1. Compute each of the following integrals.

a) $\displaystyle\int \frac{-2x^3+2x^2-x-2}{x^4+2x^3}dx = \ln x + \frac{1}{2x^2} - 3\ln|x+2| + C$

b) $\displaystyle\int (\sec x + \tan x)^2\, dx = 2\tan x + 2\sec x - x + C$

$$\begin{aligned} I &= \int \sec^2 x + \tan^2 x + 2\tan x \sec x\ dx = \int \sec^2 x\ dx + 2\int \tan x \sec x\ dx + \int \tan^2 x\ dx \\ &= \tan x + 2\sec x + \int \left(\sec^2 x - 1\right)\ dx = \tan x + 2\sec x + \tan x - x + C = 2\tan x + 2\sec x - x + C \end{aligned}$$

c) $\displaystyle\int \sqrt{4-9x^2}dx = x\sqrt{1-\frac{9}{4}x^2} + \frac{2}{3}\sin^{-1}\left(\frac{3}{2}x\right) + C$

Let $\theta = \sin^{-1}\left(\frac{3}{2}x\right)$. Then $\sin\theta = \frac{3}{2}x$ and $x = \frac{2}{3}\sin\theta$ where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Also, $dx = \frac{2}{3}\cos\theta d\theta$.

$$\begin{aligned} I &= \int \sqrt{4-9x^2}dx = \int \sqrt{4-9\left(\frac{4}{9}\sin^2\theta\right)}\left(\frac{2}{3}\cos\theta d\theta\right) = \frac{2}{3}\int \sqrt{4-4\sin^2\theta}\cos\theta d\theta \\ &= \frac{2}{3}\int 2\sqrt{1-\sin^2\theta}\cos\theta d\theta = \frac{2}{3}\int 2\left|\cos\theta\right|\cos\theta d\theta = \frac{4}{3}\int \cos^2\theta d\theta = \frac{4}{3}\int \frac{1}{2}\left(\cos 2\theta + 1\right)d\theta \\ &= \frac{2}{3}\int \cos 2\theta + 1 d\theta = \frac{2}{3}\left(\frac{1}{2}\sin 2\theta + \theta\right) + C = \frac{2}{3}\left(\frac{1}{2}\cdot 2\sin\theta\cos\theta + \theta\right) + C = \frac{2}{3}\left(\sin\theta\cos\theta + \theta\right) + C \\ &= \frac{2}{3}\left(\left(\frac{3}{2}x\right)\sqrt{1-\frac{9}{4}x^2} + \sin^{-1}\left(\frac{3}{2}x\right)\right) + C = x\sqrt{1-\frac{9}{4}x^2} + \frac{2}{3}\sin^{-1}\left(\frac{3}{2}x\right) + C \end{aligned}$$

d) $\displaystyle\int_0^{\pi/4} \sec^2 x \tan^3 x\ dx = \frac{1}{4}$

Let $u = \tan x$. Then $du = \sec^2 x dx$. When $x = 0$, then $u = 0$ and when $x = \frac{\pi}{4}$, then $u = 1$.

$$\int_0^{\pi/4} \sec^2 x \tan^3 x\ dx = \int_0^1 u^3\ du = \left.\frac{u^4}{4}\right|_0^1 = \frac{1}{4}$$

e) $\displaystyle\int_0^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}dx = 4$

Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}dx$ and so $dx = 2\sqrt{x}du$

$$\int_0^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}dx = \int_0^{\pi} \frac{\sin u}{\sqrt{x}}\left(2\sqrt{x}du\right) = 2\int_0^{\pi} \sin u du = -2\cos u\Big|_0^{\pi} = -2\left(\cos\pi - \cos 0\right) = -2\left(-1-1\right) = 4$$

f) $\displaystyle\int_0^{\infty} x^4 e^{-x^5} dx = \frac{1}{5}$

2. Let $R$ be the region bounded by the curves $y = \sin x$ and $y = \sin 2x$ between $x = 0$ and $x = \dfrac{\pi}{3}$. Compute the volume of the object we obtain by rotating $R$ about

a) the $x-$axis $\displaystyle\int_0^{\pi/3} \pi\left((\sin 2x)^2 - (\sin x)^2\right) dx = \frac{3\sqrt{3}}{16}\pi$

$$\begin{aligned}
V &= \int_0^{\pi/3} \pi\left((\sin 2x)^2 - (\sin x)^2\right) dx = \pi \int_0^{\pi/3} \left(\sin^2 2x - \sin^2 x\right) dx = \pi \int_0^{\pi/3} \left(\frac{1}{2}(1-\cos 4x) - \frac{1}{2}(1-\cos 2x)\right) dx \\
&= \pi \int_0^{\pi/3} \left(\frac{1}{2} - \frac{1}{2}\cos 4x - \frac{1}{2} + \frac{1}{2}\cos 2x\right) dx = \pi \int_0^{\pi/3} \left(\frac{1}{2}\cos 2x - \frac{1}{2}\cos 4x\right) dx = \frac{\pi}{2}\int_0^{\pi/3} (\cos 2x - \cos 4x)\, dx \\
&= \frac{\pi}{2}\left(\frac{\sin 2x}{2} - \frac{\sin 4x}{4}\Big|_0^{\pi/3}\right) = \frac{\pi}{2}\left[\left(\frac{1}{2}\sin\left(\frac{2\pi}{3}\right) - \frac{1}{4}\sin\left(\frac{4\pi}{3}\right)\right) - \left(\frac{1}{2}\sin 0 - \frac{1}{4}\sin 0\right)\right] \\
&= \frac{\pi}{2}\left(\frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{1}{4}\left(-\frac{\sqrt{3}}{2}\right)\right) = \frac{\pi}{2}\left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{8}\right) = \frac{\pi}{2}\left(\frac{3\sqrt{3}}{8}\right) = \frac{3\sqrt{3}}{16}\pi
\end{aligned}$$

b) the $y-$axis

$$V = \int_0^{\pi/3} 2\pi x(\sin 2x - \sin x)\, dx = 2\pi \int_0^{\pi/3} (x \sin 2x - x\sin x)\, dx = 2\pi \int_0^{\pi/3} x\sin 2x dx - 2\pi \int_0^{\pi/3} x \sin x dx$$

Let $u = x$ and $dv = \sin x dx$. Then $du = dx$ and $v = -\cos x$

$$\int x \sin x dx = -x\cos x - \int -\cos x dx = -x\cos x + \int \cos x dx = -x\cos x + \sin x + C$$

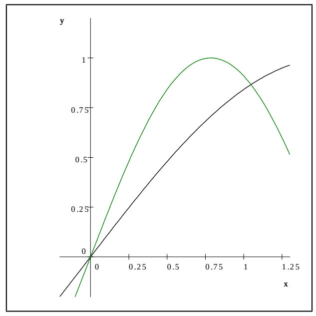
$$\begin{aligned}
V_2 &= 2\pi \int_0^{\pi/3} x\sin x dx = 2\pi(-x\cos x + \sin x)\Big|_0^{\pi/3} = 2\pi\left(\left(-\frac{\pi}{3}\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\right) - (-0\cos 0 + \sin 0)\right) \\
&= 2\pi\left(\left(-\frac{\pi}{3}\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\right) - (0)\right) = 2\pi\left(-\frac{\pi}{6} + \frac{\sqrt{3}}{2}\right) = -\frac{\pi^2}{3} + \sqrt{3}\pi
\end{aligned}$$

Let $u = x$ and $dv = \sin 2x dx$. Then $du = dx$ and $v = -\dfrac{1}{2}\cos 2x$

$$\begin{aligned}
\int x\sin 2x dx &= -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x dx = -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x dx = -\frac{1}{2}x\cos 2x + \frac{1}{2}\cdot\frac{1}{2}\sin 2x \\
&= \frac{1}{4}\sin 2x - \frac{1}{2}x\cos 2x + C
\end{aligned}$$

$$\begin{aligned}
V_1 &= 2\pi \int_0^{\pi/3} x\sin 2x dx = 2\pi\left(\frac{1}{4}\sin 2x - \frac{1}{2}x\cos 2x\right)\Big|_0^{\pi/3} \\
&= 2\pi\left[\left(\frac{1}{4}\sin\frac{2\pi}{3} - \frac{1}{2}\cdot\frac{\pi}{3}\cos\frac{2\pi}{3}\right) - \left(\frac{1}{4}\sin 0 - \frac{1}{2}\cdot 0\cos 0\right)\right] = 2\pi\left[\left(\frac{1}{4}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{\pi}{3}\left(-\frac{1}{2}\right)\right) - 0\right] \\
&= 2\pi\left(\frac{\sqrt{3}}{8} + \frac{\pi}{12}\right)
\end{aligned}$$

$$\begin{aligned} V &= V_1 - V_2 = 2\pi\left(\frac{\sqrt{3}}{8}+\frac{\pi}{12}\right) - 2\pi\left(-\frac{\pi}{6}+\frac{\sqrt{3}}{2}\right) = 2\pi\left(\frac{\sqrt{3}}{8}+\frac{\pi}{12}+\frac{\pi}{6}-\frac{\sqrt{3}}{2}\right) = 2\pi\left(\frac{3\pi}{12}-\frac{3\sqrt{3}}{8}\right) \\ &= 2\pi\left(\frac{\pi}{4}-\frac{3\sqrt{3}}{8}\right) = \frac{\pi^2}{2}-\frac{3\sqrt{3}}{4}\pi \end{aligned}$$




3. Compute the volume of the object we obtain when rotating the circle $(x-3)^2+y^2=4$ about the $y-$axis.

$$V = 2\int_1^5 2\pi x\left(\sqrt{4-(x-3)^2}\right)dx = 24\pi^2$$

4. Compute the length of the curve of $y=\dfrac{x^4}{8}+\dfrac{1}{4x^2}$ between $x=1$ and $x=2$.

$$f(x) = \frac{x^4}{8}+\frac{1}{4x^2} \qquad f'(x) = \frac{d}{dx}\left(\frac{x^4}{8}+\frac{1}{4x^2}\right) = \frac{1}{2}\left(x^3-\frac{1}{x^3}\right)$$

$$(f'(x))^2 = \left(\frac{1}{2}\left(x^3-\frac{1}{x^3}\right)\right)^2 = \frac{1}{4}\left(x^6+\frac{1}{x^6}-2\right)$$

$$(f'(x))^2+1 = \frac{1}{4}\left(x^6+\frac{1}{x^6}-2\right)+1 = \frac{1}{4}\left(x^6+\frac{1}{x^6}+2\right) = \left(\frac{1}{2}\left(x^3+\frac{1}{x^3}\right)\right)^2$$

$$\begin{aligned} L &= \int_1^2 \sqrt{(f'(x))^2+1}\,dx = \int_1^2 \sqrt{\left(\frac{1}{2}\left(x^3+\frac{1}{x^3}\right)\right)^2}dx = \int_1^2 \frac{1}{2}\left(x^3+\frac{1}{x^3}\right)dx = \frac{1}{2}\int_1^2 x^3+x^{-3}dx \\ &= \frac{1}{2}\left(\frac{x^4}{4}-\frac{x^{-2}}{2}\right)\Big|_1^2 = \frac{1}{2}\left[\left(\frac{2^4}{4}-\frac{2^{-2}}{2}\right)-\left(\frac{1^4}{4}-\frac{1^{-2}}{2}\right)\right] = \frac{1}{2}\left[\left(\frac{16}{4}-\frac{1}{2\cdot 4}\right)-\left(\frac{1}{4}-\frac{1}{2}\right)\right] \\ &= \frac{1}{2}\left[\left(4-\frac{1}{8}\right)-\left(-\frac{1}{4}\right)\right] = \frac{1}{2}\left(\frac{33}{8}\right) = \frac{33}{16} \end{aligned}$$

5. Prove that a bounded increasing sequence is convergent.

see handout

6. Determine whether each of the given series converges absolutely, converges conditionally, or diverges.

a) $\displaystyle\sum_{n=0}^{\infty} n^2\ 3^{-n^2}$ converges absolutely

ratio test:

$$\begin{aligned} \rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2\ 3^{-(n+1)^2}}{n^2\ 3^{-n^2}}\right| = \lim_{n\to\infty}\frac{(n+1)^2\ 3^{n^2-(n+1)^2}}{n^2} \\ &= \lim_{n\to\infty}\frac{(n+1)^2}{n^2}\cdot\lim_{n\to\infty}3^{n^2-n^2-2n-1} = \lim_{n\to\infty}\frac{(n+1)^2}{n^2}\cdot\lim_{n\to\infty}3^{-2n-1} = 1\cdot 0 = 0 \end{aligned}$$

b) $\displaystyle\sum_{n=0}^{\infty} \frac{3-n^2}{(n+3)^3}$ diverges

$$\sum_{n=0}^{\infty} \frac{3-n^2}{(n+3)^3} = \sum_{n=0}^{\infty} \frac{3}{(n+3)^3} - \sum_{n=0}^{\infty} \frac{n^2}{(n+3)^3} = 3\sum_{n=0}^{\infty} \frac{1}{(n+3)^3} - \sum_{n=0}^{\infty} \frac{n^2}{(n+3)^3}$$

The first series converges by the comparison test:

$$\sum_{n=0}^{\infty} \frac{1}{(n+3)^3} = \sum_{n=3}^{\infty} \frac{1}{n^3} < \sum_{n=3}^{\infty} \frac{1}{n^2} \text{ which converges}$$

the second series diverges by the comparison test:

$$\sum_{n=0}^{\infty} \frac{n^2}{(n+3)^3} > \sum_{n=0}^{\infty} \frac{n^2}{(n+n)^3} = \sum_{n=0}^{\infty} \frac{n^2}{(2n)^3} = \sum_{n=0}^{\infty} \frac{n^2}{8n^3} = \frac{1}{8}\sum_{n=0}^{\infty} \frac{1}{n} \quad \text{diverges}$$

c) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n}$ converges absolutely

ratio test:

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(n+1)!}{(n+1)^{n+1}}}{\dfrac{n!}{n^n}}\right| = \lim_{n\to\infty}\frac{(n+1)\,n!}{(n+1)(n+1)^n}\cdot\frac{n^n}{n!} = \lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty}\left(\frac{1}{\dfrac{n+1}{n}}\right)^n = \lim_{n\to\infty}\frac{1}{\left(1+\dfrac{1}{n}\right)^n} = \frac{1}{\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

7. Find the Taylor polynomial of order 5 at $x=1$ of the function $f(x) = \sqrt{x}$

$1 + \frac{1}{2}(x-1)^1 - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4 + \frac{7}{256}(x-1)^5$

We take the derivatives:

$f(x) = \sqrt{x}$ $\quad f(1) = 1$ $\quad f^{(3)}(x) = \frac{3}{8}x^{-5/2} =$ $\quad f^{(3)}(1) = -\frac{15}{16}$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\quad f'(1) = \frac{1}{2}$ $\quad \frac{3}{8}$ $\quad f^{(5)}(x) = \frac{105}{32}x^{-9/2}$ $\quad f^{(5)}(1) =$

$f''(x) = -\frac{1}{4}x^{-3/2}$ $\quad f''(1) = -\frac{1}{4}$ $\quad f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$ $\quad f^{(4)}(1) = \frac{105}{32}$

and then the $n$th term ofthe polynomial is $\dfrac{f^{(n)}(1)}{n!}(x-1)^n$ and so the answer is

$$\begin{aligned} P(x) &= 1 + \frac{1}{2}(x-1) - \frac{\frac{1}{4}}{2!}(x-1)^2 + \frac{\frac{3}{8}}{3!}(x-1)^3 - \frac{\frac{15}{16}}{4!}(x-1)^4 + \frac{\frac{105}{32}}{5!}(x-1)^5 \\ &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4 + \frac{7}{256}(x-1)^5 \end{aligned}$$

8. Find a power series representation for the function $f(x) = \dfrac{1}{x+10}$ and determine the interval of convergence

$$\frac{1}{x+10} = \frac{1}{10\left(\frac{x}{10}+1\right)} = \frac{1}{10\left(1-\left(-\frac{x}{10}\right)\right)} = \frac{\frac{1}{10}}{1-\left(-\frac{x}{10}\right)}$$

This is a geometric series with first element $\dfrac{1}{10}$ and common ratio $-\dfrac{x}{10}$.
Thus the series is

$$\frac{1}{10} - \frac{x}{100} + \frac{x^2}{1000} - \frac{x^3}{10\,000} + \frac{x^4}{100\,000} + .... = \sum_{n=0}^{\infty} \frac{1}{10}\left(-\frac{x}{10}\right)^n$$

and the radius of convergence is

$$|r| < 1 \implies -1 < -\frac{x}{10} < 1 \implies 10 > x > -10$$

9. Compute the interval of convergence for the series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x-5)^n}{3^n(n+1)}$. $(2, 8]$

Ratio test:

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\dfrac{(x-5)^{n+1}}{3^{n+1}(n+2)}}{(-1)^n\dfrac{(x-5)^n}{3^n(n+1)}}\right| = \lim_{n\to\infty}\left|(-1)^{n+1}\frac{(x-5)(x-5)^n}{3\cdot 3^n(n+2)}\cdot\frac{3^n(n+1)}{(-1)^n(x-5)^n}\right|$$

$$= \lim_{n\to\infty}\frac{|x-5|(n+1)}{3(n+2)} = \frac{|x-5|}{3}\lim_{n\to\infty}\frac{n+1}{n+2} = \frac{|x-5|}{3}$$

$$\frac{|x-5|}{3} < 1 \implies -3 < x-5 < 3 \implies 2 < x < 8$$

We check the endpoints: if $x=2$, then

$$\sum_{n=0}^{\infty}(-1)^n\frac{(x-5)^n}{3^n(n+1)} = \sum_{n=0}^{\infty}(-1)^n\frac{(2-5)^n}{3^n(n+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n(-3)^n}{3^n(n+1)} = \sum_{n=0}^{\infty}\frac{3^n}{3^n(n+1)} = \sum_{n=0}^{\infty}\frac{1}{n+1}$$

which diverges. And if $x=8$, then

$$\sum_{n=0}^{\infty}(-1)^n\frac{(x-5)^n}{3^n(n+1)} = \sum_{n=0}^{\infty}(-1)^n\frac{(8-5)^n}{3^n(n+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{3^n(n+1)} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}$$

converges conditionally.

10. Find the sum of the infinite series

a) $2^2 + 2^3 + \frac{1}{2}\cdot 2^4 + \frac{1}{6}\cdot 2^5 + \frac{1}{24}\cdot 2^6 + ....... = \displaystyle\sum_{n=2}^{\infty}\frac{2^n}{(n-2)!}$ Consider first the Taylor series for $e^x$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + ...... \quad \text{on } \mathbb{R}$$

multiply by $x^2$

$$x^2e^x = x^2 + x^3 + \frac{1}{2}x^4 + \frac{1}{6}x^5 + \frac{1}{24}x^6...... \quad \text{on } \mathbb{R}$$

and now evaluate it at $x = 2$

$$2^2e^2 = 2^2 + 2^3 + \frac{1}{2} \cdot 2^4 + \frac{1}{6} \cdot 2^5 + \frac{1}{24} \cdot 2^6 ......$$

and so the answer is $4e^2$.

b) $\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \frac{16}{81} + .....$

This is a geometric series with $a = \frac{2}{3}$ and $r = -\frac{2}{3}$, so the sum is

$$s = \frac{a}{1-r} = \frac{\frac{2}{3}}{1 - \left(-\frac{2}{3}\right)} = \frac{2}{5}$$

c) $1 - 3 + \frac{9}{2} - \frac{9}{2} + \frac{27}{8} - \frac{81}{40} + \frac{81}{80} - \frac{243}{560} + .....$

$$\begin{aligned} S &= 1 - 3 + \frac{9}{2} - \frac{9}{2} + \frac{27}{8} - \frac{81}{40} + \frac{81}{80} - \frac{243}{560} + ..... = \\ &= 1 - 3 + \frac{3^2}{2!} - \frac{9 \cdot 3}{2 \cdot 3} + \frac{27 \cdot 3}{8 \cdot 3} - \frac{81 \cdot 3}{40 \cdot 3} ..... \\ &= 1 - 3 + \frac{3^2}{2!} - \frac{3^3}{3!} + \frac{3^4}{4!} - \frac{3^5}{5!} ..... \\ &= 1 + (-3) + \frac{(-3)^2}{2!} + \frac{(-3)^3}{3!} + \frac{(-3)^4}{4!} + \frac{(-3)^5}{5!} ..... \end{aligned}$$

This $e^x$ evaluated at $x = -3$ and so the sum is $e^{-3} = \frac{1}{e^3}$.

d) $x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + ......$

We need to recognize that this is the power series for $\sinh x$.