Students must be able to correctly **state** the following theorems
Completeness Axiom, Intermediate Value Theorem (both forms), Extreme Value Theorem, Rolle's Theorem, Mean Value Theorem

Students must be able to prove the following theorems:
Differentiating functions using the definition (limit of the differential quotient); $\lim\limits_{x\to 0^+}\dfrac{\sin x}{x}=1$; $\dfrac{d}{dx}(\sin x)=\cos x$ and $\dfrac{d}{dx}(\cos x)=-\sin x$ and $\dfrac{d}{dx}(\ln x)=\dfrac{1}{x}$; If a function is differentiable at a number x, then it is continuous there. The product rule and quotient rule for derivatives; The Intermediate Value Theorem, The Mean Value Theorem

The following Sample Quiz is intended to demonstrate the difficulty level of the questions. It is not intended as a comprehensive review or list of the type of questions that can appear on the quiz.

Sample Quiz 16

1. Compute each of the following limits.

   a) $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{x}\right)^x$ b) $\lim\limits_{x\to 3^+}\dfrac{|x-3|}{x-3}$ c) $\lim\limits_{x\to 5}\dfrac{\sqrt{x-5}}{x-5}$ d) $\lim\limits_{x\to\pi}\dfrac{1+\cos x}{(x-\pi)^2}$ e) $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\tan 3x}$

2. Differentiate each of the following.

   a) $f(x)=\sin x-x\cos x$ b) $f(x)=\cos^2 x$ c) $f(r)=\tan r$ d) $f(x)=\dfrac{\ln x}{x^2+1}$

3. Simplify each of the given expressions. Use exact values.

   a) $\sin^{-1}\left(-\dfrac{1}{2}\right)$ b) $\tan^{-1}(-1)$ c) $\cos\left(\tan^{-1}\left(\dfrac{2}{3}\right)\right)$ d) $\sin\left(\cos^{-1}(x)\right)$ e) $\tan\left(\sin^{-1}(x)\right)$

4. Find all relative and absolute maximums and minimums for each of the following functions.

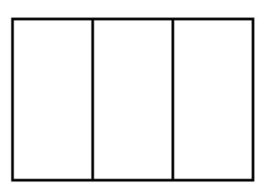
   a) $f(x)=x^4-8x^2+5$ on $[-3,2]$ b) $f(x)=x^5-5x^4+5x^3+2$ on $[-1,3]$

5. Find all values of c that satisfy the conclusion of the Mean Value Theorem.

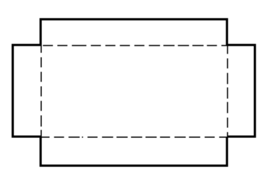
   a) $f(x)=x^3-x$ on $[-2,6]$ b) $f(x)=\sqrt{x}$ on $[0,4]$ c) $f(x)=\sin x$ on $[0,\pi]$ d) $f(x)=\dfrac{1}{x}$ on $[1,5]$

6. We know the following about an object: it has a constant acceleration of $-2\dfrac{\text{m}}{\text{s}^2}$, at $t=5\,\text{s}$ it had a velocity of $4\dfrac{\text{m}}{\text{s}}$, and at $t=10\,\text{s}$ it had a location of $120\,\text{m}$. What is the initial velocity and initial location of the object?

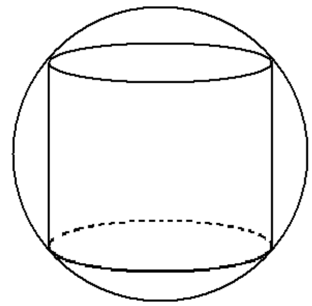
7. We have 60 meters of fencing and want to create three adjacent rectangular enclosings as shown on the figure. What is the maximal area we can enclose this way?



8. A rectangular box, open at the top, is to be constructed from a rectangular sheet of cardboard 18 cm by 36 cm by cutting out equal squares in the corners and folding up the sides. What sides squares should be cut out for the container to have maximal volume?

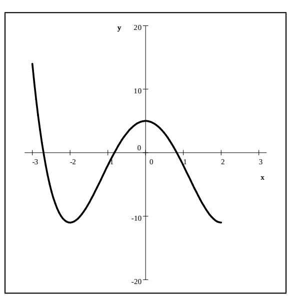


9. Find the radius and height of the greatest (by volume) cylinder that can be written into a sphere of radius 6.

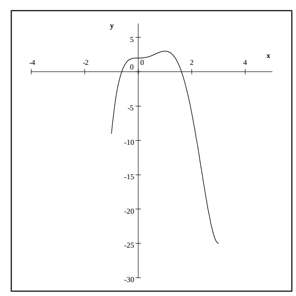


Answers

1. a) e^2 b) 1 c) undefined d) $\frac{1}{2}$ e) $\frac{5}{3}$
2. a) $x \sin x$ b) $-2 \sin x \cos x = -\sin 2x$ c) $\sec^2 r$
   d) $\frac{1}{x(x^2+1)} - \frac{2x \ln x}{(x^2+1)^2}$
3. a) $-\frac{\pi}{6}$ b) $-\frac{\pi}{4}$ c) $\frac{3\sqrt{13}}{13}$ d) $\sqrt{1-x^2}$
   e) $\frac{x}{\sqrt{1-x^2}}$
4. a) $f(x) = x^4 - 8x^2 + 5$ on $[-3, 2]$
   relative maximum: $(0, 5)$
   absolute maximum: $(-3, 14)$
   relative minimum: $(-2, -11)$
   absolute minimum: $(-2, -11)$ and $(2, -11)$



   b) $f(x) = x^5 - 5x^4 + 5x^3 + 2$ on $[-1, 3]$
   relative maximum: $(1, 3)$
   absolute maximum: $(1, 3)$
   relative minimum: none
   absolute minimum: $(3, -25)$



5. a) $\frac{2}{3}\sqrt{21}$ b) 1 c) $\frac{\pi}{2}$ d) $\sqrt{5}$
6. $v(0) = 14 \frac{\text{m}}{\text{s}}$ and $s(0) = 80\,\text{m}$
   (for detailed solutions, see next page)
7. An area of $112.5\,\text{m}^2$ if horizontal side is $15\,\text{m}$ and the vertical is $7.5\,\text{m}$.
8. $(9 - 3\sqrt{3})$ cm
9. $r_{\max} = 2\sqrt{6}$ and $h_{\max} = 4\sqrt{3}$
   (for detailed solutions, see next page)

Solutions

6. We know the following about an object: it has a constant acceleration of $-2\frac{\text{m}}{\text{s}^2}$, at $t=5\,\text{s}$ it had a velocity of $4\frac{\text{m}}{\text{s}}$, and at $t=10\,\text{s}$ it had a location of $120\,\text{m}$. What is the initial velocity and initial location of the object?

   $v(t)=\int a(t)\,dt=\int(-2)\,dt=-2t+C$ $\boxed{v(t)=-2t+C}$

   Now set $t=5\,(\text{s})$

$$\begin{aligned} v(t) &= -2t+C \qquad \text{set } t=5 \\ 4 &= v(5) \\ 4 &= -2(5)+C \quad\Longrightarrow\quad C=14 \quad \text{and so } v(t)=-2t+14 \end{aligned}$$

   Now set $t=0$

$$\begin{aligned} v(t) &= -2t+14 \\ v(0) &= -2(0)+14 \\ v(0) &= 14 \end{aligned}$$

   Now for the location function:

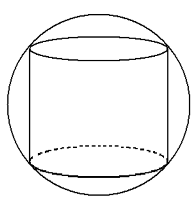
$$\begin{aligned} s(t) &= \int v(t)\,dt=\int(-2t+14)\,dt=-t^2+14t+C_2 \\ s(t) &= -t^2+14t+C_2 \end{aligned}$$

   Now set $t=10$

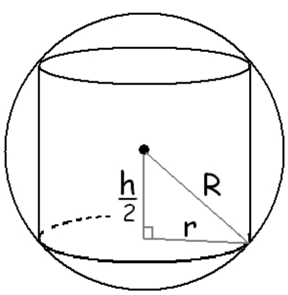
$$\begin{aligned} s(10) &= 120 \\ -(10)^2+14(10)+C_2 &= 120 \\ 40+C_2 &= 120 \\ C_2 &= 80 \end{aligned}$$

   So $s(t)=-t^2+14t+80$. The initial location is then $80\,(\text{m})$.

9. Find the radius and height of the greatest (by volume) cylinder that can be written into a sphere of radius 6.



   Solution: Let r represent the radius of the base of the cylinder, h represent the height of the cylinder, and R the radius of the sphere.



   constraint: $\left(\frac{h}{2}\right)^2+r^2=R^2$

   the quatity to be maximized: $V=\pi r^2 h$

   Solve for r^2 in the constraint: $r^2=R^2-\frac{h^2}{4}$

Now we can express the volume as a function of h

$$\begin{aligned} V &= \pi r^2 h = \pi\left(R^2 - \frac{h^2}{4}\right)h = \pi\left(R^2 - \frac{h^2}{4}\right)h = -\frac{1}{4}\pi h^3 + \pi R^2 h \\ V(h) &= -\frac{1}{4}\pi h^3 + \pi R^2 h \quad\Longrightarrow\quad V'(h) = -\frac{3}{4}\pi h^2 + \pi R^2 \end{aligned}$$

Since V' is a downward opening parabola, it's first $x-$intercept will indicate a relative minimum, and its second a relative maximum for V. Solve for the zeroes of V':

$$\begin{aligned} V'(h) &= 0 \\ -\frac{3}{4}\pi h^2 + \pi R^2 &= 0 \\ \pi R^2 &= \frac{3}{4}\pi h^2 \\ \frac{4R^2}{3} &= h^2 \qquad\Longrightarrow\qquad h = \pm\frac{2}{\sqrt{3}}R \end{aligned}$$

Or, we can just as well factor V' via the difference of squares theorem.

$$V'(h) = -\frac{3}{4}\pi h^2 + \pi R^2 = -\frac{3}{4}\pi\left(h^2 - \frac{4}{3}R^2\right) = -\frac{3}{4}\pi\left(h^2 - \left(\tfrac{2}{\sqrt{3}}R\right)^2\right) = -\frac{3}{4}\pi\left(h + \tfrac{2}{\sqrt{3}}R\right)\left(h - \tfrac{2}{\sqrt{3}}R\right)$$

So the second zero at $h = \frac{2}{\sqrt{3}}R$ indicates relative maximum. An investigation of the domain will reveal it to be an absolute, not just a relative maximum. We can compute now r:

$$r^2 = R^2 - \frac{h^2}{4} = r^2 = R^2 - \frac{\left(\frac{2}{\sqrt{3}}R\right)^2}{4} = R^2 - \frac{1}{3}R^2 = \frac{2}{3}R^2 \qquad\Longrightarrow\quad r = \pm\sqrt{\frac{2}{3}}R$$

So the greatest volume is with base radius $r = \sqrt{\frac{2}{3}}R$ and height $h = \frac{2}{\sqrt{3}}R$. If we rationalize these, we get

$$r_{\max} = \frac{\sqrt{6}}{3}R \quad\text{and}\quad h_{\max} = \frac{2\sqrt{3}}{3}R$$

These are similar expressions. The ratio between h and r here is $\dfrac{h}{r} = \dfrac{\frac{2\sqrt{3}}{3}R}{\frac{\sqrt{6}}{3}R} = \sqrt{2}$, indicating that h is exactly $\sqrt{2}$ times r. In this problem, $R = 6$. So $r_{\max} = \frac{\sqrt{6}}{3}6 = 2\sqrt{6}$ and $h_{\max} = 4\sqrt{3}$