Students must be able to correctly **state** the following theorems:
Completeness Axiom, Intermediate Value Theorem (both forms), Extreme Value Theorem,

Students must be able to prove the following theorems:

- Differentiating functions using the definition (limit of the differential quotient)
- Prove that $\dfrac{d}{dx}(\sin x) = \cos x$ and $\dfrac{d}{dx}(\cos x) = -\sin x$ and $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$
- If a function is differentiable at a number x, then it is continuous there.
- The product rule and quotient rule for derivatives.
- The Intermediate Value Theorem.

The following Sample Quiz is intended to demonstrate the difficulty level of the questions. It is not intended as a comprehensive review or list of the type of questions that can appear on the quiz.

Sample Quiz 14

1. Compute each of the following limits.

   a) $\displaystyle\lim_{x\to\infty} \frac{3^{2x-1}}{2^{3x+1}}$

   b) $\displaystyle\lim_{x\to\infty} \frac{3^{2x+1}}{2^{4x-1}}$

   c) $\displaystyle\lim_{x\to\infty} \frac{2^{x-1}-2^{-x}}{2^{x+1}+2^{-x}}$

   d) $\displaystyle\lim_{x\to\infty} x\left(\frac{1}{2}-\frac{1}{2+\frac{1}{x}}\right)$

   e) $\displaystyle\lim_{x\to\infty} \left(5^{x+1}-5^{x}\right)$

   f) $\displaystyle\lim_{x\to\infty} \frac{\cos^2 x}{x}$

   g) $\displaystyle\lim_{x\to\infty} \left(\sqrt{2x}-\sqrt{x}\right)$

   h) $\displaystyle\lim_{x\to\infty} \frac{3\cdot 2^{2x}-2^{x}+1}{-4^{x}+2^{x}-1}$

   i) $\displaystyle\lim_{x\to 0} \frac{\sin 2x}{x}$

2. Compute each of the following limits. Assume that $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$ and $\displaystyle\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x} = e$.

   a) $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)$

   b) $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}$

   c) $\displaystyle\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^{x}$

   d) $\displaystyle\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{x}$

   e) $\displaystyle\lim_{x\to\infty}\left(1-\frac{2}{x+1}\right)^{x}$

   f) $\displaystyle\lim_{x\to\infty}\left(1-\frac{4}{x}\right)^{2x}$

3. Differentiate each of the following.

   a) $f(x) = -4x^3\cos x + \dfrac{\sin x}{x}$

   b) $f(x) = \sqrt{x^5}\left(\cos^2 x + \sin^2 x\right)$

   c) $f(x) = (2\sin x - \cos x + 1)\left(3x^2 - x + 1\right)$

   d) $f(x) = \dfrac{1}{3}x^3\ln x - \dfrac{1}{9}x^3 - 3$

   e) $f(x) = \dfrac{3x^2-8x+2}{\sin x}$

   f) $f(\theta) = \tan\theta$

   g) $f(x) = \dfrac{\ln x}{\sqrt{x}}$

4. Prove that the function $f(x) = x^6 + x^4 - 1$ has at least one zero.

5. Find all relative and absolute maximums and minimums for each of the following functions.

   a) $f(x) = 3x^5 - 20x^3 - 6$ on $[-3, 3]$

   b) $f(x) = x^3 - 2x^2 - 15x + 8$ on $[-4, 5]$

   c) $f(x) = 2x^6 - 3x^4 + 1$ on $[-1, 1]$

   d) $f(x) = -3x^4 - 8x^3 + 48x^2 + 60$ on $[-5, 3]$

6. Find an equation for the tangent line drawn to the graph of $f(x) = \cos x$ at $x = \dfrac{\pi}{3}$.

7. a) Find the values of a and b so that $f(x) = 2x^3 - ax^2 + bx + 4$ has a relative maximum at $x = -2$ and a relative minimum at $x = 5$.

   b) Find the value of a if $f(x) = \dfrac{(x-a)(x-2)}{x^2}$ has a relative minimum at $x = 3$.

8. Find two nonnegative numbers whose sum is 9 and so that the product of one number and the square of the other number is the greatest possible.

9. An open rectangular box with square base is to be made from $60\,\text{ft}^2$ of material. What dimensions will result in a box with the largest possible volume?

10. A container in the shape of a right circular cylinder with no top has surface area $10\,\text{ft}^2$. What height h and base radius r will maximize the volume of the cylinder ?

11. We want to make a flower bed in the shape of a sector in a circle. If we want the area of the garden to be $100\,\text{ft}^2$, what dimensions would guarantee the least perimeter?

Answers

1. a) ∞ b) 0 c) $\dfrac{1}{4}$ d) $\dfrac{1}{4}$ e) ∞ f) 0 g) ∞ h) -3 i) 2

2. a) 1 b) e^3 c) e^4 d) $\dfrac{1}{e}$ e) e^{-2} f) e^{-8}

3. a) $f'(x) = -12x^2\cos x + 4x^3\sin x + \dfrac{1}{x}\cos x - \dfrac{1}{x^2}\sin x$ b) $f'(x) = \dfrac{5}{2}x\sqrt{x}$

   c) $f'(x) = (6x-1)(2\sin x - \cos x + 1) + (2\cos x + \sin x)\left(3x^2 - x + 1\right)$

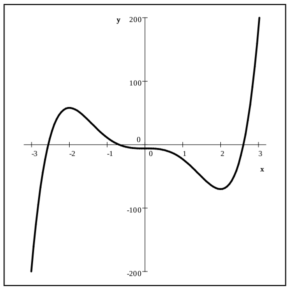
   d) $f'(x) = x^2 \ln x$ e) $f'(x) = \dfrac{\sin x\,(6x-8) - \cos x\left(3x^2 - 8x + 2\right)}{\sin^2 x}$

   f) $f'(\theta) = \sec^2\theta$ g) $f'(x) = \dfrac{2 - \ln x}{2x^{3/2}}$

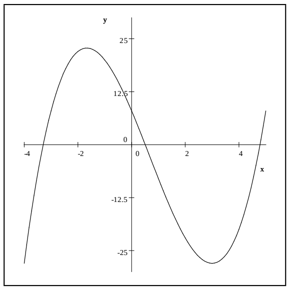
4. Claim: the function $f(x) = x^6 + x^4 - 1$ has at least one zero.

   Proof: Since f is a polynomial, it is continuous on $\mathbb{R}$. $f(0) = -1$ and $f(1) = 1$ and so by the Intermediate Value Theorem, f has a zero between 0 and 1.

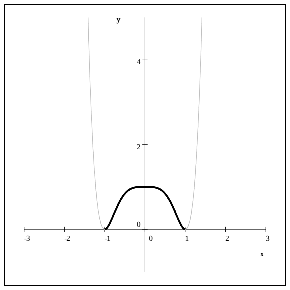
5. a) $f(x) = 3x^5 - 20x^3 - 6$ on $[-3, 3]$
   relative maximum: $(-2, 58)$
   absolute maximum: $(3, 183)$
   relative minimum: $(2, -70)$
   absolute minimum: $(-3, -195)$



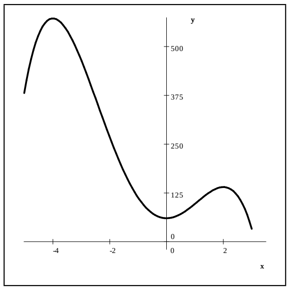
b) $f(x) = x^3 - 2x^2 - 15x + 8$ on $[-4, 5]$
relative maximum: $\left(-\frac{5}{3}, \frac{616}{27}\right)$
absolute maximum: $\left(-\frac{5}{3}, \frac{616}{27}\right)$
relative minimum: $(3, -28)$
absolute minimum: $(3, -28)$ and $(-4, -28)$



c) $f(x) = 2x^6 - 3x^4 + 1$ on $[-1, 1]$
relative maximum: $(0, 1)$
absolute maximum: $(0, 1)$
relative minimum: none
absolute minimum: $(-1, 0)$ and $(1, 0)$



d) $f(x) = -3x^4 - 8x^3 + 48x^2 + 60$ on $[-5, 3]$
relative maximum: $(-4, 572)$ and $(2, 140)$
absolute maximum: $(-4, 572)$
relative minimum: $(0, 60)$
absolute minimum: $(3, 33)$



6. $-\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) = y - \frac{1}{2}$

7. a) $a = 9, b = -60$ b) $f(x) = \frac{x^2 + x(-a-2) + 2a}{x^2} = 1 + \frac{-a-2}{x} + \frac{2a}{x^2}$

Then $f'(x) = \frac{d}{dx}\left(1 + \frac{-a-2}{x} + \frac{2a}{x^2}\right) = \frac{1}{x^2}(a+2) - 4\frac{a}{x^3}$

$f'(3) = 0 \implies \frac{1}{3^2}(a+2) - 4\frac{a}{3^3} = 0$ We solve for a and obtain 6

8. 3 and 6 with a product of 108

9. base: $2\sqrt{5}$ ft height: $\sqrt{5}$ ft $V_{\max} = 20\sqrt{5} =$ ft^3

10. $r = \sqrt{\frac{10}{3\pi}}$ $h = r$

11. $r = 10$ ft and $\theta = 2$ radian

Solution: $100 = \frac{1}{2}r^2\theta$ Solve for θ: $\theta = \frac{200}{r^2}$

The perimeter is then $P = 2r + r\theta = 2r + r\left(\frac{200}{r^2}\right) = 2r + \frac{200}{r}$

$P'(r) = 2 - \frac{200}{r^2}$ We solve for the zero of the derivative:

$2 - \frac{200}{r^2} = 0 \implies r = \pm 10$ (the negative root is ruled out) and then $\theta = 2$ radian