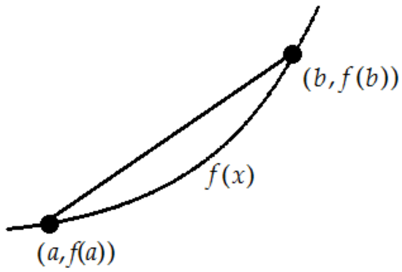
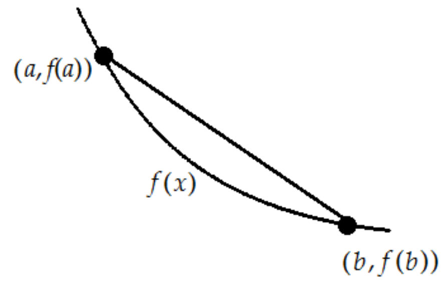
Definition: A function f is **concave up** on an interval I if for all a,b in I, the secant line segment connecting $(a, f(a))$ and $(b, f(b))$ is above the graph of the function of f on I.

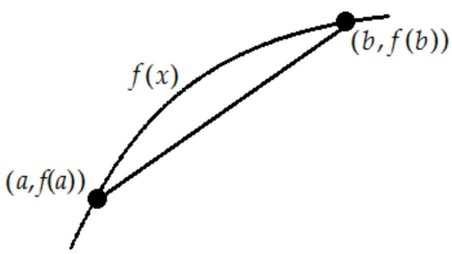


f is concave up increasing

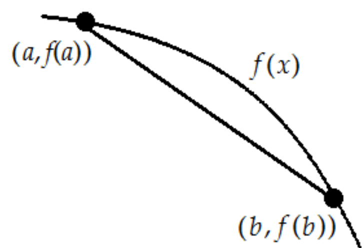


f is concave up decrasing

Definition: A function f is **concave down** on an interval I if for all a,b in I, the secant line segment connecting $(a, f(a))$ and $(b, f(b))$ is below the graph of the function of f on I.

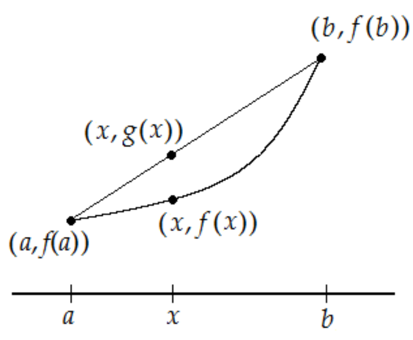


f is concave down increasing



f is concave down decrasing

Let us precisely express what it means for a function f to be concave up. Suppose that $a < b$ and x is any number with $a < x < b$. If we denote the line connecting $(a, f(a))$ and $(b, f(b))$ by g, then we have that $f(x) \leq g(x)$.



Let us write the equation of the line $g(x)$. Since it is connecting the points $(a, f(a))$ and $(b, f(b))$, its slope is $m = \dfrac{f(b) - f(a)}{b - a}$. Using the point-slope form with $(a, f(a))$, we get

$$\begin{aligned} y - f(a) &= m(x - a) \\ y &= m(x - a) + f(a) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \end{aligned}$$

and so we have that $g(x) = \dfrac{f(b) - f(a)}{b - a}(x - a) + f(a)$.

$$\begin{aligned} f(x) &\leq g(x) \\ f(x) &\leq \frac{f(b) - f(a)}{b - a}(x - a) + f(a) && \text{add } f(a) \\ f(x) - f(a) &\leq \frac{f(b) - f(a)}{b - a}(x - a) && \text{divide by } x - a \\ \frac{f(x) - f(a)}{x - a} &\leq \frac{f(b) - f(a)}{b - a} \end{aligned}$$

The last inequality compares two slopes, that of line segment between $(a, f(a))$ and $(x, f(x))$, and that of the line segment between $(a, f(a))$ and $(b, f(b))$. This is not a proof, but it suggests that in case of a concave up function, if we step to the right, the slope of the secant line increases. The limit of the slopes of these same secant lines is the derivative f'.

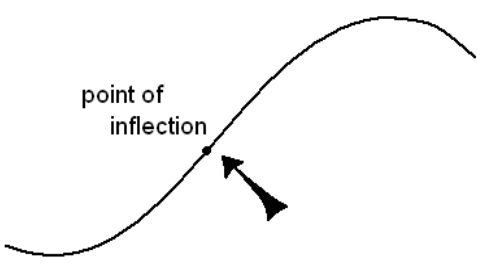
So this computation suggests that in case of a concave up function f, the derivative, f' is increasing.

Theorem: A function f is concave up if its derivative, f' is increasing. f is concave down if f' is decreasing.

It easily follows that an increasing f' means that the second derivative, f'' is positive.

$$\begin{array}{rcccc} f \text{ concave up} & \Longleftrightarrow & f' \text{ increasing} & \Longleftrightarrow & f'' \text{ positive} \\ f \text{ concave down} & \Longleftrightarrow & f' \text{ decreasing} & \Longleftrightarrow & f'' \text{ negative} \end{array}$$

Definition: A point $(x, f(x))$ is a **point of inflection** if x separates two intervals on which f behaves differently with respect to concavity.



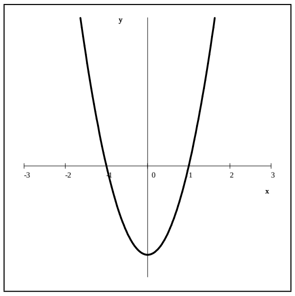
Practice Problems

In case of each of the following functions given, determine the intervals upon which the function is concave up and concave down. State the $x-$coordinate of all points of inflection.

1. $f(x) = x^4 - 6x^2 + x - 3$
2. $f(x) = x^3 + 6x^2 - 3x - 1$
3. $f(x) = x^4 - 10x^3 + 8x + 1$
4. $f(x) = 2x^4 - 8x^3 - 36x^2 - 120x + 80$
5. $f(x) = \sin x$

Answers

1. $f''(x) = 12(x^2 - 1) = 12(x+1)(x-1)$

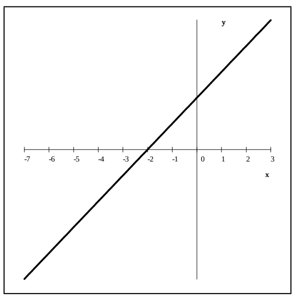


f is concave up on $(-\infty, -1)$ and on $(1, \infty)$

concave down on $(-1, 1)$

points of inflection at $x = -1$ and 1

2. $f''(x) = 6(x+2)$

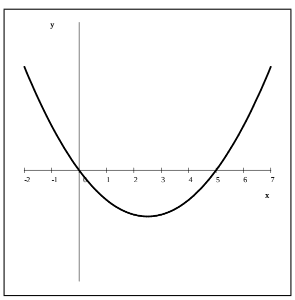


f is concave down on $(-\infty, -2)$

and concave up on $(-2, \infty)$

point of inflection at $x = -2$

3. $f''(x) = 12x(x-5)$

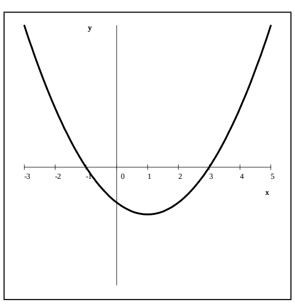


f is concave up on $(-\infty, 0)$ and on $(5, \infty)$

concave down on $(0, 5)$

points of inflection at $x = 0$ and 5

4. $f''(x) = 24x^2 - 48x - 72 = 24(x+1)(x-3)$



f is concave up on $(-\infty, -1)$ and on $(3, \infty)$

concave down on $(-1, 3)$

points of inflection at $x = -1$ and 3

5. $f''(x) = -\sin x = -f(x)$

So $\sin x$ is concave up where it is negative and concave down where it is positive. All of its zeroes are points of inflection.

Concave up: when $\pi + 2k\pi < x < 2\pi + 2k\pi$ where k is an integer

Concave down: when $2k\pi < x < \pi + 2k\pi$ where k is an integer

points of inflection: at $x = k\pi$ where k is an integer

