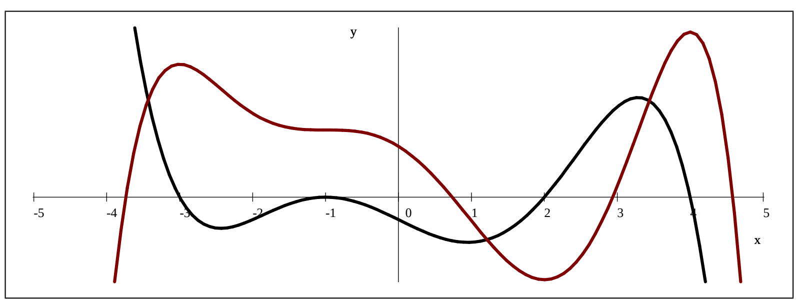
17. Find the number c that satisfies the colnclusion of the Mean Value Theorem for $f(x) = x^3 - x + 2$ on $[1, 4]$

18. An object starts from a location of $L(0) = 0$. Find the location of the object $t = 10$ if its velocity function is shown on the picture.



Answers

1. see handout
2. .



3. a) $f^{-1}(x) = (1 + \log_3 x)^2$ b) $f^{-1}(x) = 3 + 3e^x$ c) $f^{-1}(x) = \frac{x^3}{250} + \frac{1}{2}$ d) $f^{-1}(x) = \frac{8x+3}{-5x+1}$

4. a) $f'(x) = 0$ b) $f'(x) = \frac{\tan^2 x + 1}{\tan x} = \frac{1}{\cos x \sin x}$ c) $f'(x) = \pi \sec^2(\pi x)$

   d) $f'(x) = \tan x$ e) $y' = \frac{-3x^2 y + 2xy^3 + 1}{x^3 - 3x^2 y^2 + 1}$ f) $y' = \frac{-xy^2 \cos xy + y}{x^2 y \cos xy + x}$

   g) $f'(x) = -\frac{2x}{\sqrt{1 - x^4}}$ h) $f'(x) = \frac{\tan^2 x + 1}{\tan x} = \frac{1}{\sin x \cos x}$ i) $f'(x) = 2\cos x \sin x = \sin 2x$

j) $y' = \dfrac{-3x^2(x+y)+1}{3y^2(x+y)-1}$ k) $y' = \dfrac{-\cos x + y\left(\tan^2 xy + 1\right)}{-\sin y - x\left(\tan^2 xy+1\right)}$ l) $y' = \dfrac{5y^4 - 8x\left(x^2+y^2\right)^3}{-20xy^3 + 8y\left(x^2+y^2\right)^3}$

m) $y' = \dfrac{-\left(y^2+1\right)\left(\cos x\left(x^2+1\right)+1\right)}{\left(x^2+1\right)\left(\left(y^2+1\right)\sin y - 1\right)}$

5. a) $y = -7x - 12$ and $y = 7x + 17$ b) $\dfrac{1}{3}(x+2) = y - 1$

6. a) -6 b) -600 c) $-\dfrac{1}{5}$ d) $2\sin 10$ e) $\dfrac{3}{5000}$

7. $f'(x) = -(33x-79)(3x-1)^3(x-6)^6$ min at $x = \dfrac{1}{3}$ max at $x = \dfrac{79}{33}$

8. a) $f''(x) = -6x + 60 = -6(x-10)$

   f is concave up on $(-\infty, 10)$ and concave down on $(10, \infty)$, point of inflection at $x = 10$

   b) $f''(x) = 6x - 12 = 6(x-2)$

   f is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$, point of inflection at $x = 2$

   c) $f''(x) = 12x^2 - 60x = 12x(x-5)$

   f is concave up on $(-\infty, 0)$, concave down on $(0, 5)$, and concave up on $(5, \infty)$, points of inflection at $x = 0$ and 5

   d) $f''(x) = 4e^{2x} + 4xe^{2x} = 4e^{2x}(x+1)$

   f is concave down on $(-\infty, -1)$ and concave up on $(-1, \infty)$, point of inflection at $x = -1$

   e) $f''(x) = 4x^2e^{2x} + 8xe^{2x} + 2e^{2x} = 2e^{2x}\left(2x^2+4x+1\right) = 4e^{2x}\left(x+1+\dfrac{\sqrt{2}}{2}\right)\left(x+1-\dfrac{\sqrt{2}}{2}\right)$

   f is concave up on $\left(-\infty, -1-\dfrac{\sqrt{2}}{2}\right)$, concave down on $\left(-1-\dfrac{\sqrt{2}}{2}, -1+\dfrac{\sqrt{2}}{2}\right)$, and concave up on $\left(-1+\dfrac{\sqrt{2}}{2}, \infty\right)$

   points of inflection at $x = -1-\dfrac{\sqrt{2}}{2}$ and $x = -1+\dfrac{\sqrt{2}}{2}$

   f) $f''(x) = \dfrac{2x^3 - 3x}{\left(1-x^2\right)^{3/2}} = \dfrac{2x\left(x^2 - \dfrac{3}{2}\right)}{\left(1-x^2\right)^{3/2}} = \dfrac{2x\left(x - \sqrt{\dfrac{3}{2}}\right)\left(x + \sqrt{\dfrac{3}{2}}\right)}{\left(1-x^2\right)^{3/2}}$

   but $\sqrt{\dfrac{3}{2}} = \dfrac{\sqrt{6}}{2}$ is greater than 1 so these points are outside of the domain.

   f is concave up on $(-1, 0)$ and concave down on $(0, 1)$, point of inflection at $x = 0$

9. a) $\tan\theta + C$ b) $2a - 7\ln|a+3| + C$ c) $\dfrac{1}{2}x - \dfrac{3}{4}\ln|2x+1| + C$ d) $\dfrac{2^{5x-1}}{5\ln 2} + C$

   e) $-\dfrac{1}{\pi}\cos \pi x + C$ f) $-\ln|x-5| + C$ g) $\tan^{-1}x + C$ h) $x - \tan^{-1}x + C$ i) $\sin^{-1}x + C$

10. vertical: $\left(5 + 5\sqrt{3}\right)$ cm horizontal: $\left(4 + 4\sqrt{3}\right)$ cm

11. $\sqrt{2}$ by $\dfrac{\sqrt{2}}{2}$

12. see handout

13. They should spend 40 000 on research and 160 000 on marketing

14. a) $-0.24\dfrac{\text{in}}{\text{s}}$ b) $\dfrac{5}{48}\sqrt{2}\dfrac{\text{in}}{\text{s}} \approx 0.147\,313\,912\,747\,2\dfrac{\text{in}}{\text{s}}$

15. a) $-750\dfrac{\text{in}}{\text{s}}$ b) $-\dfrac{3}{500}\dfrac{\text{in}}{\text{s}} = -0.006\dfrac{\text{in}}{\text{s}}$

16. a) $-\dfrac{4}{3}\sqrt{2}\dfrac{\text{in}}{\text{s}} \approx -1.\,885\,618\,083\,164\,13\dfrac{\text{in}}{\text{s}}$ b) $-\dfrac{\sqrt{30}}{50}\dfrac{\text{in}}{\text{s}} \approx -0.109\,544\,511\dfrac{\text{in}}{\text{s}}$

17. $\sqrt{7}$

18. $L(10) = 21$

Last revised: November 13, 2013