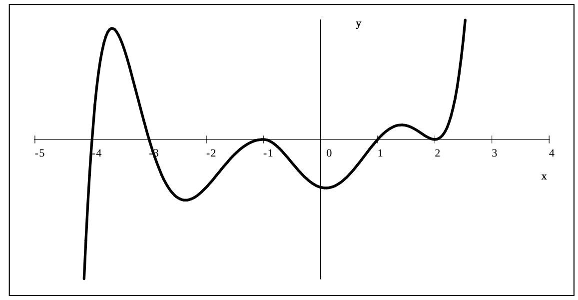
1. The picture shows the graph of a function f. Sketch the graph of each of the following.
   a) $g(x) = f'(x)$
   b) $F(x)$ where $F'(x) = f(x)$



2. Prove that $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$.

3. State the Intermediate Value Theorem.

4. Find each of the following limits.

a) $\displaystyle\lim_{x\to 5} \frac{x^2-25}{x^2-2x-15}$

b) $\displaystyle\lim_{x\to -3^-} \frac{x^2-25}{x^2-2x-15}$

c) $\displaystyle\lim_{x\to -3} \frac{x^2-25}{x^2-2x-15}$

d) $\displaystyle\lim_{x\to \infty} \frac{x^2-1}{2^x+1}$

e) $\displaystyle\lim_{x\to \infty} \left(\log_2\left(4x^2-1\right) - \log_2\left(x^2+1\right)\right)$

f) $\displaystyle\lim_{x\to 3^+} \frac{2(x+1)(x-3)(x+5)}{(x-1)(x-3)}$

g) $\displaystyle\lim_{x\to \infty} \frac{\ln x}{x}$

h) $\displaystyle\lim_{x\to 0^+} \frac{1-e^x}{(x-1)^2-1}$

i) $\displaystyle\lim_{x\to 4^+} \frac{\sqrt{x}-2}{4-x}$

j) $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{x^3}$

k) $\displaystyle\lim_{x\to 0^-} \frac{1-\cos^2 x}{\sin^3 x}$

l) $\displaystyle\lim_{x\to \infty} \left(1-\frac{2}{x}\right)^x$

m) $\displaystyle\lim_{x\to \infty} \tan^{-1} x$

5. Differentiate each of the following.

a) $f(x) = \ln \sqrt[3]{x^5-2x+4}$

b) $f(x) = 2^{\frac{x+1}{x-1}}$

c) $f(x) = \cos^{-1}(\pi x)$

d) $f(x) = -3e^{-4x^2}$

e) $f(x) = x\left(\ln 5 + \ln x^2\right) - 2x$

f) $f(x) = \ln\left(\sin^2(\pi x - 1) + 1\right)$

g) $10 = x^2y - xy^2 + y^3 - x^3$

h) $f(x) = e^{\sin x} + e^{\cos x}$

i) $f(x) = 3^x + x^3$

j) $y = \tan^{-1}\left(x^4\right)$

k) $f(\theta) = \cos^2(2\theta) + \sin^2(2\theta)$

l) $f(x) = \dfrac{e^{\sin x}}{e^{\cos x}}$

m) $g(x) = \ln(\sec x + \tan x)$

n) $f(x) = \left(\sin^{-1} x\right)^4 + x$

o) $f(x) = \dfrac{1}{3}xe^{3x} - \dfrac{1}{9}e^{3x}$

p) $f(\theta) = \sin 5\theta \cos 5\theta$

6. Differentiate each of the following.

a) $\displaystyle\frac{d}{dx}\left(\int_0^x \left(6t^2-4t+1\right) dt\right)$

b) $\displaystyle\frac{d}{dx}\left(\int_0^{x^2} \left(6t^2-4t+1\right) dt\right)$

c) $\displaystyle\frac{d}{dx}\left(\int_0^{\sin x} \left(6t^2-4t+1\right) dt\right)$

d) $\displaystyle\frac{d}{dy}\left(\int_0^y \sqrt{\ln^4 x + 10}\, dx\right)$

e) $\displaystyle\frac{d}{dt}\left(\int_0^{\sqrt{t}} \left(\frac{1}{1+x^2}\right) dx\right)$

f) $\displaystyle\frac{d}{dx}\left(\int_{x^2}^{10} \frac{y^2}{1+y^3} dy\right)$

g) $\displaystyle\frac{d}{dy}\left(\int_{\ln y}^{e} \frac{e^x}{1+e^x} dx\right)$

h) $\displaystyle\frac{d}{dx}\left(\int_{x^4}^{1} ye^{-y^2} dy\right)$

7. Compute y'' if $x^3 + y^3 = 2$.

8. Find all values of c that satisfy the conclusion of the Mean Value Theorem for each of the following.
   a) $f(x) = x^3 - x^2 + 5$ on $[1, 4]$
   b) $g(x) = \dfrac{1}{x} + x$ on $[1, 5]$

9. a) Compute the left and right Riemann Sums for $\int_0^1 e^{-x^2} dx$ using a uniform partition with $n = 5$.

b) Compute the exact values for the left and right Riemann Sums for $\int_0^2 x^3 dx$ using a uniform partition with $n = 100$.

10. Compute each of the following integrals.

a) $\int_0^1 \left(6x^2 - 4x + 5\right) \, dx$

b) $\int \frac{1}{\sqrt{1-9y^2}} dy$

c) $\int 5^x dx$

d) $\int_0^1 \frac{1}{1+y^2} \, dy$

e) $\int \frac{3x-1}{x+7} dx$

f) $\int \sin^2 \theta d\theta$

g) $\int_0^5 e^{-x} dx$

h) $\int \frac{e^{-x}}{e^{-x}+1} dx$

i) $\int \frac{e^{-x}}{\left(e^{-x}\right)^2+1} dx$

j) $\int_1^5 \sqrt{2x-1} dx$

k) $\int_0^{\pi/2} \sin x \cos x \, dx$

l) $\int_0^{\pi/4} \tan x \, dx$

m) $\int_2^{28} (x-1)^{2/3} \, dx$

n) $\int x e^{-x^2} dx$

o) $\int_0^5 x e^{-x^2} dx$

p) $\int_0^\infty x e^{-x^2} dx$

q) $\int_0^{\pi/3} \sin 5x \cos 5x dx$

r) $\int \frac{1}{2+x^2} dx$

s*) $\int \frac{1}{x\sqrt{x-1}} dx$

11. Let f be a function given by $f(x) = \ln\left(\frac{x}{x-1}\right)$

a) Find the domain of f.

b) Find the value of the derivative of f at $x = -1$.

c) Find the formula for the inverse of f.

12. We are on the surface of the Moon. The gravitational acceleration there is $g = -1.6 \frac{\mathrm{m}}{\mathrm{s}^2}$. A rock is thrown vertically upward, from an initial height of 19.2 m, with an initial velocity of $8 \frac{\mathrm{m}}{\mathrm{s}}$.

a) Find the velocity function $v(t)$ of the object.

b) Find the location function $s(t)$ of the object.

c) Find the maximal height that the rock will reach.

d) How long until the rock hits the ground?

e) What is the velocity of the rock when it hits the ground?

13. A particle starts at $t = 0$ and moves along the $x-$axis so that its position at any time $t \geq 0$ is given by $x(t) = (t-1)^3 (2t-3)$.

a) Find the velocity function $v(t)$ of the particle.

b) For what values of t is the object moving to the left?

c) Find all values of t for which the object is moving and but its acceleration is zero.

14. A function f has derivative $f'(x) = -18(x+5)^3 (x+4)^2 (x+2) x^6 (2-x)^5 (4-x)^2$.

a) Plot the graph of f'.

b) Find all critical points of f and classify each of them as a maximum, minimum, or a point of inflection.

c) How many points of inflection does f have?

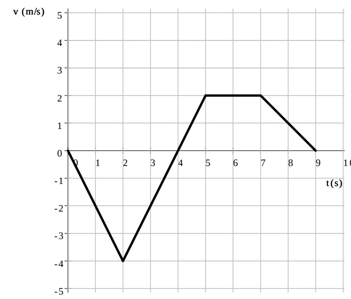
15. Write the equation of the degree 3 polynomial $P(x)$, given that

a) $P(0) = -2$, $P'(0) = 3$, $P''(0) = 12$, and $P'''(0) = -240$.

b) P has a relative minimum at $x = -3$ and a relative maximum at $x = 1$, $P(0) = 0$, and $P(1) = 5$.

16. Find an equation for all tangent lines drawn to the graph of $2x^2 + y^2 = 5y - x$ at $x = -2$.

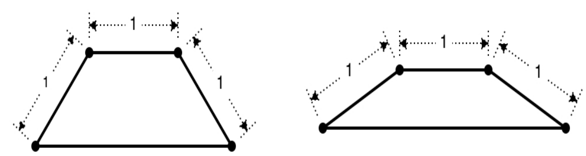
17. Graph the location function $h(t)$ of an object if $h(0) = 0$ and the velocity function, $v(t)$ is given on the graph given. Compute $h(2)$, $h(4)$, $h(5)$, $h(7)$, and $h(9)$.



18. A company estimates that the total cost of producing q units is $C(q) = q^3 - 155q^2 + 6375q + 3000$

a) What is the fixed cost?
b) At what level of production will the total cost be minimized? What is the minimal cost?
c) At what level of production will the profit be maximized, provided that we can sell every item we produce, for \$775?

19. Three sides of a symmetric trapezoid are 1 unit long as shown on the picture. Find the fourth side so that the area of the trapezoid is the greatest.



20. Let $f(x) = 2x^7 - 7x^4 + 70x + 4$. Let $g(x)$ be the inverse of $f(x)$. Find $g'(4)$.

21. A town is of a circular shape. The area of the town is growing with a constant $3\pi \frac{\text{mi}^2}{\text{y}}$ (square mile per year). How fast is its radius changing when the radius is exactly 5 miles long?

22. A tank, shaped like a cone held with its circular base upward, is being filled up with water. The top of the tank is a circle with radius 5 ft, its height is 15 ft. Water is added to the tank at the rate of $V'(t) = 2\pi \frac{\text{ft}^3}{\text{min}}$. How fast is the water level rising when the water level is 6 ft high?

23. A virus is spreading through a population in a manner that can be modeled by the function $g(t) = \dfrac{A}{1 + Be^{-t}}$ where A is the total population, $g(t)$ is the number infected at time t, and B is a constant. What proportion of the population is infected when the virus is spreading the fastest?

24. A company has \$120 000 to spend on the development and promotion of a new product. The company estimates that if x is spent on the development and y is spent on promotion, then approximately $\dfrac{x^{1/2}y^{3/2}}{400\,000}$ items of new product will be sold. Based on this estimate, what is the maximum number of products that the company can sell?

25. A company can sell 20 products if it charges \$40 per product. For each dollar decrease or increase in the price, the company can sell one more or one less product, respectively. The total cost of producing q products is $C(q) = 32q + 100$. What is the maximum profit that the company can achieve from manufacturing and selling this product?

26. Find the volume of the right circular cone of the greatest volume that can be written into a sphere with radius R.

27. Each side of a square is increasing at a rate of $6\dfrac{\text{cm}}{\text{s}}$. At what rate is the area of the square increasing when the area of the square is 25 cm?

28. The derivative of a particle's horizontal position $x(t)$ is $\dfrac{dx}{dt} = -8$, the derivative of its vertical position $y(t)$ is $\dfrac{dy}{dt} = 1$. How fast is the distance of the particle from the origin changing when the particle is at the point $(3, 4)$?

29. A kite 100 meters above the ground moves horizontally at a speed of $8\dfrac{\text{m}}{\text{s}}$. At what rate is the angle between the string and the horizontal decreasing when 200 meters of string has been let out?

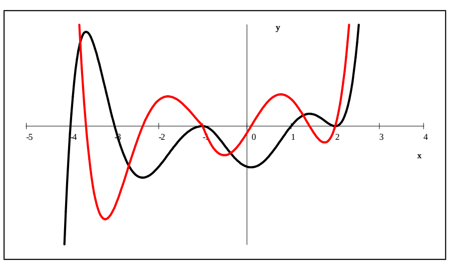
30. Let $ABCD$ be a unit square. Find the coordinates of point P on line segment CD so that the perimeter of triangle ABP is a) minimal b) maximal.

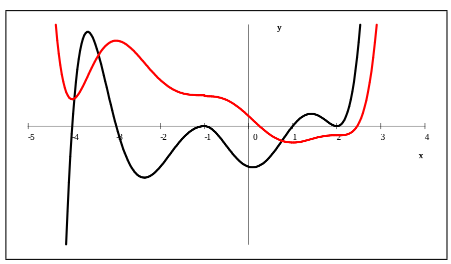
31. Let f be the function that is given by $f(x) = \dfrac{ax+b}{x^2-c}$ and that has the following properties.
i) The graph of f is symmetric with respect to the $y-$axis. ii) $\displaystyle\lim_{x\to 2+} f(x) = \infty$ iii) $f'(1) = -2$
a) Determine the values of a, b, and c.
b) Write an equation for each vertical and each horizontal asymptote of the graph of f.
c) Sketch the graph of $f(x)$.

32. Give a complete analysis of the function $f(x) = x\sqrt{1-x^2}$

Answers

1. a) $g(x) = f'(x)$ b) $F(x)$ where $F'(x) = f(x)$





2. see handout 3. see handout

4. a) $\dfrac{5}{4}$ b) $-\infty$ c) undefined d) 0 e) 2 f) 32 g) 0 h) $\dfrac{1}{2}$ i) $-\dfrac{1}{4}$ j) $-\dfrac{1}{6}$ k) $-\infty$ l) $\dfrac{1}{e^2}$ m) $\dfrac{\pi}{2}$

5. a) $\dfrac{5x^4-2}{3(x^5-2x+4)}$ b) $\ln 2 \cdot 2^{\frac{x+1}{x-1}} \cdot \dfrac{-2}{(x-1)^2}$ c) $-\dfrac{\pi}{\sqrt{1-\pi^2x^2}}$ d) $24xe^{-4x^2}$ e) $2\ln x + \ln 5$
f) $\dfrac{2\pi \sin(\pi x - 1)\cos(\pi x - 1)}{\sin^2(\pi x - 1) + 1}$ g) $y' = \dfrac{2xy - 3x^2 - y^2}{2xy - x^2 - 3y^2}$ h) $\cos x e^{\sin x} - \sin x e^{\cos x}$ i) $3x^2 + (\ln 3) 3^x$
j) $y' = \dfrac{4x^3}{x^8+1}$ k) $f'(\theta) = 0$ l) $f'(x) = (\cos x + \sin x) e^{\sin x - \cos x}$ m) $g'(x) = \sec x$
n) $f'(x) = \dfrac{4(\sin^{-1} x)^3}{\sqrt{1-x^2}} + 1$ o) $f'(x) = xe^{3x}$ p) $5\cos 10x$

6. a) 5 b) $12x^5 - 8x^3 + 2x$ c) $\left(6\sin^2 x - 4\sin x + 1\right)\cos x$ d) $\sqrt{\ln^4 y + 10}$

e) $\dfrac{1}{2\sqrt{t}\,(1+t)}$ f) $\dfrac{-2x^5}{x^6+1}$ g) $-\dfrac{1}{y+1}$ h) $-4x^7 e^{-x^8}$

7. $y'' = -\dfrac{2x}{y^2} - \dfrac{2x^4}{y^5} = -\dfrac{2x^4 + 2xy^3}{y^5}$

8. a) $\dfrac{8}{3}$ b) $\sqrt{5}$

9. a) left: $0.807\,5804$ right: 0.6811563 b) left: $3.920\,4$ right: $4.080\,4$

10. a) $15\ln 2 - 7$ b) $\dfrac{1}{3}\sin^{-1}(3y) + C$ c) $\dfrac{5^x}{\ln 5} + C$ d) $\dfrac{\pi}{4}$ e) $3x - 22\ln|x+7| + C$ f) $\dfrac{1}{2}\theta - \dfrac{1}{4}\sin 2\theta + C$

g) $1 - \dfrac{1}{e^5}$ h) $-\ln(e^{-x} + 1) + C$ i) $\tan^{-1}(e^x) + C$ j) $\dfrac{26}{3}$ k) $\dfrac{1}{2}$ l) $\dfrac{\ln 2}{2}$ m) $\dfrac{726}{5}$

n) $-\dfrac{1}{2}e^{-x^2} + C$ o) $\dfrac{1}{2} - \dfrac{1}{2e^{25}}$ p) $\dfrac{1}{2}$ q) $\dfrac{3}{40}$ r) $\dfrac{\sqrt{2}}{2}\tan^{-1}\left(\dfrac{\sqrt{2}}{2}x\right) + C$ s) $2\tan^{-1}\left(\sqrt{x-1}\right) + C$

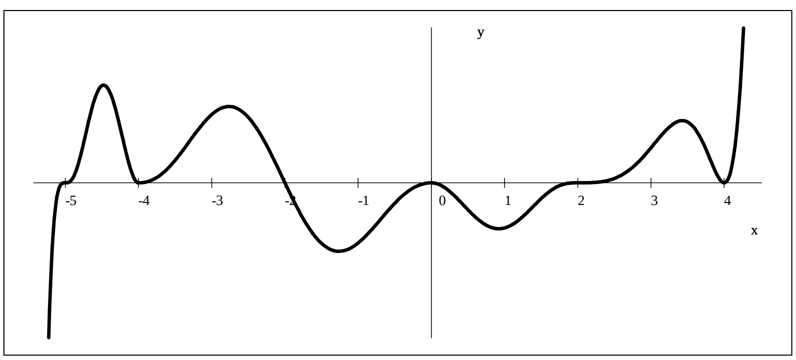
11. a) $(-\infty, 0) \cup (1, \infty)$ b) $-\dfrac{1}{2}$ c) $f^{-1}(x) = \dfrac{e^x}{e^x - 1}$

12. a) $v(t) = -1.6t + 8 = -1.6(t-5)$ b) $s(t) = -0.8t^2 + 8.0t + 19.2 = -0.8(t+2)(t-12)$

c) $39.2\,\mathrm{m}$ d) $12\,\mathrm{s}$ e) $-11.2\dfrac{\mathrm{m}}{\mathrm{s}}$

13. a) $v(t) = (t-1)^2(8t-11)$ b) $[0,1) \cup \left(1, \dfrac{11}{8}\right)$ c) $\dfrac{5}{4}$

14. a)

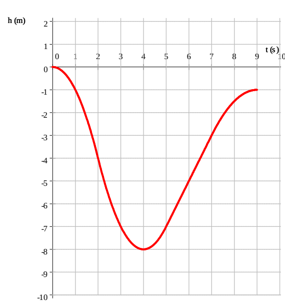


b) f has a minimum at $x = -5$, a point of inflection at $x = -4$, a maximum at $x = -2$, a point of inflection at $x = 0$, a minimum at $x = 2$, and a point of inflection at $x = 4$ c) 8

15. a) $P(x) = -40x^3 + 6x^2 + 3x - 2$ b) $P(x) = -x^3 - 3x^2 + 9x$

16. $y = -7x - 12$ and $y = 7x + 17$

17. $h(2) = -4$
$h(4) = -8$,
$h(5) = -7$
$h(7) = -3$
$h(9) = -1$



18. a) \$3000 b) \$31 125 when $q = 75$ c) $q = 80$

19. 2

20. $\dfrac{1}{70}$

21. $0.3\dfrac{\text{mi}}{\text{y}}$

22. $\dfrac{1}{2}\dfrac{\text{ft}}{\text{min}}$

23. $\dfrac{1}{2}$

24. 11 691

25. \$96

26. $\dfrac{32}{81}\pi R^3$

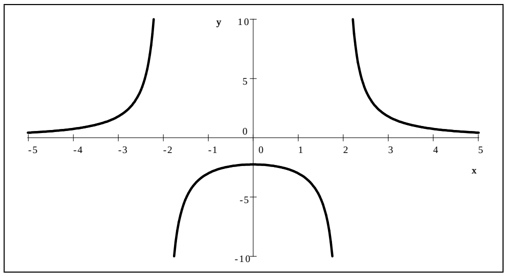
27. $60\dfrac{\text{cm}^2}{\text{s}}$

28. -4

29. $-0.02\dfrac{\text{rad}}{\text{s}}$

30. a) $\left(\dfrac{1}{2}, 1\right)$ b) $(0, 1)$ and $(1, 1)$

31. a) $a = 0$, $b = 9$, $c = 4$
b) vertical: $x = 2$, $x = -2$,
horizontal: $y = 0$



32. $f(x) = x\sqrt{1 - x^2}$ $f'(x) = \dfrac{-2x^2 + 1}{\sqrt{1 - x^2}}$ $f''(x) = \dfrac{2x^3 - 3x}{(1 - x^2)^{3/2}}$

domain: $[-1, 1]$ range: $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$; continuous on $[-1, 1]$; bounded; no asymptotes; $y-$intercept: $(0, 0)$;
x-intercepts: $(-1, 0)$, $(0, 0)$, and $(1, 0)$
decreasing on $\left(-1, -\dfrac{\sqrt{2}}{2}\right)$ and $\left(\dfrac{\sqrt{2}}{2}, 1\right)$, increasing on $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$
relative and absolute minimum: $\left(-\dfrac{\sqrt{2}}{2}, -\dfrac{1}{2}\right)$, relative and absolute maximum: $\left(\dfrac{\sqrt{2}}{2}, \dfrac{1}{2}\right)$
concave up on $(-1, 0)$, concave down on $(0, 1)$, point of inflection: $(0, 0)$
odd; $\lim\limits_{x \to -\infty} f(x)$ and $\lim\limits_{x \to \infty} f(x)$ are both undefined