Exam 1 will cover the following topics from the book:

Review of differential calculus: Chapter 3: all, Chapter 4: 4.7
Integral calculus: Chapter 5: 5.1-5.5, Chapter 7: 7.1-7.3, 7.5-7.7, Chapter 8: 8.1, 8.2, 8.3, 8.4, 8.7

Handouts to study: all posted during the first 10 classes.

Review Problems

1. Compute each of the following limits.

   a) $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x+\tan^{-1}x}{e^x-1}$

   b) $\displaystyle\lim_{x\to 1}\frac{\ln x}{x^5-x^2}$

   c) $\displaystyle\lim_{x\to 0}\frac{\cos 2x-1}{\tan^4 x}$

   d) $\displaystyle\lim_{x\to 0}\frac{\tan^{-1}3x}{\tan^{-1}4x}$

   e) $\displaystyle\lim_{x\to 0}\frac{1-\sec 3x}{x^2}$

2. a) Sketch the graph of $y=\csc x$. State its domain, range, and basic properties.

   b) Derive the formula for $\dfrac{d}{dx}\csc x$.

   c) Derive the formula for $\displaystyle\int \csc x\,dx$

   d) Sketch the graph of $y=\csc^{-1}x$. State its domain, range, and basic properties.

   e) Derive the formula for $\dfrac{d}{dx}\csc^{-1}x$.

   f) Derive the formula for $\displaystyle\int \csc^{-1}x\,dx$

3. a) Graph the function $f(x)=\tanh x$ and state its basic properties.

   b) Derive the formula for $\dfrac{d}{dx}(\tanh x)$

   c) Derive the formula for $\displaystyle\int \tanh^{-1}x\,dx$

   d) Derive the formula for $\displaystyle\int \operatorname{csch} x\,dx$

4. Differentiate each of the following.

   a) $f(x)=\ln\left(x+\sqrt{x^2+25}\right)$

   b) $f(x)=-\dfrac{1}{9}e^{-3x}-\dfrac{1}{3}xe^{-3x}$

   c) $g(x)=2^{\cos x}$

   d) $f(\theta)=\cot^{-1}\left(5\theta^2\right)$

   e) $f(x)=\tanh^{-1}x$

   f) $g(x)=\displaystyle\int_1^{x^3}\frac{1}{t^3+1}dt$

   g) $P(x)=\left[\displaystyle\int_0^x\frac{1}{t^2+1}dt\right]^3$

5. Suppose that $x>1$. Use a trigonometric substitution to compute $\displaystyle\int\frac{1}{x\sqrt{x^2-1}}dx$.

6. Compute each of the following integrals.

a) $\displaystyle\int_0^1 \ln x \; dx$

b) $\displaystyle\int e^x \sin x dx$

c) $\displaystyle\int_0^3 \frac{1}{\sqrt{x}} dx$

d) $\displaystyle\int \tanh x dx$

e) $\displaystyle\int \frac{1}{\sqrt{x^2+1}} \; dx$

f) $\displaystyle\int \cot x \; dx$

g) $\displaystyle\int \frac{1}{x^2+3} \; dx$

h) $\displaystyle\int \frac{1}{x\sqrt{x^2-1}} dx$

i) $\displaystyle\int xe^{-4x} \; dx$

j) $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}} dx$

k) $\displaystyle\int_0^{\pi/2} \cos^2 x \sin^3 x \; dx$

l) $\displaystyle\int \frac{1}{\sqrt{4-x^2}} dx$

m) $\displaystyle\int_0^{\pi/2} \sin^3 x \; dx$

n) $\displaystyle\int_3^{11} \frac{x}{\sqrt{x-2}} \; dx$

o) $\displaystyle\int \cos 5x \cos 2x \; dx$

p) $\displaystyle\int \sin^{-1} x \; dx$

q) $\displaystyle\int \frac{x}{\sqrt{25-x^2}} dx$

r) $\displaystyle\int \frac{2x^2-x+4}{x^3+4x} dx$

s) $\displaystyle\int \frac{1}{a^2+(bx)^2} dx$

t) $\displaystyle\int \frac{1-\tan\theta}{1+\tan\theta} d\theta$

u) $\displaystyle\int_0^2 x^2 \ln x dx$

v) $\displaystyle\int_1^\infty \frac{\ln x}{x^2} dx$

w) $\displaystyle\int_{\pi/4}^{\pi/2} \tan x dx$

x) $\displaystyle\int \frac{2x^3-5x^2+7x-6}{x^2-2x+1} dx$

w) $\displaystyle\int_0^\infty xe^{-4x^2} dx$

z) $\displaystyle\int_0^\infty xe^{-5x} dx$

7. Compute each of the following.

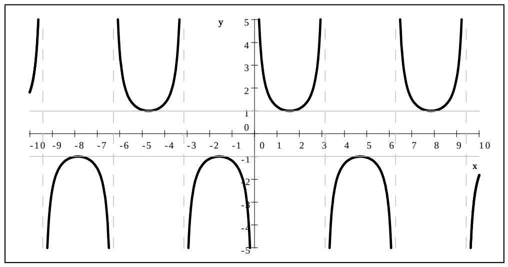
a) $\displaystyle\int \sec^3 x dx$

b) $\displaystyle\int_0^{\pi/4} \sec^3 x dx$

Answers

1. a) 2 b) $\dfrac{1}{3}$ c) $-\infty$ d) $\dfrac{3}{4}$ e) $-\dfrac{9}{2}$

2. a) $y = \csc x$
domain: $\{x : x \neq k\pi, \quad k \in \mathbb{Z}\}$ range: $(-\infty, -1] \cup [1, \infty)$ periodic with period 2π



b) Solution: see handout differentiating trigonometric functions

c) Solution 1:

$$\int \csc x dx = \int \csc x \cdot \frac{\csc x + \cot x}{\csc x + \cot x} dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx$$

Let $u = \csc x + \cot x$. Then $du = -\csc x \cot x - \csc^2 x$ and so

$$\int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx = \int \frac{1}{u} (-du) = -\int \frac{1}{u} du = -\ln |u| + C = -\ln |\csc x + \cot x| + C$$

Solution 2: This is an application of partial fractions.

$$\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx = \int \frac{\sin x}{1 - \cos^2 x} dx$$

Let $u = \cos x$. Then $du = -\sin x dx$

$$\int \frac{\sin x}{1 - \cos^2 x} dx = \int \frac{-1}{1 - \cos^2 x} (-\sin x dx) = \int \frac{-1}{1 - u^2} du = \int \frac{1}{u^2 - 1} du = \int \frac{1}{(u+1)(u-1)} du$$

We decompose the expression into partial fractions as

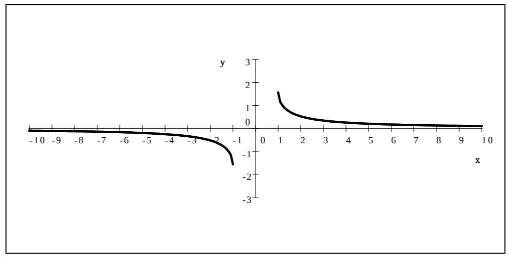
$$\frac{1}{(u+1)(u-1)} = \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1}$$

and so the integral is

$$\begin{aligned} I &= \int \frac{1}{(u+1)(u-1)} du = \frac{1}{2} \int \frac{1}{u-1} du - \frac{1}{2} \int \frac{1}{u+1} du = \frac{1}{2} \ln |u-1| - \frac{1}{2} \ln |u+1| + C \\ &= \frac{1}{2} \ln (1 - \cos x) - \frac{1}{2} \ln (1 + \cos x) + C = \frac{1}{2} \ln \frac{1 - \cos x}{1 + \cos x} + C \end{aligned}$$

d) $y = \csc^{-1} x$

domain: $(-\infty, -1] \cup [1, \infty)$ range: $\left[-\dfrac{\pi}{2}, 0\right) \cup \left(0, \dfrac{\pi}{2}\right]$
decreasing on both intervals of its domain



e) Solution: see handout differentiating trigonometric functions

f) $\int \csc^{-1} x dx = x \csc^{-1} x + \cosh^{-1} x + C$ or $x \csc^{-1} x + \ln \left| x + \sqrt{x^2 - 1} \right| + C$

Let $u = \csc^{-1} x$ and $dv = 1$. Then $du = \frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{(x^2-1)}}$ and $v = x$

$$\int \csc^{-1} x dx = x \csc^{-1} x - \int -\frac{1}{x\sqrt{x^2-1}} x dx = x \csc^{-1} x + \int \frac{1}{\sqrt{x^2-1}} dx$$

We may recognize the integral $\int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + C$. Then we are done, the answer is $I = x \csc^{-1} x + \cosh^{-1} x + C$

Otherwise, we compute $\int \frac{1}{\sqrt{x^2-1}} dx$ using a trigonometric substitution. Let $x = \sec\theta$ where $\theta \in (0, \pi)$ and $dx = \sec\theta \tan\theta d\theta$

$$\begin{aligned} I &= \int \frac{1}{\sqrt{x^2-1}} dx = \int \frac{1}{\sqrt{\sec^2\theta - 1}} \sec\theta \tan\theta d\theta = \int \frac{1}{\tan\theta} \sec\theta \tan\theta d\theta = \int \sec\theta \ d\theta \\ &= \ln|\sec\theta + \tan\theta| + C = \ln\left| x + \sqrt{x^2-1} \right| + C \end{aligned}$$

3. a) see handout b) see handout

c) $\int \tanh^{-1} x dx = x \tanh^{-1} x + \frac{1}{2} \ln\left(1 - x^2\right) + C$

Solution: we will integrate by parts. Let $u = \tanh^{-1} x$ and $dv = 1$. Then $du = \frac{1}{1-x^2} dx$ and $v = x$.

$$\begin{aligned} \int u dv &= uv - \int v du \quad \text{becomes} \\ \int \tanh^{-1} x dx &= x \tanh^{-1} x - \int x \frac{1}{1-x^2} dx \end{aligned}$$

Let $w = 1 - x^2$ then $dw = -2x dx$ and so

$$\int \frac{x}{1-x^2} dx = \int \frac{-\frac{1}{2}}{1-x^2} (-2x dx) = -\frac{1}{2} \int \frac{1}{w} dw = -\frac{1}{2} \ln|w| + C = -\frac{1}{2} \ln\left|1 - x^2\right| + C = -\frac{1}{2} \ln\left(1 - x^2\right) + C$$

and so the answer is

$$\int \tanh^{-1} x dx = x \tanh^{-1} x + \frac{1}{2} \ln\left(1 - x^2\right) + C$$

d) Solution: This is an application of partial fractions.

$$\int \operatorname{csch} x dx = \int \frac{2}{e^x - e^{-x}} dx = \int \frac{2}{e^x - \frac{1}{e^x}} \cdot \frac{e^x}{e^x} dx = \int \frac{2e^x}{(e^x)^2 - 1} dx$$

Let $u = e^x$. Then $du = e^x dx$ and

$$\int \frac{2e^x}{(e^x)^2 - 1} dx = \int \frac{2}{(e^x)^2 - 1} (e^x dx) = \int \frac{2}{u^2 - 1} du = \int \frac{2}{(u+1)(u-1)} du$$

We decompose the rational function and obtain that

$$\frac{2}{(u+1)(u-1)} = \frac{1}{u-1} - \frac{1}{u+1}$$

Then

$$\begin{aligned} \int \frac{2e^x}{(e^x)^2 - 1} dx &= \int \frac{2}{(u+1)(u-1)} du = \int \frac{1}{u-1} - \frac{1}{u+1} du = \ln|u-1| - \ln|u+1| + C \\ &= \ln|e^x - 1| - \ln|e^x + 1| + C = \ln|e^x - 1| - \ln(e^x + 1) + C \end{aligned}$$

4. a) $f'(x) = \dfrac{1}{\sqrt{x^2 + 25}}$ b) $f'(x) = xe^{-3x}$ c) $g'(x) = -(\ln 2)(\sin x) 2^{\cos x}$ d) $f'(\theta) = -\dfrac{10\theta}{25\theta^4 + 1}$

e) $f'(x) = \dfrac{1}{1 - x^2}$ f) $g'(x) = \dfrac{3x^2}{x^9 + 1}$ g) $P'(x) = 3\dfrac{(\tan^{-1} x)^2}{x^2 + 1}$

5. Let $\sec u = x$ where $0 < u < \frac{\pi}{2}$. Then $x^2 - 1 = \sec^2 u - 1 = \tan^2 u$ and $\sec u \tan u du = dx$.

$$\begin{aligned} \int \frac{1}{x\sqrt{x^2 - 1}} dx &= \int \frac{1}{\sec u \sqrt{\tan^2 u}} \sec u \tan u du = \int \frac{1}{\sec u \tan u} \sec u \tan u du = \int du = u + C \\ &= \sec^{-1} x + C \end{aligned}$$

6. a) -1 b) $\dfrac{1}{2} e^x (\sin x - \cos x) + C$ c) $2\sqrt{3}$ d) $\ln(e^x + e^{-x}) + C$

e) $\sinh^{-1} x + C$ or $\ln\left|x + \sqrt{x^2 + 1}\right| + C$ f) $\ln|\sin x| + C$ g) $\dfrac{\sqrt{3}}{3} \tan^{-1}\left(\dfrac{\sqrt{3}}{3} x\right) + C$ h) $\sec^{-1} x + C$

i) $-\dfrac{1}{4} xe^{-4x} - \dfrac{1}{16} e^{-4x} + C$ j) $\ln\left(x + \sqrt{a^2 + x^2}\right) + C$ k) $\dfrac{2}{15}$ l) $\sin^{-1}\left(\dfrac{x}{2}\right) + C$ m) $\dfrac{2}{3}$

n) $\dfrac{76}{3}$ o) $\dfrac{1}{6} \sin 3x + \dfrac{1}{14} \sin 7x + C$ p) $x \sin^{-1} x + \sqrt{1 - x^2} + C$ q) $-\sqrt{25 - x^2} + C$

r) $\ln|x| + \dfrac{1}{2} \ln(x^2 + 4) - \dfrac{1}{2} \tan^{-1} \dfrac{x}{2} + C$ s) $\dfrac{1}{ab} \tan^{-1}\left(\dfrac{b}{a} x\right) + C$ t) $\ln|\cos\theta + \sin\theta| + C$

u) $\dfrac{8}{3} \ln 2 - \dfrac{8}{9}$ v) 1 w) ∞ x) $x^2 - x + 3\ln|x - 1| + \dfrac{2}{x - 1} + C$ y) $\dfrac{1}{8}$ z) $\dfrac{1}{25}$

7. a) $\dfrac{1}{2} \sec x \tan x + \dfrac{1}{2} \ln|\sec x + \tan x| + C$ b) $\dfrac{\sqrt{2}}{2} + \dfrac{1}{2} \ln\left(\sqrt{2} + 1\right)$