Sample Problems

1. We are designing a poster to contain $50\,\text{in}^2$ of printing with a 4-inch wide margin at the top and bottom and a $2-$inch wide margin at each side. What overall dimensions will minimize the amount of paper used?
2. Let a and b be positive numbers such that $ab = 10$. Find the lowest value of $a^2 + 4b^2$.
3. A closed box with a square base is to have a volume of 250 cubic meters. The material for the top and bottom of the box costs \$2 per square meter, and the material for the sides costs \$1 per square meter. Can the box be constructed for less than \$300?
4. A company wants to manufacture cylindrical aluminum cans with a volume of 1000 cubic centimeters (one liter). What dimensions would guarantee the minimal amount of aluminum needed to produce a can?

Practice Problems

1. We are designing a poster to contain $60\,\text{in}^2$ of printing with a 3-inch wide margin at the top and bottom and a $2-$inch wide margin at each side. What overall dimensions will minimize the amount of paper used?
2. Let x and y be positive numbers such that $xy = 1$. Find the lowest possible value of $x^3 + 2y^3$.
3. We would like to construct an open box with a square base. We would like the box to have a volume of $200\,\text{in}^3$. What dimensions would guarantee that the box can be made using the least amount of material?
4. A company wants to manufacture cylindrical aluminum cans with a volume of 1000 cubic centimeters (one liter). The material for the base costs 20 cents per cm^2 and the material for the sides costs 10 cents per cm^2. What dimensions would guarantee the minimal cost of producing such a can?

Sample Problems - Answers

1. 9 inches by 18 inches
2. 40
3. no,the lowest possible cost is \$300 when the box is to be 5 m by 5 m by 10 m
4. $r = \sqrt[3]{\dfrac{500}{\pi}} \approx 5.419\,26$ and $h = 2\sqrt[3]{\dfrac{500}{\pi}} = 2r \approx 10.838\,52$

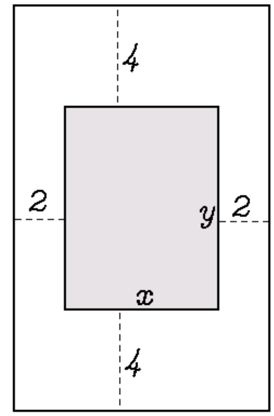
Practice Problems - Answers

1. $2\sqrt{10}$ inches by $3\sqrt{10}$ inches
2. $2\sqrt{2}$ (when $x = \sqrt[6]{2}$)
3. base: $\sqrt[3]{400}$ in by $\sqrt[3]{400}$ in height: $\dfrac{1}{2}\sqrt[3]{400}$ in
4. $r = 5\sqrt[3]{\dfrac{2}{\pi}}$ cm $\approx 4.301\,27$ cm and $h = \dfrac{20}{\pi}\sqrt[3]{\dfrac{2}{\pi}} = 5.476\,547$ cm

Sample Problems - Solutions

1. We are designing a poster to contain $50\,\text{in}^2$ of printing with a 4-inch wide margin at the top and bottom and a 2−inch wide margin at each side. What overall dimensions will minimize the amount of paper used?

Solution: Let x and y denote the vertical and horizontal sides of the printed area.



Then the printed are is $A_{\text{pr}} = xy = 50$. We solve for y in terms of x and get $y = \dfrac{50}{x}$. The dimensions of the entire paper is

$$A = (x+8)(y+4) = (x+8)\left(\frac{50}{x}+4\right)$$

This is a function of x, and we are looking for the minimum of this expression.

$$\begin{aligned} A(x) &= (x+8)\left(\frac{50}{x}+4\right) = 50 + 4x + 8\cdot\frac{50}{x} + 32 = 82 + 4x + \frac{400}{x} = 82 + 4x + 400x^{-1} \\ A'(x) &= 4 - 400x^{-2} = 4 - \frac{400}{x^2} = \frac{4x^2-400}{x^2} = \frac{4\left(x^2-100\right)}{x^2} = \frac{4(x-10)(x+10)}{x^2} \end{aligned}$$

Let us look first at the numerator. $y = 4(x-10)(x+10)$ is an upward opening parabola with $x-$intercepts at $x = -10$ and 10. This indicates a relative maximum at $x = -10$ and $x = 10$. How does the denominator x^2 affects the behavior of the function?

First of all, x^2 is never negative and so it does not have any affect on the sign of A'. But x^2 is zero at $x = 0$ which causes the function A' to be undefined at $x = 0$. However, our domain does not include zero. On our domain, $(0,\infty)$, the denominator is always positive and the only critical number number is $x = 10$. There A' changes sign from negative to positive indicating a relative minimum.

So the optimal dimensions are associated with $x = 10$. Then $y = \dfrac{50}{x} = \dfrac{50}{10} = 5$ and so that paper should be $(x+8)$ by $(y+4)$ i.e. 18 inches by 9 inches.

We should also make sure that the relative minimum we found is also an absolute minimum. The fact that A' is negative on $(0,10)$ and positive on $(10,\infty)$ indicate that A is decreasing on $(0,10)$ and increasing on $(10,\infty)$. That implies that A has an absolute minimum on $(0,\infty)$.

2. Let a and b be positive numbers such that $ab = 10$. Find the lowest value of $a^2 + 4b^2$.

Solution: We solve for a in terms of b : $a = \dfrac{10}{b}$. Then the expression $a^2 + 4b^2$ becomes

$$P(b) = \left(\frac{10}{b}\right)^2 + 4b^2 = 4b^2 + \frac{100}{b^2} = 4b^2 + 100b^{-2}$$

We differentiate this:

$$\begin{aligned} P'(b) &= 8b + 100(-2)b^{-3} = 8b - \frac{200}{b^3} = \frac{8b^4-200}{b^3} \\ &= \frac{8\left(b^4-25\right)}{b^3} = \frac{8\left(b^2+5\right)\left(b^2-5\right)}{b^3} = \frac{8\left(b^2+5\right)\left(b+\sqrt{5}\right)\left(b-\sqrt{5}\right)}{b^3} \end{aligned}$$

The critical numbers for P are $-\sqrt{5}$, 0, and $\sqrt{5}$. All relative maximums or minimums will be here. We can figure out when P' is positive and negative by sorting out the signs of each factor in the numerator and denominator.

	$b<-\sqrt{5}$	$-\sqrt{5}<b<0$	$0<b<\sqrt{5}$	$b>\sqrt{5}$
(b^2+5)	$+$	$+$	$+$	$+$
$(b+\sqrt{5})$	$-$	$+$	$+$	$+$
$(b-\sqrt{5})$	$-$	$-$	$-$	$+$
b^3	$-$	$-$	$+$	$+$
P'	$-$	$+$	$-$	$+$

Based on the signs of P' only, P has a relative minimum at $b=-\sqrt{5}$ and $\sqrt{5}$ and a relative maximum at 0. However, the function does not have a relative maximum at zero. Looking at the formula for the original function, $P(b)=4b^2+\dfrac{100}{b^2}$, we see that there is a vertical asymptote and the graph shoots up toward plus infinity on both sides of the asymptete. Not to mention tha fact that a and b must both be positive. Since b must be positive, we may consider P on the domain $(0,\infty)$. On this domain, P is continuous and differentiable everywhere, is decreasing on $\left(0,\sqrt{5}\right)$ and increasing on $\left(\sqrt{5},\infty\right)$ and so P has an absolute minimum at $b=\sqrt{5}$.

If $b=\sqrt{5}$, then

$$P\left(\sqrt{5}\right)=\left(\frac{10}{\sqrt{5}}\right)^2+4\left(\sqrt{5}\right)^2=\frac{100}{5}+4\cdot 5=20+20=40$$

Thus the smallest possible value of a^2+4b^2 is 40.

Note: It is becoming more and more laborious to sort out when the derivaive is positive and negative. Soon enough there will be a point where this becomes very difficult. In such cases, we can usually use the **second derivative test**.

The second derivative test states that if a function f is twice differentiable at a number a and

1) If $f'(a)=0$ and $f''(a)>0$ then f has a relative minimum at $x=a$.

2) If $f'(a)=0$ and $f''(a)<0$ then f has a relative maximum at $x=a$.

3) If $f'(a)=0$ and $f''(a)=0$, then the second derivative test did not yield useful information on the behavior of f at $x=a$.

We can apply the second derivative test in this problem. $P'\left(\sqrt{5}\right)=0$. We first compute the second derivative of P

$$\begin{aligned} P(b) &= 4b^2+100b^{-2} \\ P'(b) &= 8b-200b^{-3} \\ P''(b) &= 8-200(-3)\,b^{-4}=8+\frac{600}{b^4} \end{aligned}$$

We evaluate the send derivative at $\sqrt{5}$ (although it already is obvious that $P''\left(\sqrt{5}\right)$ is positive, given the formula for P'')

$$P''\left(\sqrt{5}\right)=8+\frac{600}{\left(\sqrt{5}\right)^4}=8+\frac{600}{25}=32$$

Since $P\left(\sqrt{5}\right)=0$ and $P''\left(\sqrt{5}\right)>0$, the function P has a relative minimum at $b=\sqrt{5}$.

3. A closed box with a square base is to have a volume of 250 cubic meters. The material for the top and bottom of the box costs \$2 per square meter, and the material for the sides costs \$1 per square meter. Can the box be constructed for less than \$300?

   Solution: Let x denote the side of the square base, and h denote the height of the box. Then $V = hx^2$ gives us

$$\begin{aligned} hx^2 &= 250 \\ h &= \frac{250}{x^2} \end{aligned}$$

   We now set up the cost function, $C(x)$. The top and bottom each cost \$2 per square meter, and have area x^2. The four sides each have area $xh = x\left(\frac{250}{x^2}\right) = \frac{250}{x}$ and cost \$1 per square meter. Thus

$$C(x) = 2 \cdot 2 \cdot x^2 + 4 \cdot 1 \cdot \frac{250}{x} = 4x^2 + \frac{1000}{x} = 4x^2 + 1000x^{-1}$$

   We are looking for the maximum of $C(x)$. We will differentiate C first.

$$C'(x) = 8x + 1000(-1)x^{-2} = 8x - \frac{1000}{x^2} = \frac{8x^3 - 1000}{x^2} = \frac{8\left(x^3 - 125\right)}{x^2}$$

   The numerator can be factored via the difference of cubes theorem, which states that

$$A^3 - B^3 = (A - B)\left(A^2 + AB + B^2\right)$$

   where the second factor can not be factored further as it is a sum of two squares. We can see that by completing the square

$$A^2 + AB + B^2 = \underbrace{A^2 + AB + \frac{B^2}{4}} - \frac{B^2}{4} + B^2 = \left(A + \frac{B}{2}\right)^2 + \frac{3}{4}B^2$$

   So we can factor the numerator in C' via the difference of cubes theorem

$$C'(x) = \frac{8\left(x^3 - 125\right)}{x^2} = \frac{8(x-5)\left(x^2 + 5x + 25\right)}{x^2}$$

   Recall that the domain is $(0, \infty)$. On this domain, C' is defined everywhere and has only one zero, at $x = 5$. Since both x^2 and $x^2 + 5x + 25$ are positive for all values of x, C' will change sign from negative to positive at $x = 5$, indicating a minimum of C. Thus, the lowest possible cost will be associated with $x = 5$. The actual cost is then

$$C(5) = 4 \cdot 5^2 + \frac{1000}{5} = 300$$

   Thus, we can not construct this box for less than \$300.

   Please note that instead of factoring the difference of cubes, we can apply the second derivative test.

$$\begin{aligned} C(x) &= 4x^2 + 1000x^{-1} \\ C'(x) &= 8x - 1000x^{-2} \\ C''(x) &= 8 - 1000(-2)x^{-3} = 8 + \frac{2000}{x^3} \end{aligned}$$

   Again, the formula for C'' indicates positive values on all positive numbers, so $C''(5)$ is positive. However, in case we didn't notice that,

$$C''(5) = 8 + \frac{2000}{5^3} = 24$$

   and so $C'(5) = 0$ and $C''(5) > 0$ indicates a relative minimum at $x = 5$. The adventage of factoring the sum of cubes is that it is much easier to argue that the minimum is an absolute minimum.

4. A company wants to manufacture cylindrical aluminum cans with a volume of 1000 cubic centimeters (one liter). What dimensions would guarantee the minimal amount of aluminum needed to produce a can?

Solution: Let h denote the height of the can, and r denote the radius of the base circle.

$$\pi r^2 h = 1000 \qquad h = \frac{1000}{\pi r^2}$$

The domain is $(0, \infty)$

$$\begin{aligned} S(r) &= 2\pi r h + 2\pi r^2 = 2\pi r \left(\frac{1000}{\pi r^2}\right) + 2\pi r^2 = 2\pi r^2 + \frac{2000}{r} \\ S'(r) &= 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2} \end{aligned}$$

$$\begin{aligned} 4\pi r - \frac{2000}{r^2} &= 0 \\ 4\pi r &= \frac{2000}{r^2} \\ \pi r^3 &= 500 \quad \Longrightarrow \quad r = \sqrt[3]{\frac{500}{\pi}} \approx 5.419\,26 \end{aligned}$$

and

$$h = \frac{1000}{\pi \left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{1000}{\pi \left(500^{2/3}\right)\left(\pi^{-2/3}\right)} = \frac{2 \cdot 500}{\left(500^{2/3}\right)\left(\pi^{1/3}\right)} = \frac{2 \cdot 500^{1/3}}{\left(\pi^{1/3}\right)} = 2\sqrt[3]{\frac{500}{\pi}} = 2r \approx 10.838\,521$$

But is this a minimum we found? We can still factor S' (and we should to see that the minimum is an absolute one)

$$\begin{aligned} S'(r) &= 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2} = \frac{4\pi\left(r^3 - \frac{2000}{4\pi}\right)}{r^2} = \frac{4\pi\left(r^3 - \frac{500}{\pi}\right)}{r^2} \\ &= \frac{4\pi\left(r - \sqrt[3]{\frac{500}{\pi}}\right)\left(r^2 + r\left(\sqrt[3]{\frac{500}{\pi}}\right) + \left(\sqrt[3]{\frac{500}{\pi}}\right)^2\right)}{r^2} \end{aligned}$$

The last, long factor in the numerator is always positive, and so is the denominator. So, the signs of S are determined by the linear factor: S is negative before $\sqrt[3]{\frac{500}{\pi}}$ and positive after $\sqrt[3]{\frac{500}{\pi}}$. This implies that S is decreasing before $\sqrt[3]{\frac{500}{\pi}}$ and increasing after, and so we indeed found a minimum, in fact an absolute minimum at $r = \sqrt[3]{\frac{500}{\pi}}$.

This can also be done using the second derivative: $S''(r) = 4\pi + \frac{4000}{r^3}$

Since S'' is positive on the entire domain (recall $r > 0$), S' is strictly increasing on its entire domain. This means that S' is negative before its only zero at $\sqrt[3]{\frac{500}{\pi}}$ and positive after. This implies that S is decreasing before $\sqrt[3]{\frac{500}{\pi}}$ and increasing after, and so we indeed found an absolute minimum.