Exam 1 will cover the following topics: (all handouts posted on the class's web site for classes 1-10)

- All material for Quizzes 1-4
- Differentiate any function, including logarithmic, exponential, and inverse trigonometric functions.
- Apply the fundamental theorem to compute definite integrals and differentiate functions defined using definite integrals.
- Approximate logarithmic expressions using the definition of $\ln x$ as $\displaystyle\int_1^x \frac{1}{t}dt$.
- Graph trigonometric functions (all 12 of them) and state their basic propoerties, differentiate them and integrate them. (Exception: we did not yet integrate $\sec^{-1} x$ and $\csc^{-1} x$, so you don't need to know those.)
- Integrate using substitution, trigonometric substitution, integration by parts, and partial fractions. Integrate trigonometric and inverse trigonometric functions.
- Evaluate limits using L'Hopital's Rule.
- Evaluate improper integrals.
- Correctly and precisely **state**
    - the Fundamental Theorem of Calculus (both parts)
    - the definition of $\ln x$
- **Prove** each of the following
    - differentiation formulas of all trigonometric functions, including all inverse trigonometric functions, for example, derive the formula for $\dfrac{d}{dx}\left(\sec^{-1}(x)\right)$
    - For all $x, y > 0$, $\ln(xy) = \ln x + \ln y$ and $\ln x - \ln y = \ln\left(\dfrac{x}{y}\right)$
    - $\displaystyle\lim_{x\to\infty} \ln x = \infty$

Review Problems

1. Compute each of the following limits.

a) $\displaystyle\lim_{x\to 0} \frac{\sin^{-1} x + \tan^{-1} x}{e^x - 1}$

b) $\displaystyle\lim_{x\to 1} \frac{\ln x}{x^5 - x^2}$

c) $\displaystyle\lim_{x\to 0} \frac{\cos 2x - 1}{\tan^4 x}$

d) $\displaystyle\lim_{x\to 0} \frac{\tan^{-1} 3x}{\tan^{-1} 4x}$

e) $\displaystyle\lim_{x\to 0} \frac{1 - \sec 3x}{x^2}$

f) $\displaystyle\lim_{x\to 0} (1 + \sin 3x)^{2/x}$

2. a) Sketch the graph of $y = \csc x$. State its domain, range, and basic properties.

b) Derive the formula for $\dfrac{d}{dx} \csc x$.

c) Derive the formula for $\displaystyle\int \csc x dx$

d) Sketch the graph of $y = \csc^{-1} x$. State its domain, range, and basic properties.

e) Derive the formula for $\dfrac{d}{dx} \csc^{-1} x$.

3. Differentiate each of the following.

a) $f(x) = \ln\left(x + \sqrt{x^2 + 25}\right)$

b) $f(x) = -\dfrac{1}{9}e^{-3x} - \dfrac{1}{3}xe^{-3x}$

c) $g(x) = 2^{\cos x}$

d) $f(\theta) = \cot^{-1}\left(5\theta^2\right)$

e) $f(x) = \dfrac{2x+5}{x-3}$

f) $g(x) = \displaystyle\int_1^{x^3} \frac{1}{t^3+1} dt$

g) $P(x) = \left[\displaystyle\int_0^x \frac{1}{t^2+1} dt\right]^3$

4. Suppose that $x > 1$. Use a trigonometric substitution to compute $\displaystyle\int \frac{1}{x\sqrt{x^2-1}} dx$.

5. Compute each of the following integrals.

a) $\displaystyle\int_0^1 \ln x \, dx$

b) $\displaystyle\int e^x \sin x dx$

c) $\displaystyle\int_0^3 \frac{1}{\sqrt{x}} dx$

d) $\displaystyle\int \tan^{-1} x dx$

e) $\displaystyle\int \frac{1}{\sqrt{x^2+1}} \, dx$

f) $\displaystyle\int \cot x \, dx$

g) $\displaystyle\int \frac{1}{x^2+3} \, dx$

h) $\displaystyle\int \frac{1}{x\sqrt{x^2-1}} dx$

i) $\displaystyle\int xe^{-4x} \, dx$

j) $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}} dx$

k) $\displaystyle\int_0^{\pi/2} \cos^2 x \sin^3 x \, dx$

l) $\displaystyle\int \frac{1}{\sqrt{4-x^2}} dx$

m) $\displaystyle\int_0^{\pi/2} \sin^3 x \, dx$

n) $\displaystyle\int_3^{11} \frac{x}{\sqrt{x-2}} \, dx$

o) $\displaystyle\int \cos 5x \cos 2x \, dx$

p) $\displaystyle\int \sin^{-1} x \, dx$

q) $\displaystyle\int \frac{x}{\sqrt{25-x^2}} dx$

r) $\displaystyle\int \frac{2x^2-x+4}{x^3+4x} dx$

s) $\displaystyle\int \frac{1}{a^2+(bx)^2} dx$

t) $\displaystyle\int \frac{1-\tan\theta}{1+\tan\theta} d\theta$

u) $\displaystyle\int_0^2 x^2 \ln x dx$

v) $\displaystyle\int_1^\infty \frac{\ln x}{x^2} dx$

w) $\displaystyle\int_{\pi/4}^{\pi/2} \tan x dx$

x) $\displaystyle\int \frac{2x^3-5x^2+7x-6}{x^2-2x+1} dx$

y) $\displaystyle\int_0^\infty xe^{-4x^2} dx$

z) $\displaystyle\int_0^\infty xe^{-5x} dx$

aa) $\displaystyle\int \sec^3 x dx$

ab) $\displaystyle\int_0^{\pi/4} \sec^3 x dx$

6. Define the functions $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ and $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$. Prove each of the following.

a) $\cosh^2 x - \sinh^2 x = 1$ for all x

b) $\sinh 2x = 2\sinh x \cosh x$ for all x

c) $\cosh 2x = \cosh^2 x + \sinh^2 x$

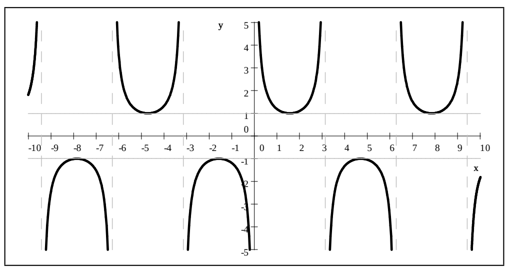
d) $\frac{d}{dx}(\sinh x) = \cosh x$ and $\frac{d}{dx}(\cosh x) = \sinh x$.

7. Prove that $\ln 5$ is greater than 1.283.

Answers

1. a) 2 b) $\frac{1}{3}$ c) $-\infty$ d) $\frac{3}{4}$ e) $-\frac{9}{2}$ f) e^6

2. a) $y = \csc x$
domain: $\{x : x \neq k\pi,\ \ k \in \mathbb{Z}\}$ range: $(-\infty, -1] \cup [1, \infty)$ periodic with period 2π



b) Solution: see handout differentiating trigonometric functions

c) Solution 1:

$$\int \csc x dx = \int \csc x \cdot \frac{\csc x + \cot x}{\csc x + \cot x} dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx$$

Let $u = \csc x + \cot x$. Then $du = -\csc x \cot x - \csc^2 x$ and so

$$\int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx = \int \frac{1}{u}(-du) = -\int \frac{1}{u} du = -\ln|u| + C = -\ln|\csc x + \cot x| + C$$

Solution 2: This is an application of partial fractions.

$$\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx = \int \frac{\sin x}{1 - \cos^2 x} dx$$

Let $u = \cos x$. Then $du = -\sin x dx$

$$\int \frac{\sin x}{1 - \cos^2 x} dx = \int \frac{-1}{1 - \cos^2 x}(-\sin x dx) = \int \frac{-1}{1 - u^2} du = \int \frac{1}{u^2 - 1} du = \int \frac{1}{(u+1)(u-1)} du$$

We decompose the expression into partial fractions as

$$\frac{1}{(u+1)(u-1)} = \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1}$$

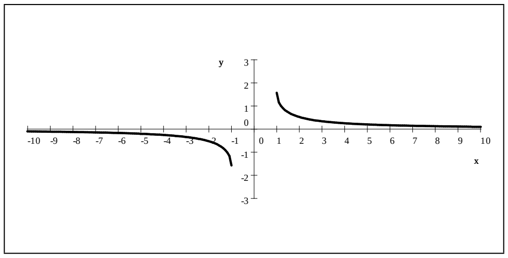
and so the integral is

$$\begin{aligned} I &= \int \frac{1}{(u+1)(u-1)} du = \frac{1}{2}\int \frac{1}{u-1} du - \frac{1}{2}\int \frac{1}{u+1} du = \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C \\ &= \frac{1}{2}\ln(1-\cos x) - \frac{1}{2}\ln(1+\cos x) + C = \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x} + C \end{aligned}$$

d) $y = \csc^{-1} x$

domain: $(-\infty, -1] \cup [1, \infty)$ range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

decreasing on both intervals of its domain



e) Solution: see handout differentiating trigonometric functions

3. a) $f'(x) = \dfrac{1}{\sqrt{x^2+25}}$ b) $f'(x) = xe^{-3x}$ c) $g'(x) = -(\ln 2)(\sin x) 2^{\cos x}$ d) $f'(\theta) = -\dfrac{10\theta}{25\theta^4+1}$

e) $f'(x) = -\dfrac{11}{(x-3)^2}$ f) $g'(x) = \dfrac{3x^2}{x^9+1}$ g) $P'(x) = 3\dfrac{\left(\tan^{-1} x\right)^2}{x^2+1}$

4. Let $\sec u = x$ where $0 < u < \frac{\pi}{2}$. Then $x^2 - 1 = \sec^2 u - 1 = \tan^2 u$ and $\sec u \tan u du = dx$.

$$\int \frac{1}{x\sqrt{x^2-1}} dx = \int \frac{1}{\sec u\sqrt{\tan^2 u}} \sec u \tan u du = \int \frac{1}{\sec u \tan u} \sec u \tan u du = \int du = u + C = \sec^{-1} x + C$$

5. a) -1 b) $\dfrac{1}{2}e^x(\sin x - \cos x) + C$ c) $2\sqrt{3}$ d) $x\tan^{-1} x - \dfrac{1}{2}\ln\left(x^2+1\right) + C$ e) $\ln\left(x + \sqrt{x^2+1}\right) + C$

f) $\ln|\sin x| + C$ g) $\dfrac{\sqrt{3}}{3}\tan^{-1}\left(\dfrac{\sqrt{3}}{3}x\right) + C$ h) $\sec^{-1} x + C$ i) $-\dfrac{1}{4}xe^{-4x} - \dfrac{1}{16}e^{-4x} + C$

j) $\ln\left(x + \sqrt{a^2+x^2}\right) + C$ k) $\dfrac{2}{15}$ l) $\sin^{-1}\left(\dfrac{x}{2}\right) + C$ m) $\dfrac{2}{3}$ n) $\dfrac{76}{3}$

o) $\dfrac{1}{6}\sin 3x + \dfrac{1}{14}\sin 7x + C$ p) $x\sin^{-1} x + \sqrt{1-x^2} + C$ q) $-\sqrt{25-x^2} + C$

r) $\ln|x| + \dfrac{1}{2}\ln\left(x^2+4\right) - \dfrac{1}{2}\tan^{-1}\dfrac{x}{2} + C$ s) $\dfrac{1}{ab}\tan^{-1}\left(\dfrac{b}{a}x\right) + C$ t) $\ln|\cos\theta + \sin\theta| + C$

u) $\dfrac{8}{3}\ln 2 - \dfrac{8}{9}$ v) 1 w) ∞ x) $x^2 - x + 3\ln|x-1| + \dfrac{2}{x-1} + C$ y) $\dfrac{1}{8}$ z) $\dfrac{1}{25}$

aa) $\dfrac{1}{2}\sec x \tan x + \dfrac{1}{2}\ln|\sec x + \tan x| + C$ ab) $\dfrac{\sqrt{2}}{2} + \dfrac{1}{2}\ln\left(\sqrt{2}+1\right)$

6. a) $\cosh^2 x - \sinh^2 x = 1$ for all x

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4} \\ &= \frac{e^{2x} + e^{-2x} + 2 - \left(e^{2x} + e^{-2x} - 2\right)}{4} = \frac{4}{4} = 1\end{aligned}$$

b) $\sinh 2x = 2\sinh x \cosh x$ for all x

$$\begin{aligned}\sinh 2x &= \frac{e^{2x} - e^{-2x}}{2} = \frac{(e^x)^2 - (e^{-x})^2}{2} = \frac{(e^x + e^{-x})(e^x - e^{-x})}{2} = 2 \cdot \frac{(e^x + e^{-x})(e^x - e^{-x})}{4} \\ &= 2 \cdot \frac{(e^x + e^{-x})}{2} \cdot \frac{(e^x - e^{-x})}{2} = 2\sinh x \cosh x\end{aligned}$$

c) $\cosh 2x = \cosh^2 x + \sinh^2 x$

$$\begin{aligned}\cosh^2 x + \sinh^2 x &= \left(\frac{(e^x + e^{-x})}{2}\right)^2 + \left(\frac{(e^x - e^{-x})}{2}\right)^2 = \frac{(e^x + e^{-x})^2 + (e^x - e^{-x})^2}{4} \\ &= \frac{e^{2x} + e^{-2x} + 2 + e^{2x} + e^{-2x} - 2}{4} = \frac{2e^{2x} + 2e^{-2x}}{4} = \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x\end{aligned}$$

d) $\dfrac{d}{dx}(\sinh x) = \cosh x$ and $\dfrac{d}{dx}(\cosh x) = \sinh x$.

$$\begin{aligned}(\sinh x)' &= \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{(e^x - e^{-x})'}{2} = \frac{(e^x - (-e^{-x}))}{2} = \frac{e^x + e^{-x}}{2} = \cosh x \\ (\cosh x)' &= \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{(e^x + e^{-x})'}{2} = \frac{(e^x + (-e^{-x}))}{2} = \frac{e^x - e^{-x}}{2} = \sinh x\end{aligned}$$

7. The definition of $\ln 5$ is the are under the graph of $f(x) = \dfrac{1}{x}$ between 1 and 5. In short, $\ln 5 = \displaystyle\int_1^5 \frac{1}{x}dx$ The function $f(x) = \dfrac{1}{x}$ is decreasing, so right sums underestimate the value of the definite integral. The right sum using a uniform partition with $n = 4$ is $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} = \dfrac{77}{60} = 1.28\overline{3}$. So $\ln 5 \geq 1.28\overline{3} > 1.283$