

Exam 3 will cover the following topics. All topics covered by Quizzes 1-8 and Exams 1,2. These topics include quadratic inequalities, functions and their graphs, inverse functions, exponents and logarithms, limits (including those related to  $e$ ), differentiation by using the definition, the sum rule, the constant multiplier rule, the generalized power rule, the product rule, chain rule and quotient rule, derivative of trigonometric, exponential, logarithmic, and inverse trigonometric functions; applications of the derivative: increasing and decreasing functions, relative and absolute extrema, optimization, tangent lines, related rates, and antiderivatives.

The statement and proof of theorems will also be covered on the exam.

Students must be able to correctly state the following statements:

Least Upper Bound Property of real Numbers, Intermediate Value Theorem, Rolle's Theorem, Mean Value Theorem, Second Derivative Test.

Students must be able to prove the following theorems: the Intermediate Value Theorem, the Mean Value Theorem, differentiating functions using the definition (limit of the differential quotient). These include  $\sin x$ ,  $\cos x$ ,  $\ln x$ ,  $\log_a x$ ,  $a^x$ ,  $\sin^{-1} x$ ,  $\cos^{-1} x$ ,  $\tan^{-1} x$ . If a function is differentiable at a number  $x$ , then it is continuous there. The product and quotient rule for derivatives.  $\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$

## Review Problems

1. Consider the function  $f(x) = (x+1)^7(x-3)^8$

- Sketch the graph of  $f$ .
- Compute the derivative of  $f$  and factor it as much as possible.
- Sketch the graph of the derivative.
- Find the  $x$ -coordinate of all relative extrema for  $f(x)$ .

2. Find each of the following limits.

a)  $\lim_{x \rightarrow 1^+} \frac{x^2 - 1}{x^2 - 3x + 2}$

e)  $\lim_{x \rightarrow 3} \frac{x^2 - 3x}{x^2 - 9}$

i)  $\lim_{x \rightarrow 7} \frac{\sqrt{x+2} - 3}{x - 7}$

n)  $\lim_{x \rightarrow 0} \frac{x^2}{1 - \cos 3x}$

b)  $\lim_{x \rightarrow 2^+} \frac{x^2 - 1}{x^2 - 3x + 2}$

f)  $\lim_{x \rightarrow 3^+} \frac{x^2 + 3x}{x^2 - 9}$

j)  $\lim_{x \rightarrow \infty} \left(1 + \frac{2}{x}\right)^x$

o)  $\lim_{x \rightarrow \infty} \left(1 - \frac{2}{x}\right)^x$

c)  $\lim_{x \rightarrow \infty} \frac{2x^2 + 3x + 1}{3x^2 - 5x + 2}$

g)  $\lim_{x \rightarrow 0} \frac{\sqrt{16+x} - 4}{x}$

k)  $\lim_{x \rightarrow \infty} \frac{3 + 2 \ln(x^3)}{2 - \ln 7x}$

p)  $\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 - 1}}$

d)  $\lim_{x \rightarrow \infty} \frac{-x^3 + 2x + 1}{x - 3}$

h)  $\lim_{x \rightarrow 0} \frac{\tan x}{x}$

l)  $\lim_{x \rightarrow \infty} \frac{3 \cdot 2^{2x-1}}{5 \cdot 3^{x+2}}$

q)  $\lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 2x}$

m)  $\lim_{x \rightarrow \infty} \tan^{-1} x$

3. Find the inverse for each of the following functions.

a)  $f(x) = \sqrt[3]{3x-5}$

b)  $f(x) = 2^{3x-1} + 1$

c)  $f(x) = 2 \ln(x+8)$

4. Differentiate each of the following.

a)  $f(x) = 7^x - x^7 + \sqrt{7x} - e^7$

d)  $f(x) = \frac{8x^2 - 20x + 1}{4x - 10}$

h)  $f(x) = \tan(3x)$

b)  $f(x) = \frac{e^x - e^{-x}}{2}$

e)  $f(x) = \log_x(x^2 + 1)$

i)  $f(x) = \frac{1}{1+x^6}$

f)  $f(x) = \ln(\ln(x^5))$

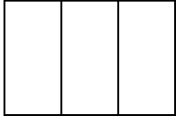
j)  $f(x) = \frac{5^{2x-1}}{5^{x-4}}$

c)  $f(x) = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}$

g)  $f(x) = \log_5\left(\sqrt[3]{5x^3 - 7x^2 + 8}\right)$

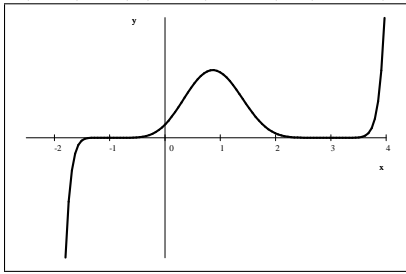
k)  $f(\theta) = \tan^{-1}(\theta^2)$

- l)  $f(x) = \sin^2 x + \cos^2 x$       n)  $f(\theta) = \sin^{-1}(10\theta^3)$   
 m)  $g(x) = -\frac{1}{4x^2} - \frac{\ln x}{2x^2}$
5. Find the second derivative for the function  $f(x) = \frac{5x-1}{5x+1}$ .
6. Find all values of  $x$  for which the tangent line drawn to the graph of  $f(x) = \frac{1}{2}x^2 - \ln x + 8$  is perpendicular to  $x + 2y = -7$ .
7. For each of the following functions, answer the following questions.
- For what interval(s) is  $f$  increasing and decreasing?
  - Find both coordinates of all relative extrema.
  - Find both coordinates of all absolute extrema.
- a)  $f(x) = x^3 - 3x^2 - 9x + 5$  on  $[-4, 5]$   
 b)  $f(x) = x(x-4)^4$  on  $[-1, 5]$   
 c)  $f(x) = x^3 e^{2x}$  on  $[-2, 1]$
8. Compute each of the following indefinite integrals.
- a)  $\int \frac{x-3}{x+2} dx$       e)  $\int e^{2x} dx$       i)  $\int 5^{2x-3} dx$       m)  $\int \frac{1}{1+x^2} dx$   
 b)  $\int x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} dx$       f)  $\int 5^x - x^5 dx$       j)  $\int (3x-5)^{10} dx$       n)  $\int \frac{x^2}{1+x^2} dx$   
 c)  $\int x^5 - 2ax - a^2 dx$       g)  $\int \sin 5x dx$       k)  $\int \sqrt{7x+1} dx$       o)  $\int \tan^2 x dx$   
 d)  $\int x^5 - 2ax - a^2 da$       h)  $\int \frac{2x-1}{x+3} dx$       l)  $\int \frac{3}{\sqrt[7]{x^4}} dx$       p)  $\int \frac{1}{\sqrt{1-x^2}} dx$
9. Find the equation of  $F(x)$  if  $F'(x) = 10x^4 - 6x^2 + 4x - 5$  and  $F(-1) = 8$ .
10. Find the equation of the tangent line drawn to  $x^3y^2 - x^3 + y^2 = 2xy - 1$  at the point  $P(-2, 1)$ .
11. An object is moving along a vertical line. Its acceleration, as a function of time, is  $a(t) = 24t - 10$ . After 5 seconds, the velocity of the object is  $v(5) = -20$  and its height is  $h(5) = 25$ .
- Find the velocity function  $v(t)$ .
  - Find the location function  $h(t)$ .
12. A street light is at the top of a 20 ft tall pole. A woman 6 ft tall walks away from the pole with a speed of  $5\frac{\text{ft}}{\text{s}}$  along a straight path. How fast is the tip of her shadow moving when she is 30 ft from the base of the pole?
13. We want to build a flowerbed in shape of a circular sector. If the perimeter is to be 20 feet, what radius and central angle would guarantee the maximal area for this flowerbed?
14. A box manufacturer plans to create an open box with a surface area of  $30\text{ft}^2$ . What is the maximum size volume that can be formed by bending this material into a box? The box is to have a square base and rectangular sides.
15. Let  $ABCD$  be a unit square. Find the coordinates of point  $P$  on line segment  $CD$  so that the perimeter of triangle  $ABP$  is a) minimal    b) maximal.

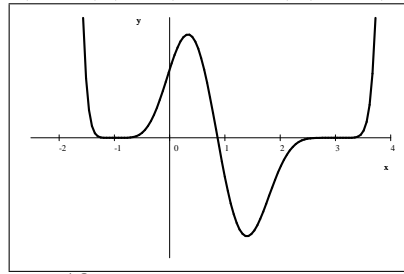
16. Let  $P(x, y)$  be a point on the graph of  $y = -x^2 + 1$  with  $0 \leq x \leq 1$ . Let  $PQRS$  be a rectangle with one side on the  $x$ -axis and two vertices on the graph, as shown on the picture below. Find the exact value of the greatest possible area of such a rectangle.
17. Find two non-negative numbers  $x$  and  $y$  for which  $2x + y = 30$ , such that  $x^5y^3$  is maximized.
18. Find an equation of the tangent line drawn to the graph of  $f(x) = \frac{1}{(3x-1)^2}$  at  $x = 1$ .
19. Which point on  $y = 1 - x^2$  is closest to the origin?
20. We have  $P$  meters of fencing and want to create three adjacent rectangular enclosures as shown on the figure. What is the maximal area we can enclose this way?
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21. Suppose that  $g$  is a function with  $g(5) = 2$  and  $g'(5) = -3$ . Compute  $f'(5)$  if  $f(x)$  is defined as
- a)  $f(x) = [g(x)]^4$       b)  $f(x) = \ln(g(x))$       c)  $f(x) = \frac{1}{g(x)}$       d)  $f(x) = \sqrt{g(x)}$
22. Find the exact values of  $a$  and  $b$  so that the curve  $y = x^2 + ax + b$  will be tangent to the line  $y = 2x + 1$  at the point  $(1, 3)$ .
23. Find a polynomial function  $P(x)$  such that  $P$  is of degree four (or less) and  $f(0) = P(0)$ ,  $f'(0) = P'(0)$ ,  $f''(0) = P''(0)$ ,  $f'''(0) = P'''(0)$ , and  $f^{(4)}(0) = P^{(4)}(0)$  if  $f(x)$  is defined as
- a)  $f(x) = \sin x$       b)  $f(x) = e^{-x/2}$
24. Suppose that  $f$  is a function with derivative  $f'(x) = (9 - x^2)(x + 2)^2(x - 3)(1 - x)x$
- a) Graph  $f'$       e) Plot the graph of  $f$  in the same coordinate system with  $f'$ .
- b) Find all  $x$  for which  $f$  has a relative maximum.      f) Is it possible that  $f$  does not have any  $x$ -intercept?
- c) Find all  $x$  for which  $f$  has a relative minimum.      g) Graph  $f''$  in the same coordinate system with  $f'$ .
- d) How many points of inflection does  $f$  have?
25. Car A is traveling west at  $40 \frac{\text{mi}}{\text{h}}$  and car B is traveling north at  $45 \frac{\text{mi}}{\text{h}}$ . Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?
26. A company finds that if  $x$  is the amount of money spent on research and  $y$  is the amount of money spent on advertising, then they can make a profit of  $\sqrt[4]{xy^3}$ . The company has \$50 000 to spend on these two activities. How much money should the company spend on research and on advertising to maximize its profit?
27. A company finds that if they produce  $q$  units of a product, then they can sell them all at a price of  $p(q) = \sqrt{6300 - 0.02q}$ . What production level would guarantee the maximum revenue?
28. Give a complete analysis of the function  $f(x) = x\sqrt{1-x^2}$
29. Each side of a square is increasing at a rate of  $6 \frac{\text{cm}}{\text{s}}$ . At what rate is the area of the square increasing when the area of the square is 25 cm<sup>2</sup>?
30. Prove the Mean Value Theorem.

## Answers

1.) a)  $f(x) = (x+1)^7(x-3)^8$



b-c)  $f'(x) = (15x-13)(x+1)^6(x-3)^7$

d) relative min at  $x = 3$  and relative max at  $x = \frac{13}{15}$ 

2.) a)  $-2$  b)  $\infty$  c)  $\frac{2}{3}$  d)  $-\infty$  e)  $\frac{1}{2}$  f)  $\infty$  g)  $\frac{1}{8}$  h)  $1$  i)  $\frac{1}{6}$  j)  $e^2$  h)  $-6$  l)  $\infty$

m)  $\frac{\pi}{2}$  n)  $\frac{2}{9}$  o)  $\frac{1}{e^2}$  p)  $1$  q)  $\frac{3}{2}$

3.) a)  $f^{-1}(x) = \frac{x^7+5}{3}$  b)  $f^{-1}(x) = \frac{1}{3}(\log_2(x-1)+1)$  c)  $f^{-1}(x) = e^{x/2} - 8$

4.) a)  $7^x \ln 7 - 7x^6 + \frac{7}{2\sqrt{7x}}$  b)  $\frac{e^x + e^{-x}}{2}$  c)  $xe^{4x}$  d)  $\frac{8x^2 - 40x + 49}{4x^2 - 20x + 25} = 2 - \frac{1}{(2x-5)^2}$

e)  $\frac{\frac{2x \ln x}{x^2+1} - \ln(x^2+1) \left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{2x}{\ln x(x^2+1)} - \frac{\ln(x^2+1)}{x(\ln x)^2}$  f)  $\frac{1}{x \ln x} - \frac{15x^2 - 14x}{3 \ln 5 (5x^3 - 7x^2 + 8)}$

h)  $3 \tan^2 3x + 3 = \frac{3}{\cos^2(3x)}$  i)  $\frac{-6x^5}{(x^6+1)^2}$  j)  $\ln 5 \cdot 5^{x+3}$  k)  $\frac{2\theta}{\theta^4+1}$

l)  $0$  m)  $\frac{\ln x}{x^3}$  n)  $\frac{30\theta^2}{\sqrt{1-100\theta^6}}$  5.)  $-\frac{100}{(5x+1)^3}$  6.)  $1 + \sqrt{2}$

7.) a)  $f(x) = x^3 - 3x^2 - 9x + 5$  on the interval  $[-4, 5]$

i)  $f$  increases on  $(-4, -1)$ , decreases on  $(-1, 3)$ , increases on  $(3, 5)$  ii) rel. min:  $(3, -22)$ , rel. max:  $(-1, 10)$ iii) abs. min:  $(-4, -71)$  abs max:  $(-1, 10)$  and  $(5, 10)$ 

b)  $f(x) = x(x-4)^4$  on the interval  $[-1, 5]$

i) increases on  $\left(-1, \frac{4}{5}\right)$ , decreases on  $\left(\frac{4}{5}, 4\right)$ , and increases on  $(4, 5)$ ii) rel. min:  $(4, 0)$ , rel. max:  $\left(\frac{4}{5}, 83.886\right)$  iii) abs. min:  $(-1, -625)$ , abs. max:  $\left(\frac{4}{5}, 83.886\right)$ 

c)  $f(x) = x^3 e^{2x}$  on  $[-2, 1]$

i)  $f$  decreases on  $\left(-2, -\frac{3}{2}\right)$  and increases on  $\left(-\frac{3}{2}, 1\right)$  ii) rel. min:  $\left(-\frac{3}{2}, -\frac{27}{8}e^{-3}\right)$  there is no rel. max.iii) abs. min:  $\left(-\frac{3}{2}, -\frac{27}{8}e^{-3}\right)$  abs max:  $(1, e^2)$ 

8.) a)  $x - 5 \ln|x+2| + C$  b)  $\frac{x^3}{3} + \frac{x^2}{2} + x + \ln x - \frac{1}{x} + C$  c)  $\frac{x^6}{6} - ax^2 - a^2x + C$

d)  $x^5 a - xa^2 - \frac{a^3}{3} + C$  e)  $\frac{1}{2}e^{2x} + C$  f)  $\frac{5^x}{\ln 5} - \frac{x^6}{6} + C$  g)  $-\frac{1}{5} \cos 5x + C$

h)  $2x - 7 \ln|x + 3| + C$     i)  $\frac{5^{2x-3}}{2 \ln 5} + C$     j)  $\frac{(3x-5)^{11}}{33} + C$     k)  $\frac{2}{21}(7x+1)^{3/2} + C$     l)  $7x^{3/7} + C$

m)  $\tan^{-1} x + C$     n)  $x - \tan^{-1} x + C$     o)  $\tan x - x + C$     p)  $\sin^{-1} x + C$

9.)  $F(x) = 2x^2 - 5x - 2x^3 + 2x^5 + 1$     10.) a)  $v(t) = 12t^2 - 10t - 270$     b)  $h(t) = 4t^3 - 5t^2 - 270t + 1000$

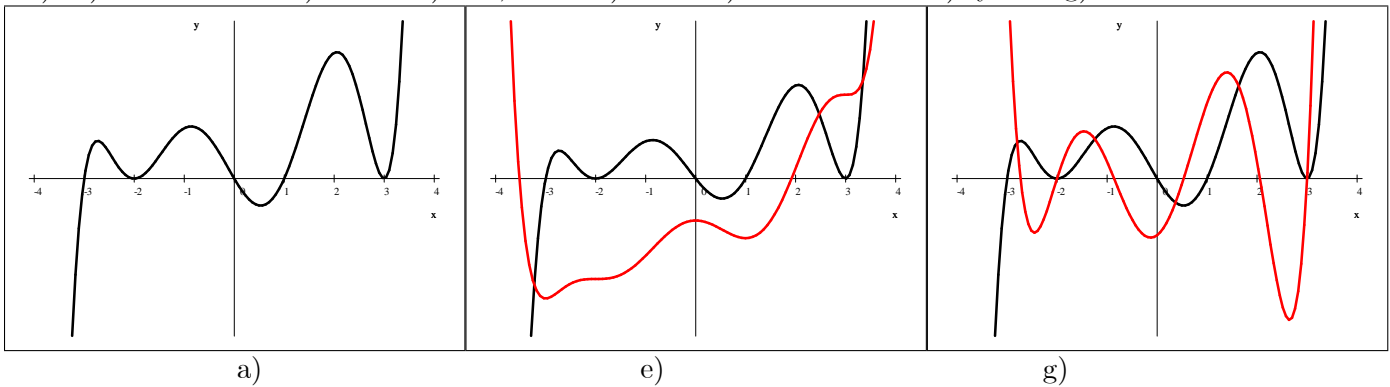
11.)  $-\frac{1}{5}(x+2) = y - 1$  or  $y = -\frac{1}{5}x + \frac{3}{5}$     12.)  $\frac{50 \text{ ft}}{7 \text{ s}}$

15.) a)  $\left(\frac{1}{2}, 1\right)$     b)  $(0, 1)$  and  $(1, 1)$     16.)  $\frac{4}{9}\sqrt{3}$     17.)  $x = \frac{75}{8}$      $y = \frac{45}{4}$     18.)  $y = -\frac{3}{4}x + 1$

19.)  $\left(-\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$  and  $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$     20.)  $\frac{P^2}{32}$     21.) a)  $-96$     b)  $-\frac{3}{2}$     c)  $\frac{3}{4}$     d)  $-\frac{3}{4}\sqrt{2}$

22.)  $a = 0, b = 2$     23.) a)  $x - \frac{1}{6}x^3$     b)  $1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4$

24.) a) see below    b) 0    c)  $-3, 1$     d) 6    e) see below    f) yes    g) see below



25.) distance decreases by  $60 \frac{\text{mi}}{\text{h}}$     26.) \$12 500 on research and \$37 500 on advertising.    27.) 210 000 units

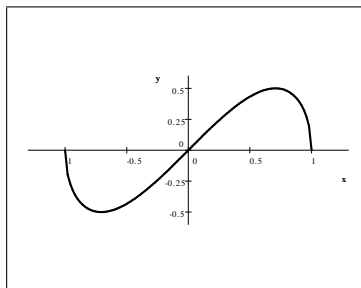
28.)  $f(x) = x\sqrt{1-x^2}$      $f'(x) = \frac{-2x^2+1}{\sqrt{1-x^2}}$      $f''(x) = \frac{2x^3-3x}{(1-x^2)^{3/2}}$

domain:  $[-1, 1]$     range:  $\left[-\frac{1}{2}, \frac{1}{2}\right]$ ; continuous on  $[-1, 1]$ ;  $y$ -intercept:  $(0, 0)$ ;

$x$ -intercepts:  $(-1, 0)$ ,  $(0, 0)$ , and  $(1, 0)$

decreasing on  $\left(-1, -\frac{\sqrt{2}}{2}\right)$  and  $\left(\frac{\sqrt{2}}{2}, 1\right)$ , increasing on  $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

relative and absolute minimum:  $\left(-\frac{\sqrt{2}}{2}, -\frac{1}{2}\right)$ , relative and absolute maximum:  $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$



29.)  $60 \frac{\text{cm}^2}{\text{s}}$     30.) see handout