Quiz 9 will cover the following material: (all handouts posted on the web site so far)

1. All material for Quizzes 1-8 and Exams 1, 2
2. Sequences, Limits of sequences
3. Definition of series, convergence of series
4. Sum of Geometric Series.

Sample Quiz 9

1. Compute the limit of each of the following sequences or state if it diverges. Justify your answer.

   a) $\displaystyle\lim_{n\to\infty}\left(1+\frac{2}{\ln n}\right)^n$ b) $\displaystyle\lim_{n\to\infty}\frac{3^n-\sin n}{3^n}$

2. Find the sum of each of the following geometric series or state if it diverges.

   a) $\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^n}{5^{n+1}}$ b) $\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^{2n+1}}{5^{n+1}}$ c) $\displaystyle\sum_{n=0}^{\infty}\frac{\pi^{n-1}}{e^{n+1}}$

3. Prove that if a sequence converges to a positive number, then only finitely many terms of the sequence are negative.

© copyright Hidegkuti 2013

Answers

1. a) $\lim_{n\to\infty}\left(1+\dfrac{2}{\ln n}\right)^n$ diverges to infinity

proof: $\left(1+\dfrac{2}{\ln n}\right)^n = e^{\ln\left(1+\frac{2}{\ln n}\right)^n} = e^{n\ln\left(1+\frac{2}{\ln n}\right)} = e^{\lim\limits_{n\to\infty} n\ln\left(1+\frac{2}{\ln n}\right)}$

Consider now the exponent, $\lim_{n\to\infty} n\ln\left(1+\dfrac{2}{\ln n}\right)$. We apply l'Hopital's Rule.

$$\lim_{n\to\infty} n\ln\left(1+\frac{2}{\ln n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{2}{\ln n}\right)}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\frac{1}{1+\frac{2}{\ln n}}\cdot 2\left(-\ln n\right)^{-2}\cdot\frac{1}{n}}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{2n^2}{\left(1+\frac{2}{\ln n}\right)n\left(\ln n\right)^2}$$

$$= \lim_{n\to\infty}\frac{2n}{\left(\left(\ln n\right)^2+2\ln n\right)} = \lim_{n\to\infty}\frac{2}{\left(2\left(\ln n\right)\frac{1}{n}+\frac{2}{n}\right)} = \lim_{n\to\infty}\frac{2n}{\left(2\ln n+2\right)} = \lim_{n\to\infty}\frac{2}{\frac{2}{n}}$$

$$= \lim_{n\to\infty} n = \infty$$

b) $\lim_{n\to\infty}\dfrac{3^n-\sin n}{3^n}$ converges to 1 by the Sandwich rule

$$\frac{3^n-1}{3^n}\leq\frac{3^n-\sin n}{3^n}\leq\frac{3^n+1}{3^n} \quad \text{and}$$

$$\lim_{n\to\infty}\frac{3^n-1}{3^n} = \lim_{n\to\infty}\frac{1-\frac{1}{3^n}}{1} = \frac{1-0}{1} = 1 \quad \text{and} \quad \lim_{n\to\infty}\frac{3^n+1}{3^n} = \lim_{n\to\infty}\frac{1+\frac{1}{3^n}}{1} = \frac{1+0}{1} = 1$$

2. a) $\dfrac{1}{7}$ b) -2 c) diverges

3. Claim: if a sequence converges to a positive number, then only finitely many terms of the sequence are negative.

proof: Suppose that $\{a_n\}$ converges to $L>0$. Let $\varepsilon = \dfrac{L}{3}$. Since a_n converges to L, there exists $N\in\mathbb{N}$ so that for all $n>N$

$$L-\varepsilon < a_n < L+\varepsilon$$

Focusing on the left-hand side only: $L-\varepsilon = L-\dfrac{L}{3} = \dfrac{2}{3}L > 0$ and so

$$0 < \frac{2}{3}L = L-\varepsilon < a_n$$

thus a_n is positive. The only terms that can be negative are $a_1, a_2, \ldots, a_N$.

© copyright Hidegkuti 2013