Exam 1 will cover the following topics from the book:

Review of differential calculus: Chapter 3: all, Chapter 4: 4.5, 4.8
Integral calculus: Chapter 5: all, Chapter 7: 7.1-7.3, Chapter 8: 8.1, 8.2, 8.3, 8.4, 8.7

Handouts to study: all posted during the course.

Review Problems

1. Compute each of the following limits.

   a) $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x+\tan^{-1}x}{e^x-1}$

   b) $\displaystyle\lim_{x\to 1}\frac{\ln x}{x^5-x^2}$

   c) $\displaystyle\lim_{x\to 0}\frac{\cos 2x-1}{\tan^4 x}$

   d) $\displaystyle\lim_{x\to 0}\frac{\tan^{-1}3x}{\tan^{-1}4x}$

   e) $\displaystyle\lim_{x\to 0}\frac{1-\sec 3x}{x^2}$

2. a) Sketch the graph of $y=\csc x$. State its domain, range, and basic properties.

   b) Derive the formula for $\dfrac{d}{dx}\csc x$.

   c) Derive the formula for $\displaystyle\int \csc x\,dx$

   d) Sketch the graph of $y=\csc^{-1}x$. State its domain, range, and basic properties.

   e) Derive the formula for $\dfrac{d}{dx}\csc^{-1}x$.

   f) Derive the formula for $\displaystyle\int \csc^{-1}x\,dx$

3. a) Graph the function $f(x)=\tanh x$ and state its basic properties.

   b) Derive the formula for $\dfrac{d}{dx}(\tanh x)$

   d) Derive the formula for $\displaystyle\int \operatorname{csch} x\,dx$

   c) Derive the formula for $\displaystyle\int \tanh^{-1}x\,dx$

4. Differentiate each of the following.

   a) $f(x)=\ln\left(x+\sqrt{x^2+25}\right)$

   b) $f(x)=-\dfrac{1}{9}e^{-3x}-\dfrac{1}{3}xe^{-3x}$

   c) $g(x)=2^{\cos x}$

   d) $f(\theta)=\cot^{-1}\left(5\theta^2\right)$

   e) $f(x)=\tanh^{-1}x$

   f) $g(x)=\displaystyle\int_1^{x^3}\frac{1}{t^3+1}\,dt$

   g) $P(x)=\left[\displaystyle\int_0^x\frac{1}{t^2+1}\,dt\right]^3$

5. Suppose that $x>1$. Use a trigonometric substitution to compute $\displaystyle\int\frac{1}{x\sqrt{x^2-1}}\,dx$.

6. Compute each of the following integrals.

a) $\displaystyle\int_0^1 \ln x \, dx$

b) $\displaystyle\int e^x \sin x dx$

c) $\displaystyle\int_0^3 \frac{1}{\sqrt{x}} dx$

d) $\displaystyle\int \tanh x dx$

e) $\displaystyle\int \frac{1}{\sqrt{x^2+1}} \, dx$

f) $\displaystyle\int \cot x \, dx$

g) $\displaystyle\int \frac{1}{x^2+3} \, dx$

h) $\displaystyle\int \frac{1}{x\sqrt{x^2-1}} dx$

i) $\displaystyle\int xe^{-4x} \, dx$

j) $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}} dx$

k) $\displaystyle\int_0^{\pi/2} \cos^2 x \sin^3 x \, dx$

l) $\displaystyle\int \frac{1}{\sqrt{4-x^2}} dx$

m) $\displaystyle\int_0^{\pi/2} \sin^3 x \, dx$

n) $\displaystyle\int_3^{11} \frac{x}{\sqrt{x-2}} \, dx$

o) $\displaystyle\int \cos 5x \cos 2x \, dx$

p) $\displaystyle\int \sin^{-1} x \, dx$

q) $\displaystyle\int \frac{x}{\sqrt{25-x^2}} dx$

r) $\displaystyle\int \frac{2x^2-x+4}{x^3+4x} dx$

s) $\displaystyle\int \frac{1}{a^2+(bx)^2} dx$

t) $\displaystyle\int \frac{1-\tan\theta}{1+\tan\theta} d\theta$

u) $\displaystyle\int_0^2 x^2 \ln x dx$

v) $\displaystyle\int_1^\infty \frac{\ln x}{x^2} dx$

w) $\displaystyle\int_{\pi/4}^{\pi/2} \tan x dx$

x) $\displaystyle\int \frac{2x^3-5x^2+7x-6}{x^2-2x+1} dx$

w) $\displaystyle\int_0^\infty xe^{-4x^2} dx$

z) $\displaystyle\int_0^\infty xe^{-5x} dx$

7. Compute each of the following.

a) $\displaystyle\int \sec^3 x dx$

b) $\displaystyle\int_0^{\pi/4} \sec^3 x dx$

8. Compute the area between

a) the graphs of $f(x) = \cos x$ and $g(x) = \sin x$ between $x = 0$ and $x = \dfrac{\pi}{4}$.

b) the graphs of $y = \dfrac{1}{3}x$ and $y = \sqrt{x}$ between $x = 0$ and $x = 9$.

c) the graphs of $y = x^2 + 4x - 19$ and $y = 4x - 3$. (You need to find where these intersect.)

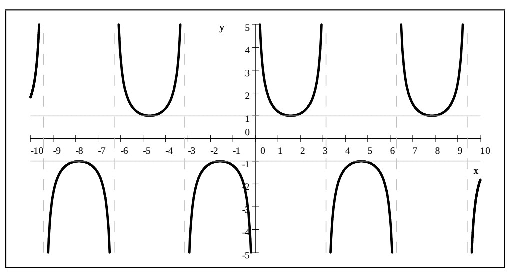
9. Consider $\int_0^1 \sqrt{1+x^4}dx$. Compute each of the following Riemann sums to estimate the area under the graph. Present your answer as a decimal, rounded to four (or more) decimal places.
   a) The left-sum with a uniform partition of $n=5$.
   b) The right-sum with a uniform partition of $n=5$.
   c) The trapezoidal sum with a uniform partition of $n=5$.

10. Prove that $\ln 5$ is greater than 1.283.

Answers

1. a) 2 b) $\frac{1}{3}$ c) $-\infty$ d) $\frac{3}{4}$ e) $-\frac{9}{2}$

2. a) $y = \csc x$
   domain: $\{x : x \neq k\pi, \quad k \in \mathbb{Z}\}$ range: $(-\infty, -1] \cup [1, \infty)$ periodic with period 2π



   b) Solution: see handout differentiating trigonometric functions
   c) Solution 1:

$$\int \csc x dx = \int \csc x \cdot \frac{\csc x + \cot x}{\csc x + \cot x} dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx$$

Let $u = \csc x + \cot x$. Then $du = -\csc x \cot x - \csc^2 x$ and so

$$\int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx = \int \frac{1}{u}(-du) = -\int \frac{1}{u} du = -\ln|u| + C = -\ln|\csc x + \cot x| + C$$

Solution 2: This is an application of partial fractions.

$$\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx = \int \frac{\sin x}{1 - \cos^2 x} dx$$

Let $u = \cos x$. Then $du = -\sin x dx$

$$\int \frac{\sin x}{1-\cos^2 x} dx = \int \frac{-1}{1-\cos^2 x}(-\sin x dx) = \int \frac{-1}{1-u^2} du = \int \frac{1}{u^2-1} du = \int \frac{1}{(u+1)(u-1)} du$$

We decompose the expression into partial fractions as

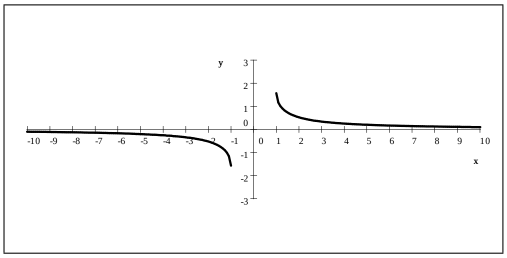
$$\frac{1}{(u+1)(u-1)} = \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1}$$

and so the integral is

$$\begin{aligned} I &= \int \frac{1}{(u+1)(u-1)} du = \frac{1}{2}\int \frac{1}{u-1} du - \frac{1}{2}\int \frac{1}{u+1} du = \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C \\ &= \frac{1}{2}\ln(1-\cos x) - \frac{1}{2}\ln(1+\cos x) + C = \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x} + C \end{aligned}$$

d) $y = \csc^{-1} x$

domain: $(-\infty, -1] \cup [1, \infty)$ range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
decreasing on both intervals of its domain



e) Solution: see handout differentiating trigonometric functions

f) $\int \csc^{-1} x dx = x\csc^{-1} x + \cosh^{-1} x + C$ or $x\csc^{-1} x + \ln\left|x + \sqrt{x^2-1}\right| + C$

3. see handouts

4. a) $f'(x) = \frac{1}{\sqrt{x^2+25}}$ b) $f'(x) = xe^{-3x}$ c) $g'(x) = -(\ln 2)(\sin x) 2^{\cos x}$ d) $f'(\theta) = -\frac{10\theta}{25\theta^4+1}$

e) $f'(x) = \frac{1}{1-x^2}$ f) $g'(x) = \frac{3x^2}{x^9+1}$ g) $P'(x) = 3\frac{\left(\tan^{-1} x\right)^2}{x^2+1}$

5. Let $\sec u = x$ where $0 < u < \frac{\pi}{2}$. Then $x^2 - 1 = \sec^2 u - 1 = \tan^2 u$ and $\sec u \tan u du = dx$.

$$\begin{aligned} \int \frac{1}{x\sqrt{x^2-1}} dx &= \int \frac{1}{\sec u\sqrt{\tan^2 u}} \sec u \tan u du = \int \frac{1}{\sec u \tan u} \sec u \tan u du = \int du = u + C \\ &= \sec^{-1} x + C \end{aligned}$$

6. a) -1 b) $\frac{1}{2}e^x(\sin x - \cos x) + C$ c) $2\sqrt{3}$ d) $\ln(e^x + e^{-x}) + C$

e) $\sinh^{-1} x + C$ or $\ln\left|x + \sqrt{x^2+1}\right| + C$ f) $\ln|\sin x| + C$ g) $\frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{\sqrt{3}}{3}x\right) + C$ h) $\sec^{-1} x + C$

i) $-\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x} + C$ j) $\ln\left(x + \sqrt{a^2+x^2}\right) + C$ k) $\frac{2}{15}$ l) $\sin^{-1}\left(\frac{x}{2}\right) + C$ m) $\frac{2}{3}$

n) $\frac{76}{3}$ o) $\frac{1}{6}\sin 3x + \frac{1}{14}\sin 7x + C$ p) $x\sin^{-1} x + \sqrt{1-x^2} + C$ q) $-\sqrt{25-x^2} + C$

r) $\ln|x| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{2}\tan^{-1}\frac{x}{2} + C$ s) $\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}x\right) + C$ t) $\ln|\cos\theta + \sin\theta| + C$

u) $\frac{8}{3}\ln 2 - \frac{8}{9}$ v) 1 w) ∞ x) $x^2 - x + 3\ln|x-1| + \frac{2}{x-1} + C$ y) $\frac{1}{8}$ z) $\frac{1}{25}$

7. a) $\frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$ b) $\frac{\sqrt{2}}{2} + \frac{1}{2}\ln\left(\sqrt{2}+1\right)$

8. a) $\sqrt{2}-1$ b) $\frac{9}{2}$ c) $\frac{256}{3}$

9. a) $1.052\,722\,195\,833\,78$ b) $1.135\,564\,908\,308\,40$ c) $1.094\,143\,552\,071\,09$

10. The definition of $\ln 5$ is the are under the graph of $f(x) = \frac{1}{x}$ between 1 and 5. In short, $\ln 5 = \int_1^5 \frac{1}{x}dx$ The function $f(x) = \frac{1}{x}$ is decreasing, so right sums underestimate the value of the definite integral. The right sum using a uniform partition with $n = 4$ is $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} = 1.28\overline{3}$. So $\ln 5 \geq 1.28\overline{3} > 1.283$