1. Differentiate each of the following.

a) $f(x) = 3x^4 - x^3 + 4x^2 - x + 7$

b) $f(x) = \sqrt[5]{2x^6 + 5x^2 + 1}$

c) $f(x) = \cos x + x \sin x$

d) $f(x) = \cos\left(2x - \dfrac{\pi}{2}\right)$

e) $f(x) = (3x-1)^{100}$

f) $f(x) = \dfrac{-x+5}{\sqrt{x^2+1}}$

g) $f(x) = \dfrac{x^2+1}{\sin 5x}$

h) $f(x) = \sin^{-1}\left(x^2\right)$

i) $f(x) = \sqrt{1+\sqrt{x}}$

j) $f(x) = \tan 3x$

k) $f(x) = \sec x + \tan x$

2. Find the exact value of each of the following.

a) $\sin\left(\dfrac{-7\pi}{3}\right)$

b) $\sin\left(\cos^{-1}\left(-\dfrac{1}{2}\right)\right)$

c) $\tan^{-1}\left(\tan\left(\dfrac{7\pi}{3}\right)\right)$

3. What is the exact value of $\sin x$ if $\tan x = -2$?

4. Assume that for all real numbers x and y,

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \sin(x+y) &= \sin x \cos y + \cos x \sin y \quad \text{and} \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \end{aligned}$$

Prove each of the following.

a) $\sin(x-y) = \sin x \cos y - \cos x \sin y$

b) $\cos 2x = 2\cos^2 x - 1$

c) $\cos 2x = 1 - 2\sin^2 x$

d) $\sin x = \pm\sqrt{\dfrac{1-\cos 2x}{2}}$

e) $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

f) $\sec^2 x = 1 + \tan^2 x$

g) $\cos^2 x = \dfrac{1}{2}(\cos 2x + 1)$

h) $\dfrac{\sin x}{1+\cos x} = \dfrac{1-\cos x}{\sin x}$

5. Simplify each of the following

a) $\sin\left(\sin^{-1} x\right)$

b) $\cos\left(\sin^{-1} x\right)$

c) $\sin\left(\tan^{-1} x\right)$

d) $\tan\left(\cos^{-1} x\right)$

6. Prove that the following expressions are all equivalent.

$$A = \sqrt{\frac{1+\sin x}{1-\sin x}} \qquad B = \frac{1+\sin x}{\cos x} \qquad C = \sec x + \tan x \qquad D = \frac{\cos x}{1-\sin x} \qquad E = \frac{1}{\sec x - \tan x}$$

(Hint: prove that $A = B$ and that $B = C$ and that $B = D$ and then $D = E$)

7. Prove each of the following.

a) $\log_{24} 90 = \dfrac{\ln 2 + 2\ln 3 + \ln 5}{3\ln 2 + \ln 3}$

b) $2\log_{10}(2x) + \log_{10}(25x) - 3\log_{10}(0.1x) = 5$

c) $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8 = 3$

d) $\log_3 |\tan x| = -\log_3 |\cot x|$

8. Compute each of the following limits.

a) $\displaystyle\lim_{x\to\infty}\frac{\sin 5x}{x}$

b) $\displaystyle\lim_{x\to 0}\frac{\sin 5x}{x}$

c) $\displaystyle\lim_{x\to 0}\frac{\sqrt{9+x}-3}{x}$

d) $\displaystyle\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}$

e) $\displaystyle\lim_{x\to 2}\frac{1}{x^2-4}$

f) $\displaystyle\lim_{x\to 2^-}\frac{1}{x^2-4}$

g) $\displaystyle\lim_{x\to 0}\frac{\cos x+1}{x-\pi}$

h) $\displaystyle\lim_{x\to 0^+}\log_3 x$

i) $\displaystyle\lim_{x\to\infty}\tan^{-1}x$

j) $\displaystyle\lim_{x\to 1^-}\frac{x^2+x-2}{x^2-1}$

k) $\displaystyle\lim_{x\to -1^-}\frac{x^2+x-2}{x^2-1}$

l) $\displaystyle\lim_{x\to\infty}\frac{x^2+x-2}{x^2-1}$

9. Use implicit differentiation to differentiate each of the following.

a) $\left(x^2-y^2\right)^4=2xy^2$

b) $(x+y)^3=\sin x-\sin y$

10. Prove that $\dfrac{d\left(x^2-x\right)}{dx}=2x-1$, using the definition of the derivative as the limit of the differential quotient.

11. Prove that if $f(x)=\sin^{-1}x$ then $f'(x)=\dfrac{1}{\sqrt{1-x^2}}$

12. Evaluate each of the following integrals.

a) $\displaystyle\int\frac{1}{x^2+1}\,dx$

b) $\displaystyle\int\frac{x^2}{x^2+1}\,dx$

c) $\displaystyle\int\sin 5x\,dx$

d) $\displaystyle\int\frac{1}{-x+2}\,dx$

e) $\displaystyle\int(3x-1)^{10}\,dx$

f) $\displaystyle\int\frac{1}{\sqrt{1-9x^2}}\,dx$

g) $\displaystyle\int\frac{1}{x-5}\,dx$

h) $\displaystyle\int 3-\frac{2}{x-5}\,dx$

i) $\displaystyle\int\frac{3x-17}{x-5}\,dx$

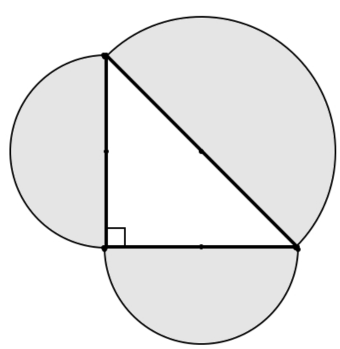
13. Express each of the following in terms of the variable given.

a) Let a be the side of a regular triangle. Express its area in terms of a.

b) Let h be the height of a regular triangle. Express its area in terms of h.

c) Let A be the area of a square. Express its perimeter in terms of A.

d) Let c be the hypotenuse of an isosceles right triangle. We write a semicircle on each of its sides as shown on the picture below. Express the area of the shaded region in terms of c.



Answers

1. a) $f'(x) = 12x^3 - 3x^2 + 8x - 1$ b) $f'(x) = \dfrac{12x^5 + 10x}{5\left(2x^6 + 5x^2 + 1\right)^{4/5}} = \dfrac{1}{5}\dfrac{12x^5 + 10x}{2x^6 + 5x^2 + 1}\sqrt[5]{2x^6 + 5x^2 + 1}$

   c) $f'(x) = x\cos x$ d) $f'(x) = -2\sin\left(2x - \dfrac{\pi}{2}\right)$ e) $f'(x) = 300\,(3x-1)^{99}$

   f) $f'(x) = \dfrac{-5x - 1}{\left(x^2+1\right)^{3/2}}$ g) $f'(x) = \dfrac{2x\sin 5x - 5\left(x^2+1\right)\cos 5x}{\sin^2 5x}$ h) $f'(x) = \dfrac{2x}{\sqrt{1 - x^4}}$

   i) $f'(x) = \dfrac{1}{4\sqrt{x}\sqrt{\sqrt{x}+1}}$ j) $f'(x) = 3\sec^2 3x = 3\tan^2 3x + 3$

   k) $f'(x) = \sec x \tan x + \sec^2 x = \tan^2 x + 1 + \dfrac{\sin x}{\cos^2 x}$

2. a) $-\dfrac{\sqrt{3}}{2}$ b) $\dfrac{\sqrt{3}}{2}$ c) $\dfrac{\pi}{3}$

3. $\pm\dfrac{2\sqrt{5}}{5}$

4. see solutions

5. a) x b) $\sqrt{1-x^2}$ c) $\dfrac{x}{\sqrt{x^2+1}}$ d) $\dfrac{\sqrt{1-x^2}}{x}$

6. see solutions

7. see solutions

8. a) 0 b) 5 c) $\dfrac{1}{6}$ d) $-\dfrac{1}{4}$ e) undefined f) $-\infty$ g) $-\dfrac{2}{\pi}$ h) $-\infty$ i) $\dfrac{\pi}{2}$ j) $\dfrac{3}{2}$
   k) $-\infty$ l) 1

9. a) $y' = \dfrac{-y^2 + 4x\left(x^2 - y^2\right)^3}{2xy + 4y\left(x^2 - y^2\right)^3}$ b) $y' = \dfrac{\cos x - 3\,(x+y)^2}{\cos y + 3\,(x+y)^2}$

10. see solutions

11. see solutions

12. a) $\tan^{-1} x + C$ b) $x - \tan^{-1} x + C$ c) $-\dfrac{1}{5}\cos 5x + C$ d) $-\ln|-x+2| + C$
    e) $\dfrac{(3x-1)^{11}}{33} + C$ f) $\dfrac{1}{3}\sin^{-1}(3x) + C$ g) $\ln|x-5| + C$ h) $3x - 2\ln|x-5| + C$
    i) $3x - 2\ln|x-5| + C$
    Solution for b):
    $$\int \frac{x^2}{x^2+1}dx = \int \frac{x^2+1-1}{x^2+1}dx = \int \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}dx = \int 1 - \frac{1}{x^2+1}dx = \int 1dx - \int \frac{1}{x^2+1}dx = x - \tan^{-1} x + C$$

13. a) $\dfrac{\sqrt{3}}{4}a^2$ b) $\dfrac{\sqrt{3}}{3}h^2$ c) $4\sqrt{A}$ d) $\dfrac{1}{4}\pi c^2$

