Quiz 10 will cover the following material:

All material covered in Classes 1-12

The following Sample Quiz is intended to demonstrate the difficulty level of the questions. It is not intended as a comprehensive review or list of the type of questions that can appear on the quiz.

Sample Quiz 10

1. a) Differentiate $f(x) = \sqrt{1-x^2}$ by computing the limit of the difference quotient.

   b) Use your result to prove that the tangent line drawn to the upper half of the unit circle is perpendicular to the tangnet line drawn to the point of tangency.

2. Compute each ofthe following limits.

   a) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$ b) $\lim\limits_{x\to 0} \dfrac{\sqrt{1-\cos x}}{x}$ c) $\lim\limits_{x\to \pi} \dfrac{\sin x}{x-\pi}$

3. a) Prove that if a function f is differentiable at a number a, then it is also continuous there.

   b) Prove that $\dfrac{d}{dx}(\sin x) = \cos x$

   c) Prove the sum rule of derivatives.

4. Differentiate each of the following.

   a) $f(x) = \sin x - \dfrac{1}{x}$

   b) $g(x) = x^3 + \sqrt[3]{x} + \dfrac{1}{x^3}$

   c) $f(x) = \dfrac{5x^3-2x}{5x^2}$

   d) $y = x^4 - e^4 + \cos x - \dfrac{1}{3}\sin x$

5. Find all values of a and b for which the line $y = -4x + 19$ is a tangent line to $y = -x^3 + ax^2 + bx - 8$ at $x = 3$.

6. Suppose that an object's location function is given by $L(t) = t^3 - 6t^2 + 3t - 10$. Find the moment when the object is moving downward with the greatest speed. What is that greatest speed?

7. Suppose that P is a polynomial with degree 3. Then we can write the polynomial as $P(x) = ax^3 + bx^2 + cx + d$ where a, b, c, and d are real numbers. Find the values of a, b, c, and d if we kow that $P(0) = -2$, $P'(0) = 2$, $P''(0) = 10$ and $P'''(0) = -24$.

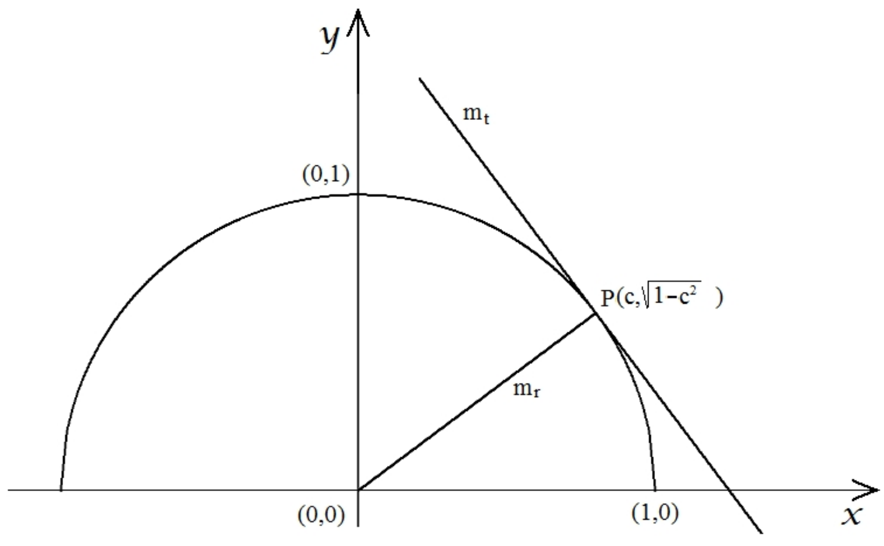
Answers

1. a) Claim: $\dfrac{d}{dx}\left(\sqrt{1-x^2}\right) = -\dfrac{x}{\sqrt{1-x^2}}$

Proof:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{\sqrt{1-(x+h)^2}-\sqrt{1-x^2}}{h} = \lim_{h\to 0}\frac{\sqrt{1-(x+h)^2}-\sqrt{1-x^2}}{h}\cdot\frac{\sqrt{1-(x+h)^2}+\sqrt{1-x^2}}{\sqrt{1-(x+h)^2}+\sqrt{1-x^2}} \\
&= \lim_{h\to 0}\frac{\left[1-(x+h)^2\right]-(1-x^2)}{h\left(\sqrt{1-(x+h)^2}+\sqrt{1-x^2}\right)} = \lim_{h\to 0}\frac{\left[1-\left(x^2+2xh+h^2\right)\right]-\left(1-x^2\right)}{h\left(\sqrt{1-(x+h)^2}+\sqrt{1-x^2}\right)} \\
&= \lim_{h\to 0}\frac{1-x^2-2xh-h^2-1+x^2}{h\left(\sqrt{1-(x+h)^2}+\sqrt{1-x^2}\right)} = \lim_{h\to 0}\frac{-2xh-h^2}{h\left(\sqrt{1-(x+h)^2}+\sqrt{1-x^2}\right)} \\
&= \lim_{h\to 0}\frac{h(-2x-h)}{h\left(\sqrt{1-(x+h)^2}+\sqrt{1-x^2}\right)} = \lim_{h\to 0}\frac{-2x-h}{\sqrt{1-(x+h)^2}+\sqrt{1-x^2}} \\
&= \frac{-2x}{\sqrt{1-x^2}+\sqrt{1-x^2}} = \frac{-2x}{2\sqrt{1-x^2}} = \boxed{\frac{-x}{\sqrt{1-x^2}}}
\end{aligned}$$

b) Let $f(x) = \sqrt{1-x^2}$ - the upper half of the unit circle. Let c be a number between 0 and 1. We will look at the tangent line drawn at the point $\left(c, \sqrt{1-c^2}\right)$.



First, let us figure out the slope of the radius. That can be easily done via the slope formula between $\left(c, \sqrt{1-c^2}\right)$ and $(0,0)$.

$$m_r = \frac{y_2-y_1}{x_2-x_1} = \frac{\sqrt{1-c^2}-0}{c-0} = \frac{\sqrt{1-c^2}}{c}$$

Recall that the derivative measures the slope of the tangent line. The derivative of $f\left(x\right)=\sqrt{1-x^2}$ is $-\dfrac{x}{\sqrt{1-x^2}}$. So, the tangent line drawn at $x=c$ will have slope $-\dfrac{c}{\sqrt{1-c^2}}$ and so

$$m_t=-\frac{c}{\sqrt{1-c^2}}$$

Now the product of the two slopes is

$$m_r\cdot m_t=\frac{\sqrt{1-c^2}}{c}\cdot\left(-\frac{c}{\sqrt{1-c^2}}\right)=-1$$

and so the two lines are perpendicular.

2. a) $\dfrac{1}{2}$ b) undefined c) -1
3. see handout
4. a) $f'\left(x\right)=\cos x+\dfrac{1}{x^2}$ b) $g'\left(x\right)=3x^2+\dfrac{\sqrt[3]{x}}{3x}-\dfrac{3}{x^4}$ c) $f'\left(x\right)=\dfrac{2}{5x^2}+1$ d) $y'=4x^3-\sin x-\dfrac{1}{3}\cos x$
5. $a=3, b=5$
6. $f'\left(2\right)=-9$
7. $a=-4$, $b=5$, $c=2$, $d=-2$

Last revised: October 5, 2017