1. a) Graph the function $f(x) = \operatorname{sech} x$ and state its basic properties.

   b) Derive the formula for $\dfrac{d}{dx}\sinh^{-1} x$ c) Derive the formula for $\displaystyle\int \operatorname{csch} x\,dx$

2. a) Compute $\displaystyle\int \frac{1}{x\sqrt{x^2-2}}dx$. Assume $x > 0$.

   b) Compute $\displaystyle\int \sqrt{25-x^2}dx$. Assume $-5 \le x \le 5$.

3. Compute each of the following integrals. Assume that $0 < a < x$

a) $\displaystyle\int_0^5 \frac{x}{\sqrt{25-x^2}}dx$

b) $\displaystyle\int e^{\sqrt{x}}dx$

c) $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}dx$

d) $\displaystyle\int \sinh^{-1} x\,dx$

e) $\displaystyle\int_0^\infty \frac{1}{x^2+2}dx$

f) $\displaystyle\int_3^\infty \frac{1}{x^2-4}dx$

g) $\displaystyle\int \frac{4x^3-6x^2+2x-3}{x^2+x^4}dx$

h) $\displaystyle\int \frac{2x^3-9x^2+15x-11}{(x-2)^2}dx$

i) $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}dx$

j) $\displaystyle\int e^x \sin x\,dx$

k) $\displaystyle\int \frac{1}{a^2+(bx)^2}dx$

l) $\displaystyle\int \frac{1-\tan\theta}{1+\tan\theta}d\theta$

m) $\displaystyle\int_0^2 x^2 \ln x\,dx$

n) $\displaystyle\int_1^\infty \frac{\ln x}{x^2}dx$

o) $\displaystyle\int_{\pi/4}^{\pi/2} \tan x\,dx$

p) $\displaystyle\int_0^\infty xe^{-x^2}dx$

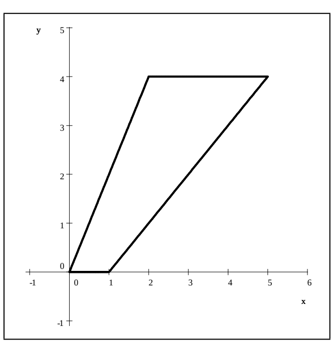
q) $\displaystyle\int_0^\infty xe^{-5x}dx$

4. Consider the ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$.

   a) Set up an integral expressing the length of the arc of the elipse between $x = 0$ and $x = 1$.

   b) Use Simpson's Rule with $n = 6$ to approximate the arc length.

5. Let R be the region bounded by the graphs of $y = 2x$; $y = 4$; $y = x - 1$; and $y = 0$.



Compute the volume of the object we obtain when revolving R about

a) the $x-$axis

b) the $y-$axis

6. Let R be the region bounded by the graphs of $y = -x^2 + 4$ and $y = -x + 2$. Compute the volume of the solid obtained by revolving R about the line $x = 4$.

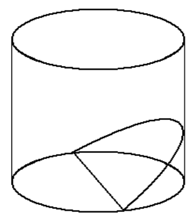
7. Compute the arc length of $f(x) = \ln(2\cos x)$ between $x = -\dfrac{\pi}{3}$ and $x = \dfrac{\pi}{3}$.

8. Compute the volume of the solid with a circular base with radius r if cross sections perpendicular to the base are

   a) squares
   b) equilateral triangles
   c) isosceles right triangles with the hypotenuse lying on the base.
   d) isosceles right triangles with the shorter side on the base.

9. A circe with radius 5 meters is cut into two parts by a straight line that is at a distance of 2 meters from its center. Use integral calculus to compute the area of the smaller piece.

10. A wedge is cut out of a circular cylinder of radius R by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Compute the volume of the wedge.



11. Let R be the region bounded by the circle $(x-3)^2 + y^2 = 1$. Find the volume of the torus we obtain by rotating R about

    a) the $y-$axis
    b) the line $x = 1$
    c) the line $x = 2$

12. State the definition of a sequence converging to L.

13. Show that the sequence $\{a_n\}$ defined by $a_n = \dfrac{n}{n+1}$ is convergent by

    a) proving that it is increasing and bounded from above
    b) by an $\epsilon - N$ proof.

14. Prove that a convergent sequence is bounded.

15. In each case, determine whether the sequence is convergent or divergent. If it converges, then compute its limit.

    a) $a_n = \left(1 + \dfrac{1}{n}\right)^{2n}$
    b) $a_n = \left(1 + \dfrac{1}{n}\right)^{n^2}$
    c) $a_n = \left(\dfrac{5n+2}{5n+1}\right)^{n+1}$
    d) $a_n = \left(\dfrac{2}{3} + \dfrac{1}{n}\right)^{n}$
    e) $a_n = \left(\dfrac{n+3}{n-2}\right)^{n}$
    f) $a_n = \dfrac{\sqrt{n^2+1} + \sqrt{n}}{n + \sqrt{2}}$
    g) $a_n = \dfrac{3^n}{4^n+1} \sin \dfrac{1}{n}$
    h) $a_n = \dfrac{(-3)^{n+1}}{4^{n-1} + 5^n}$
    i) $a_n = \dfrac{1 - 5^{n+1}}{5^n + 1}$
    j) $a_n = \dfrac{n^2 + 2^n}{n!}$
    k) $a_n = \sqrt[n]{1+n^2}$

Answers

1. a) see handout b) see handout

   c)
$$\int \operatorname{csch} x dx = \int \frac{2}{e^x - e^{-x}} dx = \int \frac{2e^x}{(e^x)^2 - 1} dx$$

   Let $u = e^x$. Then $du = e^x dx$
$$\int \frac{2}{(e^x)^2 - 1} e^x dx = \int \frac{2}{u^2 - 1} du = \int \frac{2}{(u+1)(u-1)} du$$

   We decompose this expression into partial fractions as, we compute that $\dfrac{2}{u^2-1} = \dfrac{1}{u-1} - \dfrac{1}{u+1}$
$$\begin{aligned} \int \frac{2}{(u+1)(u-1)} du &= \int \frac{1}{u-1} - \frac{1}{u+1} du = \int \frac{1}{u-1} du - \int \frac{1}{u+1} du \\ &= \ln|u-1| - \ln|u+1| + C = \ln|e^x - 1| - \ln|e^x + 1| + C \end{aligned}$$

2. a) $\dfrac{1}{\sqrt{2}} \operatorname{arcsec}\left(\dfrac{x}{\sqrt{2}}\right) + C$ b) $\dfrac{25}{2}\left(\arcsin\left(\dfrac{x}{5}\right) + \dfrac{x\sqrt{25-x^2}}{25}\right) + C$

3. a) 5 b) $2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$ c) $\ln\left(x + \sqrt{x^2 - a^2}\right) + C$ or $\cosh^{-1}\dfrac{x}{a} + C$ d) $x \sinh^{-1} x - \sqrt{x^2+1} + C$

   e) $\dfrac{\sqrt{2}}{4}\pi$ f) $\dfrac{1}{4}\ln 5$ g) $2\ln|x| + \dfrac{3}{x} + \ln\left(x^2+1\right) - 3\arctan x + C$ h) $x^2 - x + 3\ln|x-2| + \dfrac{1}{x-2} + C$

   i) $\ln\left(x + \sqrt{a^2 + x^2}\right) + C$ j) $\dfrac{1}{2}(\sin x) e^x - \dfrac{1}{2}(\cos x) e^x + C$ k) $\dfrac{1}{ab}\tan^{-1}\left(\dfrac{b}{a}x\right) + C$

   l) $\ln|\cos\theta + \sin\theta| + C$ m) $\dfrac{8}{3}\ln 2 - \dfrac{8}{9}$ n) 1 o) ∞ p) $\dfrac{1}{2}$ q) $\dfrac{1}{25}$

4. a) $\displaystyle\int_0^1 \sqrt{\frac{5x^2+16}{16-4x^2}} dx$ b) $1.101\,391\,684$

5. a) $\dfrac{112}{3}\pi$ b) 36π

6. $\dfrac{63\pi}{2}$

7. $2\ln\left(2+\sqrt{3}\right)$

8. a) $\dfrac{16}{3}r^3$ b) $\dfrac{4}{3}\sqrt{3}r^3$ c) $\dfrac{4}{3}r^3$ d) $\dfrac{8}{3}r^3$

9. $25\sin^{-1}\left(\dfrac{\sqrt{21}}{5}\right) - 2\sqrt{21} \approx 19.816\,835\,63$

10. $\dfrac{2\sqrt{3}}{9}R^3$

11. a) $6\pi^2$ b) $4\pi^2$ c) $2\pi^2$

12. A sequence $\{a_n\}$ converges to $L \in \mathbb{R}$ if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for all $n > N$,

$$|a_n - L| < \epsilon$$

Also correct:

A sequence $\{a_n\}$ converges to $L \in \mathbb{R}$ if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for all $n > N$,

$$L - \epsilon < a_n < L + \epsilon$$

13. a) $a_{n+1} > a_n$ is equivalent to $a_{n+1} - a_n > 0$

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} = \frac{n^2 + 2n + 1 - n^2 - 2n}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)}$$

Since n is positive, this expression is clearly positive and so the sequence is increasing. On the other hand, we will show that for all $n \in \mathbb{N}$, $a_n < 1$.

$$1 - a_n = 1 - \frac{n}{n+1} = \frac{n+1-n}{n+1} = \frac{1}{n+1}$$

$$1 - a_n > 0 \quad \Longrightarrow 1 > a_n$$

b) We will show that the limit is 1. Let $\epsilon > 0$ be given. Let $N = \left\lceil \frac{1}{\epsilon} \right\rceil$

$$|a_n - 1| = \left| \frac{n}{n+1} - 1 \right| = \left| \frac{n - (n+1)}{n+1} \right| = \left| \frac{-1}{n+1} \right| = \frac{1}{n+1}$$

If $n > N$, then $\frac{1}{n+1} < \frac{1}{n} < \frac{1}{N} < \epsilon$

$$|a_n - 1| < \epsilon$$

Note: $|a_n - 1| < \epsilon$ is equivalent to $1 - \epsilon < a_n < 1 + \epsilon$ since

$$|a_n - 1| < \epsilon \quad \Longrightarrow \quad -\epsilon < a_n - 1 < \epsilon \quad \Longrightarrow \quad 1 - \epsilon < a_n < 1 + \epsilon$$

14. see handout

15. a) converges to e^2 b) diverges to infinity c) converges to $\sqrt[5]{e}$ d) converges to 0 e) converges to e^5 f) converges to 1 g) converges to 0 h) converges to 0 i) converges to -5 j) converges to 0 k) converges to 1

