The exam will cover the following sections from the book: Chapter 5: all Chapter 6: all except 6.4 Chapter 7: all except 7.4 Chapter 8: all except 8.5 Chapter 10: 10.1 Appendix: A.6

1. a) Graph the function $f(x) = \operatorname{sech} x$ and state its basic properties.

   b) Derive the formula for $\dfrac{d}{dx} \sinh^{-1} x$

   c) Derive the formula for $\displaystyle\int \operatorname{csch} x dx$

2. a) Compute $\displaystyle\int \frac{1}{x\sqrt{x^2-2}} dx$. Assume $x > 0$.

   b) Compute $\displaystyle\int \sqrt{25-x^2} dx$. Assume $-5 \le x \le 5$.

3. Use the definition of $\ln x$ to prove that $\ln 10$ is a number between 1.8 and 3.

4. Compute each of the following integrals. Assume that $0 < a < x$

   a) $\displaystyle\int_0^5 \frac{x}{\sqrt{25-x^2}} dx$

   b) $\displaystyle\int e^{\sqrt{x}} dx$

   c) $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}} dx$

   d) $\displaystyle\int \sinh^{-1} x dx$

   e) $\displaystyle\int_0^\infty \frac{1}{x^2+2} dx$

   f) $\displaystyle\int_3^\infty \frac{1}{x^2-4} dx$

   g) $\displaystyle\int \frac{4x^3-6x^2+2x-3}{x^2+x^4} dx$

   h) $\displaystyle\int \frac{2x^3-9x^2+15x-11}{(x-2)^2} dx$

   i) $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}} dx$

   j) $\displaystyle\int e^x \sin x dx$

   k) $\displaystyle\int \frac{1}{a^2+(bx)^2} dx$

   l) $\displaystyle\int \frac{1-\tan\theta}{1+\tan\theta} d\theta$

   m) $\displaystyle\int_0^2 x^2 \ln x dx$

   n) $\displaystyle\int_1^\infty \frac{\ln x}{x^2} dx$

   o) $\displaystyle\int_{\pi/4}^{\pi/2} \tan x dx$

   p) $\displaystyle\int_0^\infty x e^{-x^2} dx$

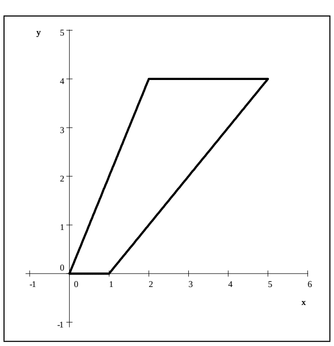
   q) $\displaystyle\int_0^\infty x e^{-5x} dx$

5. Consider the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

   a) Set up an integral expressing the length of the arc of the ellipse between $x = 0$ and $x = 1$.

   b) Use Simpson's Rule with $n = 6$ to approximate the arc length.

6. Let R be the region bounded by the graphs of $y = 2x$; $y = 4$; $y = x - 1$; and $y = 0$.



Compute the volume of the object we obtain when revolving R about

a) the $x-$axis

b) the $y-$axis

c) Compute the center of mass of the object assuming uniform density.

7. Let R be the region bounded by the graphs of $y = -x^2 + 4$ and $y = -x + 2$. Compute the volume of the solid obtained by revolving R about the line $x = 4$.

8. Compute the arc length of $f(x) = \ln(2\cos x)$ between $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{3}$.

9. Compute the volume of the solid with a circular base with radius r if cross sections perpendicular to the base are

   a) squares

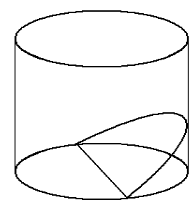
   b) equilateral triangles

   c) isosceles right triangles with the hypotenuse lying on the base.

   d) isosceles right triangles with the shorter side on the base.

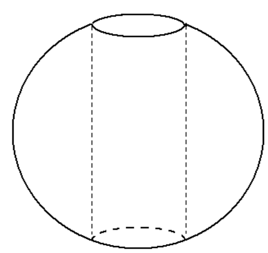
10. A circe with radius 5 meters is cut into two parts by a straight line that is at a distance of 2 meters from its center. Use integral calculus to compute the area of the smaller piece.

11. A wedge is cut out of a circular cylinder of radius R by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Compute the volume of the wedge.



12. Let R be the region bounded by the circle $(x-3)^2 + y^2 = 1$. Find the volume of the torus we obtain by rotating R about a) the y-axis b) the line $x = 1$ c) the line $x = 2$

13. Suppose that $R > r > 0$. A hole of radius r is bored through the center of a sphere of radius R. Find the volume of the remaining portion of the sphere.



14. Compute the center of mass of the region bounded by $y = e^{-x}$, $x = 0$, and $y = 0$.

15. Two children are playing with a 10 meter long uniform bar that has a mass of 20 kg. One child has a mass of 40 kg and the other a mass of 50 kg. If they want to sit at the opposite ends of a bar and they want to balance it, where should they place the support under the bar?

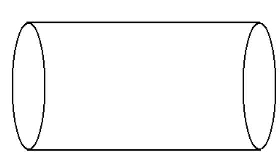
16. a) Redo the previous problem but this time the bar weighs only 10 kg.

    b) Redo the previous problem but this time the bar weighs only 5 kg.

    c) Redo the previous problem but this time the bar weighs M.

17. A ship's anchor, weighing 1200 N, is attached to a chain that weighs 100 N per meter. Find the work done by the winch when the anchor is pulled in from a height of 20 meters.

18. A water tank is of the shape of a cylinder with base radius of 2 m and a height of 10 m. It is positioned so that the circular cross sections are vertical and is full of water. Find the work required to pums all water out of the tank through the top. The density of water is $1000\frac{\text{kg}}{\text{m}^3}$ and $g = 9.81\frac{\text{m}}{\text{s}^2}$.



19. State the definition of a sequence converging to L.

20. Show that the sequence $\{a_n\}$ defined by $a_n = \dfrac{n}{n+1}$ is convergent by

  a) proving that it is increasing and bounded from above

  b) by an $\varepsilon - N$ proof.

21. Prove that a convergent sequence is bounded.

22. In each case, determine whether the sequence is convergent or divergent. If it converges, then compute its limit.

  a) $a_n = \left(1 + \dfrac{1}{n}\right)^{2n}$

  b) $a_n = \left(1 + \dfrac{1}{n}\right)^{n^2}$

  c) $a_n = \left(\dfrac{5n+2}{5n+1}\right)^{n+1}$

  d) $a_n = \left(\dfrac{2}{3} + \dfrac{1}{n}\right)^{n}$

  e) $a_n = \left(\dfrac{n+3}{n-2}\right)^{n}$

  f) $a_n = \dfrac{\sqrt{n^2+1} + \sqrt{n}}{n + \sqrt{2}}$

  g) $a_n = \dfrac{3^n}{4^n+1} \sin \dfrac{1}{n}$

  h) $a_n = \dfrac{(-3)^{n+1}}{4^{n-1} + 5^n}$

  i) $a_n = \dfrac{1 - 5^{n+1}}{5^n + 1}$

  j) $a_n = \dfrac{n^2 + 2^n}{n!}$

  k) $a_n = \sqrt[n]{1+n^2}$

23. Assume that the given sequences are all convergent. Find the value of the limit.

  a) $a_1 = 5$ and $a_{n+1} = 3 - \dfrac{2}{a_n}$

  b) $a_1 = 10$ and $a_{n+1} = 1 + \dfrac{1}{a_n}$

  c) $a_1 = 24$ and $a_{n+1} = 0.8a_n + 10$

  d) $a_1 = 1$ and $a_{n+1} = \dfrac{a_n + 1}{a_n + 5}$

24. Find the value for each of the following.

  a) $\sqrt{6 + \sqrt{6 + \sqrt{6} + \sqrt{6 + \ldots}}}$

  b) $2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}}$

Answers

1. a) see handout b) see handout c) see handout

2. a) $\dfrac{1}{\sqrt{2}}\operatorname{arcsec}\left(\dfrac{x}{\sqrt{2}}\right)+C$ b) $\dfrac{25}{2}\left(\arcsin\left(\dfrac{x}{5}\right)+\dfrac{x\sqrt{25-x^2}}{25}\right)+C$

3. $\ln 10$ is the area under $f(x)=\dfrac{1}{x}$ from $x=1$ to $x=10$. Using a left-hand sum with a uniform partition with $n=9$
we have that $\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4}+\dfrac{1}{5}+\dfrac{1}{6}+\dfrac{1}{7}+\dfrac{1}{8}+\dfrac{1}{9}+\dfrac{1}{10}\approx 1.928$ This Riemann sum underestimates the area. The right-hand sum for the same partition is $1+\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4}+\dfrac{1}{5}+\dfrac{1}{6}+\dfrac{1}{7}+\dfrac{1}{8}+\dfrac{1}{9}\approx 2.828\,968$. This sum overestimates the area under the graph and so $1.98\le\ln 10\le 2.83$.

4. a) 5 b) $2\sqrt{x}e^{\sqrt{x}}-2e^{\sqrt{x}}+C$ c) $\ln\left(x+\sqrt{x^2-a^2}\right)+C$ or $\cosh^{-1}\dfrac{x}{a}+C$ d) $x\sinh^{-1}x-\sqrt{x^2+1}+C$

   e) $\dfrac{\sqrt{2}}{4}\pi$ f) $\dfrac{1}{4}\ln 5$ g) $2\ln|x|+\dfrac{3}{x}+\ln\left(x^2+1\right)-3\arctan x+C$ h) $x^2-x+3\ln|x-2|+\dfrac{1}{x-2}+C$

   i) $\ln\left(x+\sqrt{a^2+x^2}\right)+C$ j) $\dfrac{1}{2}(\sin x)e^x-\dfrac{1}{2}(\cos x)e^x+C$ k) $\dfrac{1}{ab}\tan^{-1}\left(\dfrac{b}{a}x\right)+C$

   l) $\ln|\cos\theta+\sin\theta|+C$ m) $\dfrac{8}{3}\ln 2-\dfrac{8}{9}$ n) 1 o) ∞ p) $\dfrac{1}{2}$ q) $\dfrac{1}{25}$

5. a) $\displaystyle\int_0^1\sqrt{\frac{5x^2+16}{16-4x^2}}dx$ b) $1.101\,391\,684$

6. a) $\dfrac{112}{3}\pi$ b) 36π c) $\left(\dfrac{9}{4},\dfrac{7}{3}\right)$

7. $\dfrac{63\pi}{2}$

8. $2\ln\left(2+\sqrt{3}\right)$

9. a) $\dfrac{16}{3}r^3$ b) $\dfrac{4}{3}\sqrt{3}r^3$ c) $\dfrac{4}{3}r^3$ d) $\dfrac{8}{3}r^3$

10. $25\sin^{-1}\left(\dfrac{\sqrt{21}}{5}\right)-2\sqrt{21}\approx 19.816\,835\,63$

11. $\dfrac{2\sqrt{3}}{9}R^3$

12. a) $6\pi^2$ b) $4\pi^2$ c) $2\pi^2$

13. $\dfrac{4}{3}\pi\left(R^2-r^2\right)^{3/2}$

14. $\left(1,\dfrac{1}{4}\right)$

15. $\dfrac{50}{11}=4.\overline{54}$ meters away from the heavier child and $\dfrac{60}{11}=5.\overline{54}$ from the lighter child

16. a) 4.5 meters from the heavier child and 5.5 meters from the other child

b) $\dfrac{85}{19}$ meters from the heavier child and $\dfrac{66}{19}$ meters from the other child

c) $\dfrac{5M+400}{M+90}$ meters from the heavier child and $\dfrac{5M+500}{M+90}$ meters from the other child

17. $44\,000\text{ J}$

18. $784\,800\pi\text{ J} \approx 2465\,521.914\,537\,27\text{ J}$

19. see handout

20. a) $a_{n+1} > a_n$ is equivalent to $a_{n+1} - a_n > 0$

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} = \frac{n^2+2n+1-n^2-2n}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)}$$

Since n is positive, this expression is clearly positive and so the sequence is increasing. On the other hand, we will show that for all $n \in \mathbb{N}$, $a_n < 1$.

$$1 - a_n = 1 - \frac{n}{n+1} = \frac{n+1-n}{n+1} = \frac{1}{n+1}$$

$$1 - a_n > 0 \quad \Longrightarrow 1 > a_n$$

b) We will show that the limit is 1. Let $\varepsilon > 0$ be given. Let $N = \left\lceil \dfrac{1}{\varepsilon} \right\rceil$

$$|a_n - 1| = \left| \frac{n}{n+1} - 1 \right| = \left| \frac{n-(n+1)}{n+1} \right| = \left| \frac{-1}{n+1} \right| = \frac{1}{n+1}$$

If $n > N$, then $\dfrac{1}{n+1} < \dfrac{1}{n} < \dfrac{1}{N} < \varepsilon$ so we have

$$|a_n - 1| < \varepsilon$$

21. see handout

22. a) converges to e^2 b) diverges to infinity c) converges to $\sqrt[5]{e}$ d) converges to 0
e) converges to e^5 f) converges to 1 g) converges to 0 h) converges to 0 i) converges to -5
j) converges to 0 k) converges to 1

23. a) 2 b) $\dfrac{\sqrt{5}+1}{2}$ c) 50 d) $\sqrt{5}-2$

24. a) 3 b) $\sqrt{2}+1$

