Quiz 10 will cover the following material: (all handouts posted on the web site so far)

1. All material for Quizzes 1-9 and Exams 1, 2
2. Sum of Geometric Series and telescoping sums
3. Comparison test, nth term test, and integral test.

Sample Quiz 10

1. Prove that convergent sequences have unique limits: if $\{a_n\}$ is a sequence with $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} a_n = B$, then $A = B$
2. Compute the limit of each of the following sequences or state if it diverges. Justify your answer.

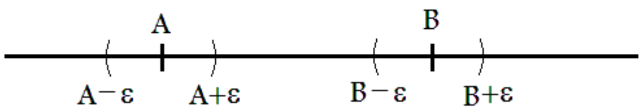
   a) $\lim\limits_{n\to\infty} \dfrac{n!}{3^n}$ b) $\lim\limits_{n\to\infty} \dfrac{\ln n}{\sqrt[5]{n}}$
3. In case of each of the following series given, determine whether it converges or diverges. Show computation and explain which test you are using. (Note that answers may vary.)

   a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n + 2^{3n-1}}{5^{n+1}}$ b) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{(n+2)!}$ c) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n\ln n}$ d) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2-3}{2n^2+5n-1}$
4. Find all values of $p > 0$ so that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent.

Answers

1. Prove that convergent sequences have unique limits: if $\{a_n\}$ is a sequence with $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} a_n = B$, then $A = B$

   Proof. The basic idea here is that if ε is selected to be small enough, then the ε neighborhood of A will be disjoint of the ε neighborhood of B and so a_n can not be in both intervals.



   Suppose that $A \neq B$. We may assume that $A < B$. (Otherwise just re-label them so that the larger number is denoted by B.) Define $\varepsilon = \dfrac{B-A}{2}$. Since $\{a_n\}$ converges to A, there exists N_A so that for all $n > N_A$,

$$A - \varepsilon < a_n < A + \varepsilon$$

   Similarly, since $\{a_n\}$ converges to B, there exists N_B so that for all $n > N_B$,

$$B - \varepsilon < a_n < B + \varepsilon$$

   Now let $n > \max(N_A, N_B)$, so both conditions hold. Then

$$A - \varepsilon < a_n < A + \varepsilon \quad \text{and} \quad B - \varepsilon < a_n < B + \varepsilon$$

   We will only need the right-hand side of the first inequality and the left-hand side of the other:

$$\begin{aligned}
a_n &< A + \varepsilon \quad \text{and} \quad B - \varepsilon < a_n \qquad \text{recall that } \varepsilon = \frac{B-A}{2} \\
a_n &< A + \varepsilon \quad \text{and} \quad B - \varepsilon < a_n \\
a_n &< A + \frac{B-A}{2} \quad \text{and} \quad B - \frac{B-A}{2} < a_n \\
a_n &< \frac{2A}{2} + \frac{B-A}{2} \quad \text{and} \quad \frac{2B}{2} - \frac{B-A}{2} < a_n \\
a_n &< \frac{2A+B-A}{2} \quad \text{and} \quad \frac{2B-B+A}{2} < a_n \\
a_n &< \frac{A+B}{2} \quad \text{and} \quad \frac{A+B}{2} < a_n
\end{aligned}$$

   These two can not be true at the same time. This is a contradiction, so $A \neq B$ is impossible. This completes our proof.

2. Compute the limit of each of the following sequences or state if it diverges. Justify your answer.

   a) $\displaystyle\lim_{n\to\infty} \frac{n!}{3^n}$ diverges

   b) $\displaystyle\lim_{n\to\infty} \frac{\ln n}{\sqrt[5]{n}} = 0$

3. In case of each of the following series given, determine whether it converges or diverges. Show computation and explain which test you are using. (Note that answers may vary.)

a) $\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^n+2^{3n-1}}{5^{n+1}}$ diverges

$\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^n+2^{3n-1}}{5^{n+1}}=\sum_{n=0}^{\infty}\frac{(-2)^n}{5^{n+1}}+\frac{2^{3n-1}}{5^{n+1}}=\sum_{n=0}^{\infty}\frac{1}{5}\left(-\frac{2}{5}\right)^n+\sum_{n=0}^{\infty}\frac{1}{10}\left(\frac{8}{5}\right)^n$ the first geometric series converges since $|r|=\dfrac{2}{5}<1$ and the second diverges because $|r|=\dfrac{5}{8}>1$. Thus the entire series diverges.

b) $\displaystyle\sum_{n=0}^{\infty}\frac{n!}{(n+2)!}$ converges by the comparison test $\dfrac{n!}{(n+2)!}=\dfrac{1}{(n+1)(n+2)}<\dfrac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges

c) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n\ln n}$ diverges by the integral test

d) $\displaystyle\sum_{n=0}^{\infty}\frac{n^2-3}{2n^2+5n-1}$ diverges by the nth term test

4. Find all values of $p>0$ so that the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent.

Solution: We will use the integral test. First we find the antiderivative of $\dfrac{1}{x^p}$

$$\int\frac{1}{x^p}dx=\begin{cases}\dfrac{x^{-p+1}}{-p+1}+C & \text{if } p\neq 1\\ \ln|x|+C & \text{if } p=1\end{cases}$$

Case 1. Suppose that $p=1$. We already know that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

Case 2. Suppose that $p\neq 1$. Then

$$\int_1^{\infty}\frac{1}{x^p}dx=\lim_{N\to\infty}\int_1^N\frac{1}{x^p}dx=\lim_{N\to\infty}\left(\frac{x^{-p+1}}{-p+1}\bigg|_1^N\right)=\lim_{N\to\infty}\left(\frac{N^{-p+1}}{-p+1}-\left(\frac{1^{-p+1}}{-p+1}\right)\right)$$

$$=\frac{1}{1-p}\lim_{N\to\infty}\left(N^{-p+1}-1\right)=\frac{1}{1-p}\lim_{N\to\infty}\left(\frac{1}{N^{p-1}}-1\right)=\frac{1}{1-p}\left(\frac{1}{\lim\limits_{N\to\infty}\left(N^{p-1}\right)}-1\right)$$

Recall that $p>0$. If $p<1$, then $\dfrac{1}{\lim\limits_{N\to\infty}\left(N^{p-1}\right)}$ diverges to infinity. If $p>1$, then $\dfrac{1}{\lim\limits_{N\to\infty}\left(N^{p-1}\right)}=0$ and so the improper integral is $\dfrac{1}{p-1}$ and so the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges.

In summary, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if and only if $p>1$.