Differentiating inverse Trig functions

Induction

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6 inverse trig functions:

arcsinx same as $\sin^{-1}x$	different from $\frac{1}{\sin x} = \csc x$

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① $\frac{d}{dx}(\sin^{-1}x) = ?$

$$\sin(\sin^{-1}(x)) = x$$

$$\cos(\sin^{-1}x) \cdot \frac{d}{dx}(\sin^{-1}(x)) = 1$$

$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\cos(\sin^{-1}x)} = \frac{1}{\sqrt{1-x^2}}$$

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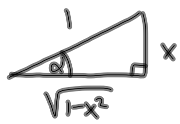
Simplify $\cos(\underbrace{\sin^{-1}x}_{\alpha})$

Let $\alpha = \sin^{-1}x$

$\sin\alpha = x$

$\cos\alpha = ?$



$$\cos(\sin x) = \sqrt{1-x^2}$$

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So $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

domain: $[-1,1]$ domain: $(-1,1)$

What happened at $x = \pm 1$?

EC

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② $\frac{d}{dx}(\cos^{-1}x) = ?$

$$\cos(\cos^{-1}x) = x$$

$$-\sin(\cos^{-1}x) \cdot \frac{d}{dx}(\cos^{-1}x) = 1$$

$$\frac{d}{dx}(\cos^{-1}x) = \frac{1}{-\sin(\cos^{-1}x)} = \frac{-1}{\sqrt{1-x^2}}$$

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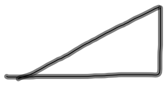
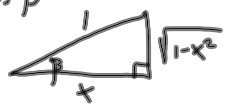


$$\sin(\underbrace{\cos^{-1}x}_{\beta}) = ?$$

$\sin\beta = ?$

$\cos\beta = x$



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We found that the derivatives for $\sin^{-1}x$ and $\cos^{-1}x$ are opposites.

For extra credit, present another proof for this.

Hint: what does it mean for f and g that $f' = -g'$?

Hint 2. First think within the 1st quadrant $(0, \frac{\pi}{2})$

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$$\frac{d}{dx}(\tan^{-1}x) = ?$$

$$\tan(\tan^{-1}(x)) = x$$

$$\left[1 + \tan^2(\tan^{-1}(x))\right] \cdot \frac{d}{dx}(\tan^{-1}x) = 1$$

$$(1+x^2) \cdot \frac{d}{dx}(\tan^{-1}x) = 1$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

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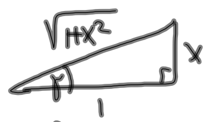
OR $\tan(\tan^{-1}(x)) = x$

$$\sec^2(\tan^{-1}x) \cdot \frac{d}{dx}(\tan^{-1}x) = 1$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{\sec^2(\underbrace{\tan^{-1}x}_{\gamma})} = \frac{1}{1+x^2}$$

$\tan\gamma = x$

$\sec^2\gamma = ?$



$\cos\gamma = \frac{1}{\sqrt{1+x^2}}$ $\sec^2\gamma = 1+x^2$

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④ $\frac{d}{dx}(\cot^{-1}x) = ?$

First compute $\frac{d}{dx}(\cot x)$

$$\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} = \frac{-1}{\sin^2 x}$$

OR $\frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-\sin^2 x}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x}$ $= -\csc^2 x$

$$= -1 - \cot^2 x$$

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$$\cot(\cot^{-1}x) = x$$

$$\left[-1 - \cot^2(\cot^{-1}x)\right] \frac{d}{dx}(\cot^{-1}x) = 1$$

$$\frac{d}{dx}(\cot^{-1}x) = \frac{1}{-1-x^2}$$

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⑤ $\frac{d}{dx}\left(\sec^{-1}x\right) = ?$

$$\sec\left(\sec^{-1}x\right) = x$$

$$\sec\left(\sec^{-1}x\right)\cdot\tan\left(\sec^{-1}x\right)\cdot\frac{d}{dx}\left(\sec^{-1}x\right) = 1$$

$$\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{x\cdot\tan\left(\sec^{-1}x\right)} = \frac{1}{|x|\sqrt{x^2-1}}$$

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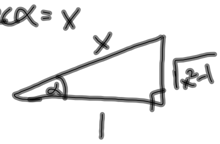
$$\tan(\underbrace{\sec^{-1}x}_{\alpha})$$

$$= \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$$

$\tan\alpha = ?$

$\sec\alpha = x$



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① $\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}$

② $\frac{d}{dx}\left(\cos^{-1}x\right) = \frac{-1}{\sqrt{1-x^2}}$

③ $\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$

④ $\frac{d}{dx}\left(\cot^{-1}x\right) = \frac{-1}{1+x^2}$

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On a related note,

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$$

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Induction

$$1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$n=1$ $\quad LHS = 1^2$ $\quad RHS = \frac{1\cdot2\cdot3}{6} = 1$

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$n=2$ $\quad LHS = 1^2 + 2^2 = 5$

$RHS = \frac{2\cdot3\cdot5}{6} = 5$

$n=3$ $\quad LHS = 1^2 + 2^2 + 3^2 = 14$

$RHS = \frac{3\cdot4\cdot7}{6} = 14$

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$n=4$ $LHS = 1^2+2^2+3^2+4^2 = 30$

$RHS = \frac{4\cdot5\cdot9}{6} = 30$

Induction: Part 1. Check for $n=1,2,3,4$

Part 2. We prove: if it is true for k, then it will also be true for $k+1$

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for part 2, see handout

Claim: $1+2+3+\ldots+n = \frac{n(n+1)}{2}$

$n=1$ $1 = \frac{1\cdot2}{2}$ ✓

$n=2$ $1+2 \stackrel{?}{=} \frac{2\cdot3}{2}$ ✓

$n=3$ $1+2+3 \stackrel{?}{=} \frac{3\cdot4}{2}$ ✓

$n=4$ $1+2+3+4=10 \stackrel{?}{=} \frac{4\cdot5}{2}=10$ ✓

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Part 2:

Suppose

$$1+2+\ldots+k \stackrel{IH}{=} \frac{k(k+1)}{2} \qquad / \text{Add } k+1$$

$$1+2+\ldots+k+k+1 = \frac{k(k+1)}{2} + k+1$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

Want: $1+2+\ldots+k+k+1 = \frac{(k+1)(k+2)}{2}$

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Claim: Any 10-power can be written as a sum of 2 squares.

$10 = 1^2+3^2$

$100 = 8^2+6^2$

$1000 = 10^2+30^2$

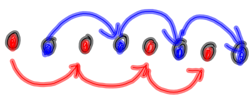
$10\,000 = 80^2+60^2$

$10^k = x^2+y^2$

$10^k \cdot 100$

$10^{k+2} = 100x^2 + 100y$

$= (10x)^2+(10y)^2$



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