Math 99 **Midterm Review** Solutions

1. Perform the operations as indicated.

(a) $\sqrt{(-1)^4 - 6\left(2^2 - (-3)^2\right) - (-1)^3 + 10 \div 5 \cdot 2} =$ 6

Solution: We apply order of operations. Parentheses first.

$$= \sqrt{(-1)^4 - 6\left(2^2 - (-3)^2\right) - (-1)^3 + 10 \div 5 \cdot 2} = \sqrt{(-1)^4 - 6(4-9) - (-1)^3 + 10 \div 5 \cdot 2}$$
$$= \sqrt{(-1)^4 - 6(-5) - (-1)^3 + 10 \div 5 \cdot 2}$$

Now exponents:

$$\sqrt{(-1)^4 - 6(-5) - (-1)^3 + 10 \div 5 \cdot 2} = \sqrt{1 - 6(-5) - (-1) + 10 \div 5 \cdot 2}$$

Now multiplications, divisions, left to right.

$$\sqrt{1 - 6(-5) - (-1) + 10 \div 5 \cdot 2} = \sqrt{1 - (-30) - (-1) + 2 \cdot 2} = \sqrt{1 - (-30) - (-1) + 4}$$

Now additions, subtractions, left to right.

$$\sqrt{1 - (-30) - (-1) + 4} = \sqrt{1 + 30 - (-1) + 4} = \sqrt{31 - (-1) + 4} = \sqrt{31 + 1 + 4} = \sqrt{32 + 4} = \sqrt{36}$$

Now we take the square root. (The fact that this operation is last is because the long square root is a case of an "invisible parentheses")

$$\sqrt{36} = 6$$

Thus the solution is 6.

(b) $\dfrac{-3^2 - (-3)^2 - 16 \div (-2) \cdot (-2) + (-2)^2}{|(-4)(-7) - (-2)|} =$ -1

Solution: We apply order of operations. The big bar is an "invisible parentheses". It means that we have to completely work out the numerator and the denominator and then apply the division. In the numerator, there is no parentheses, so we start with exponents, left to right. Notice that $-3^2 = -9$ and not 9.

$$-3^2 - (-3)^2 - 16 \div (-2) \cdot (-2) + (-2)^2 = -9 - 9 - 16 \div (-2) \cdot (-2) + 4$$

We now perform all multiplications and divisions, left to right.

$$-9 - 9 - 16 \div (-2) \cdot (-2) + 4 = -9 - 9 - (-8) \cdot (-2) + 4 = -9 - 9 - 16 + 4$$

Now we perform all additions, subtractins, left to right.

$$-9 - 9 - 16 + 4 = -18 - 16 + 4 = -34 + 4 = -30$$

Now the denominator. It has a parentheses, since the absolute value sign also functions as parentheses. We start with the multiplication.

$$|(-4)(-7) - (-2)| = |28 - (-2)| = |28 + 2| = |30| = 30$$

The answer is thus $\dfrac{-30}{30} = -1$

(c) $\dfrac{(-1)^2 - \left(-\dfrac{1}{2}\right)^2}{5\dfrac{5}{8}} + \dfrac{1}{5} = \dfrac{1}{3}$

Solution: We apply order of operations. We will keep all negative signs in the numerator. We start with exponents, left to right. Every step will be shown.

$$\frac{(-1)^2 - \left(\frac{-1}{2}\right)^2}{5\frac{5}{8}} + \frac{1}{5} = \qquad (-1)^2 = -1\,(-1) = 1$$

$$\frac{1 - \left(\frac{-1}{2}\right)^2}{5\frac{5}{8}} + \frac{1}{5} = \qquad \left(\frac{-1}{2}\right)^2 = \frac{-1}{2}\left(\frac{-1}{2}\right) = \frac{1}{4}$$

$$\frac{1 - \frac{1}{4}}{5\frac{5}{8}} + \frac{1}{5} =$$

Subtraction in the numerator is next.

$$\frac{1 - \frac{1}{4}}{5\frac{5}{8}} + \frac{1}{5} = \qquad 1 - \frac{1}{4} = \frac{1}{1} - \frac{1}{4} = \frac{4}{4} - \frac{1}{4} = \frac{3}{4}$$

$$\frac{\frac{3}{4}}{5\frac{5}{8}} + \frac{1}{5} =$$

Converting the mixed number to improper fraction is an addition:

$$\frac{\frac{3}{4}}{5\frac{5}{8}} + \frac{1}{5} = \qquad 5 + \frac{5}{8} = \frac{5}{1} + \frac{5}{8} = \frac{40}{8} + \frac{5}{8} = \frac{45}{8}$$

$$\frac{\frac{3}{4}}{\frac{45}{8}} + \frac{1}{5} =$$

To divide is to multiply by the reciprocal.

$$\frac{\frac{3}{4}}{\frac{45}{8}} = \frac{3}{4} \cdot \frac{8}{45} = \frac{24}{180} = \frac{2}{15}$$

$$= \frac{2}{15} + \frac{1}{5} = \frac{2}{15} + \frac{3}{15} = \frac{5}{15} = \frac{1}{3}$$

(d) $\frac{2}{3}-\frac{3}{5}\left(-\frac{1}{3}\right)^2 = \frac{3}{5}$

Solution: We apply the order of operations agreement. We start with the exponentiation.

$$\begin{aligned}
\frac{2}{3}-\frac{3}{5}\left(-\frac{1}{3}\right)^2 &= && \left(-\frac{1}{3}\right)^2=\frac{-1}{3}\cdot\frac{-1}{3}=\frac{1}{9}\\
\frac{2}{3}-\frac{3}{5}\cdot\frac{1}{9} &= && \text{multiplication: } \frac{3}{5}\cdot\frac{1}{9}=\frac{3\cdot 1}{5\cdot 9}=\frac{3}{45}=\frac{1}{15}\\
\frac{2}{3}-\frac{1}{15} &= && \text{common denominator is 15}\\
\frac{2\cdot 5}{3\cdot 5}-\frac{1}{15} &= \\
\frac{10}{15}-\frac{1}{15} &= \frac{9}{15}=\frac{\not{3}\cdot 3}{\not{3}\cdot 5}=\frac{3}{5}
\end{aligned}$$

(e) $\left|\left|2-3^3\right|-4^2\right| = 9$

Solution: one trick here is to understand how the absolute value signs are paired. The first two can not be a pair, since there is nothin between them. Thus they must be the beginning of two different pairs. Then, as always, the first one to open is the last one to close.

$$\begin{aligned}
\left|\left|2-3^3\right|-4^2\right| &= && \text{exponent in innermost parentheses}\\
\left|\left|2-27\right|-4^2\right| &= && \text{subtraction in innermost parentheses}\\
\left|\left|-25\right|-4^2\right| &= && \text{the absolute value of } -25 \text{ is } 25\\
\left|25-4^2\right| &= && \text{exponent}\\
\left|25-16\right| &= && \text{subtraction}\\
\left|9\right| &= && \text{the absolute value of 9 is 9}\\
&= 9
\end{aligned}$$

(f) $-\left|-5\right| = -5$

Solution: Two negatives do not always make a positive. This reads: the opposite of the absolute value of -5. Since the absolute value of -5 is 5, we have the opposite of 5, which is -5.

2. Evaluate $15-\left|-x-x^2+5\right|$ if

(a) $x=0$ 10

Solution: We first re-write the expression replacing the variables with parentheses. Then we copy the value of x into the parenthes. Then we work out the order of operations problem.

$$\begin{aligned}
15-\left|-x-x^2+5\right| &= \\
15-\left|-(\ \ )-(\ \ )^2+5\right| &= \\
15-\left|-(0)-(0)^2+5\right| &= && \text{exponent}\\
15-\left|0-0+5\right| &= && \text{subtraction}\\
15-\left|0+5\right| &= && \text{addition}\\
15-\left|5\right| &= && \text{absolute value}\\
15-5 &= 10
\end{aligned}$$

(b) $x = 2$ 14

Solution: We first re-write the expression replacing the variables with parentheses. Then we copy the value of x into the parenthes. Then we work out the order of operations problem.

$$\begin{aligned}
15 - \left|-x - x^2 + 5\right| &= \\
15 - \left|-(\ \ ) - (\ \ )^2 + 5\right| &= \\
15 - \left|-(2) - (2)^2 + 5\right| &= && \text{exponent} \\
15 - |-2 - 4 + 5| &= && \text{subtraction} \\
15 - |-6 + 5| &= && \text{addition} \\
15 - |-1| &= && \text{absolute value} \\
15 - 1 &= 14
\end{aligned}$$

(c) $x = -2$ 12

Solution: We first re-write the expression replacing the variables with parentheses. Then we copy the value of x into the parenthes. Then we work out the order of operations problem.

$$\begin{aligned}
15 - \left|-x - x^2 + 5\right| &= \\
15 - \left|-(\ \ ) - (\ \ )^2 + 5\right| &= \\
15 - \left|-(-2) - (-2)^2 + 5\right| &= && \text{exponent} \\
15 - |2 - 4 + 5| &= && \text{subtraction} \\
15 - |-2 + 5| &= && \text{addition} \\
15 - |3| &= && \text{absolute value} \\
15 - 3 &= 12
\end{aligned}$$

(d) $x = \frac{1}{2}$ $\frac{43}{4}$

Solution: We first re-write the expression replacing the variables with parentheses. Then we copy the value of x into the parenthes. Then we work out the order of operations problem.

$$\begin{aligned}
15 - \left|-x - x^2 + 5\right| &= \\
15 - \left|-(\ \ ) - (\ \ )^2 + 5\right| &= \\
15 - \left|-\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 + 5\right| &= && \text{exponent: } \left(\frac{1}{2}\right)^2 = \frac{1}{2}\cdot\frac{1}{2} = \frac{1\cdot 1}{2\cdot 2} = \frac{1}{4} \\
15 - \left|-\frac{1}{2} - \frac{1}{4} + 5\right| &= && \text{subtraction: } -\frac{1}{2} - \frac{1}{4} = \frac{-2}{4} - \frac{1}{4} = \frac{-1-2}{4} = \frac{-3}{4} \\
15 - \left|-\frac{3}{4} + 5\right| &=
\end{aligned}$$

We will now perform the addition $-\frac{3}{4} + 5$:

$$\begin{aligned}
-\frac{3}{4} + 5 &= \frac{-3}{4} + \frac{5}{1} && \text{the common denominator is 4} \\
&= \frac{-3}{4} + \frac{5\cdot 4}{1\cdot 4} \\
&= \frac{-3}{4} + \frac{20}{4} = \frac{-3+20}{4} = \frac{17}{4}
\end{aligned}$$

So now we have

$$\begin{aligned} 15-\left|\frac{17}{4}\right| &= && \text{absolute value} \\ 15-\frac{17}{4} &= && \\ \frac{15}{1}-\frac{17}{4} &= && \text{common denominator is } 4 \\ \frac{15\cdot 4}{1\cdot 4}-\frac{17}{4} &= && \\ \frac{60}{4}-\frac{17}{4} &= \frac{60-17}{4}=\frac{43}{4} && \end{aligned}$$

3. Evaluate $\dfrac{3ab+2a^2-2b^2}{a+2b}$ if

(a) $a=2$ and $b=-3$ 7

Solution: We need to plug in $a=2$ and $b=-3$ into the expression given and then evaluate it by applying order of operations.

$$\begin{aligned} \frac{3ab+2a^2-2b^2}{a+2b} &= \frac{3(2)(-3)+2(2)^2-2(-3)^2}{(2)+2(-3)} = \qquad \text{exponents} \\ &= \frac{3(2)(-3)+2\cdot 4-2\cdot 9}{(2)+2(-3)} \end{aligned}$$

Now we perform all multiplications and divisions, left to right

$$\begin{aligned} \frac{3(2)(-3)+2\cdot 4-2\cdot 9}{(2)+2(-3)} &= \frac{6(-3)+2\cdot 4-2\cdot 9}{(2)+2(-3)} = \frac{-18+2\cdot 4-2\cdot 9}{(2)+2(-3)} = \\ &= \frac{-18+8-2\cdot 9}{(2)+2(-3)} = \frac{-18+8-18}{(2)+2(-3)} = \frac{-18+8-18}{(2)+(-6)} \end{aligned}$$

Now we perform all additions and subtractions, left to right. Finally we divide.

$$\frac{-18+8-18}{(2)+(-6)} = \frac{-10-18}{(2)+(-6)} = \frac{-28}{(2)+(-6)} = \frac{-28}{-4} = 7$$

(b) $a=-1$ and $b=-2$ 0

Solution: We need to plug in $a=-1$ and $b=-2$ into the expression given and then evaluate it by applying order of operations. Since there is no parentheses, we start with exponents.

$$\frac{3ab+2a^2-2b^2}{a+2b} = \frac{3(-1)(-2)+2(-1)^2-2(-2)^2}{(-1)+2(-2)} = \frac{3(-1)(-2)+2\cdot 1-2\cdot 4}{(-1)+2(-2)}$$

Now we perform all multiplications and divisions, left to right (we individually show them)

$$\begin{aligned} \frac{3(-1)(-2)+2\cdot 1-2\cdot 4}{(-1)+2(-2)} &= \frac{(-3)(-2)+2\cdot 1-2\cdot 4}{(-1)+2(-2)} = \frac{6+2\cdot 1-2\cdot 4}{(-1)+2(-2)} = \\ &= \frac{6+2-2\cdot 4}{(-1)+2(-2)} = \frac{6+2-8}{(-1)+2(-2)} = \frac{6+2-8}{(-1)+(-4)} \end{aligned}$$

Now we perform all additions and subtractions, left to right. Finally we divide.

$$\frac{6+2-8}{(-1)+(-4)} = \frac{8-8}{-1+(-4)} = \frac{0}{-1+(-4)} = \frac{0}{-5} = 0$$

(c) $a=-6$ and $b=3$. undefined

Solution: We need to plug in $a=-6$ and $b=3$ into the expression given and then evaluate it by applying order of operations. Since there is no parentheses, we start with exponents.

$$\frac{3ab+2a^2-2b^2}{a+2b}=\frac{3(-6)(3)+2(-6)^2-2(3)^2}{(-6)+2(3)}=\frac{3(-6)(3)+2\cdot 36-2\cdot 9}{(-6)+2(3)}$$

Now we perform all multiplications and divisions, left to right (we individually show them)

$$\begin{aligned}\frac{3(-6)(3)+2\cdot 36-2\cdot 9}{(-6)+2(3)} &= \frac{(-18)(3)+2\cdot 36-2\cdot 9}{(-6)+2(3)}=\frac{-54+2\cdot 36-2\cdot 9}{(-6)+2(3)}= \\ &= \frac{-54+72-2\cdot 9}{(-6)+2(3)}=\frac{-54+72-18}{(-6)+2(3)}=\frac{-54+72-18}{(-6)+6}\end{aligned}$$

Now we perform all additions and subtractions, left to right. Finally we divide (IF WE CAN).

$$\frac{-54+72-18}{(-6)+6}=\frac{18-18}{(-6)+6}=\frac{0}{0}=\text{undefined}$$

since division by 0 is not allowed. The answer is: undefined

(d) $a=-\frac{1}{2}$ and $b=\frac{3}{4}$ $\quad -\frac{7}{4}$

Solution: We need to plug in $a=-\frac{1}{2}$ and $b=\frac{3}{4}$ into the expression given and then evaluate it by applying order of operations.

$$\frac{3ab+2a^2-2b^2}{a+2b}=\frac{3\left(-\frac{1}{2}\right)\left(\frac{3}{4}\right)+2\left(-\frac{1}{2}\right)^2-2\left(\frac{3}{4}\right)^2}{\left(-\frac{1}{2}\right)+2\left(\frac{3}{4}\right)}$$

Since there is no parentheses, we start with exponents. We proceed left to right. Keep the negative signs in the numerator.

$$\frac{3\left(\frac{-1}{2}\right)\left(\frac{3}{4}\right)+2\left(\frac{-1}{2}\right)^2-2\left(\frac{3}{4}\right)^2}{\left(\frac{-1}{2}\right)+2\left(\frac{3}{4}\right)}= \qquad \left(\frac{-1}{2}\right)^2=\left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right)=\frac{1}{4}$$

$$\frac{3\left(\frac{-1}{2}\right)\left(\frac{3}{4}\right)+2\left(\frac{1}{4}\right)-2\left(\frac{3}{4}\right)^2}{\left(\frac{-1}{2}\right)+2\left(\frac{3}{4}\right)}= \qquad \left(\frac{3}{4}\right)^2=\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)=\frac{9}{16}$$

$$\frac{3\left(\frac{-1}{2}\right)\left(\frac{3}{4}\right)+2\left(\frac{1}{4}\right)-2\left(\frac{9}{16}\right)}{\left(\frac{-1}{2}\right)+2\left(\frac{3}{4}\right)}=$$

Now we perform all multiplications and divisions, left to right (we individually show them)

$$\frac{3\left(\frac{-1}{2}\right)\left(\frac{3}{4}\right)+2\left(\frac{1}{4}\right)-2\left(\frac{9}{16}\right)}{\left(\frac{-1}{2}\right)+2\left(\frac{3}{4}\right)} = \qquad 3\left(-\frac{1}{2}\right)=\frac{3}{1}\cdot\frac{-1}{2}=\frac{-3}{2}$$

$$\frac{\frac{-3}{2}\left(\frac{3}{4}\right)+2\left(\frac{1}{4}\right)-2\left(\frac{9}{16}\right)}{\left(\frac{-1}{2}\right)+2\left(\frac{3}{4}\right)} =$$

$$\frac{\frac{-3}{2}\left(\frac{3}{4}\right)+2\left(\frac{1}{4}\right)-2\left(\frac{9}{16}\right)}{\left(\frac{-1}{2}\right)+2\left(\frac{3}{4}\right)} = \qquad \frac{-3}{2}\left(\frac{3}{4}\right)=\frac{-9}{8}$$

$$\frac{\frac{-9}{8}+2\left(\frac{1}{4}\right)-2\left(\frac{9}{16}\right)}{\left(\frac{-1}{2}\right)+2\left(\frac{3}{4}\right)} = \qquad 2\left(\frac{1}{4}\right)=\frac{2}{1}\cdot\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$$

$$\frac{\frac{-9}{8}+\frac{1}{2}-2\left(\frac{9}{16}\right)}{\left(\frac{-1}{2}\right)+2\left(\frac{3}{4}\right)} = \qquad 2\left(\frac{9}{16}\right)=\frac{2}{1}\cdot\frac{9}{16}=\frac{18}{16}=\frac{9}{8}$$

$$\frac{\frac{-9}{8}+\frac{1}{2}-\frac{9}{8}}{\left(\frac{-1}{2}\right)+2\left(\frac{3}{4}\right)} = \qquad 2\left(\frac{3}{4}\right)=\frac{2}{1}\cdot\frac{3}{4}=\frac{6}{4}=\frac{3}{2}$$

$$= \frac{\frac{-9}{8}+\frac{1}{2}-\frac{9}{8}}{\left(\frac{-1}{2}\right)+\frac{3}{2}}$$

Now we perform all additions and subtractions, left to right.

$$\frac{\frac{-9}{8}+\frac{1}{2}-\frac{9}{8}}{\left(\frac{-1}{2}\right)+\frac{3}{2}} = \qquad \frac{-9}{8}+\frac{1}{2}=\frac{-9}{8}+\frac{4}{8}=\frac{-9+4}{8}=\frac{-5}{8}$$

$$\frac{\frac{-5}{8}-\frac{9}{8}}{\left(\frac{-1}{2}\right)+\frac{3}{2}} = \qquad \frac{-5}{8}-\frac{9}{8}=\frac{-5-9}{8}=\frac{-14}{8}=\frac{-7}{4}$$

$$\frac{\frac{-7}{4}}{\left(\frac{-1}{2}\right)+\frac{3}{2}} = \qquad \left(\frac{-1}{2}\right)+\frac{3}{2}=\frac{-1+3}{2}=\frac{2}{2}=1$$

$$= \frac{\left(\frac{-7}{4}\right)}{1} = -\frac{7}{4}$$

4. Consider the equation $-x^2+2x^3+3=-4x(x-2)$. For each of the numbers given, determine whether it is a solution of the equation or not.

(a) $x=-2$ $\quad -17 \neq -32 \implies$ no

Solution: We evaluate both sides with $x=-2$

$$\begin{aligned}
\text{LHS} &= -x^2+2x^3+3 = -(\ \ )^2+2(\ \ )^3+3 = -(-2)^2+2(-2)^3+3\\
&= -4+2(-8)+3 = -4-16+3 = -20+3 = -17\\
\text{RHS} &= -4x(x-2) = -4(\ \ )((\ \ )-2) = -4(-2)((-2)-2)\\
&= -4(-2)(-4) = 8(-4) = -32
\end{aligned}$$

Since $-17 \neq -32$, the number -2 is not a solution.

(b) $x=-\frac{1}{2}$ $\quad \frac{5}{2} \neq -5 \implies$ no

Solution: We evaluate both sides with $x=-\frac{1}{2}$

$$\begin{aligned}
\text{LHS} &= -x^2+2x^3+3 = -(\ \ )^2+2(\ \ )^3+3\\
&= -\left(-\frac{1}{2}\right)^2+2\left(-\frac{1}{2}\right)^3+3 & \left(\frac{-1}{2}\right)^2=\frac{-1}{2}\cdot\frac{-1}{2}=\frac{1}{4}\\
&= -\frac{1}{4}+2\left(-\frac{1}{2}\right)^3+3 & \left(\frac{-1}{2}\right)^3=\frac{-1}{2}\cdot\frac{-1}{2}\cdot\frac{-1}{2}=\frac{-1}{8}\\
&= -\frac{1}{4}+2\left(-\frac{1}{8}\right)+3 & 2\cdot\frac{-1}{8}=\frac{2}{1}\cdot\frac{-1}{8}=\frac{-2}{8}=\frac{-1}{4}\\
&= -\frac{1}{4}+\left(-\frac{1}{4}\right)+3 & \frac{-1}{4}+\frac{-1}{4}=\frac{-2}{4}=\frac{-1}{2}\\
&= -\frac{1}{2}+3 & \frac{-1}{2}+\frac{3}{1}=\frac{-1}{2}+\frac{6}{2}=\frac{-1+6}{2}=\frac{5}{2}\\
&= \frac{5}{2}\\
\text{RHS} &= -4x(x-2) = -4(\ \ )((\ \ )-2)\\
&= -4\left(-\frac{1}{2}\right)\left(\left(-\frac{1}{2}\right)-2\right) & \frac{-1}{2}-2=\frac{-1}{2}-\frac{2}{1}=\frac{-1}{2}-\frac{4}{2}=\frac{-1-4}{2}=\frac{-5}{2}\\
&= -4\left(\frac{-1}{2}\right)\left(\frac{-5}{2}\right) & -4\left(\frac{-1}{2}\right)=\frac{-4}{1}\cdot\frac{-1}{2}=\frac{4}{2}=2\\
&= 2\left(\frac{-5}{2}\right)=\frac{2}{1}\cdot\frac{-5}{2}=\frac{-10}{2}=-5
\end{aligned}$$

Since $\frac{5}{2} \neq -5$, the number $-\frac{1}{2}$ is not a solution.

(c) $x = \frac{1}{2}$ $3 = 3 \implies$ yes

Solution: We evaluate both sides with $x = \frac{1}{2}$

$$\begin{aligned}
\text{LHS} &= -x^2 + 2x^3 + 3 = -(\ )^2 + 2(\ )^3 + 3 \\
&= -\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^3 + 3 \\
&= -\frac{1}{4} + 2\left(\frac{1}{2}\right)^3 + 3 \\
&= -\frac{1}{4} + 2\left(\frac{1}{8}\right) + 3 \\
&= -\frac{1}{4} + \frac{1}{4} + 3 = 0 + 3 = 3
\end{aligned}$$

$$\left(\frac{1}{2}\right)^2 = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$
$$\left(\frac{1}{2}\right)^3 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$$
$$2\cdot\frac{1}{8} = \frac{2}{1}\cdot\frac{1}{8} = \frac{2}{8} = \frac{1}{4}$$

$$\begin{aligned}
\text{RHS} &= -4x(x-2) = -4(\ )((\ )-2) \\
&= -4\left(\frac{1}{2}\right)\left(\left(\frac{1}{2}\right) - 2\right) \\
&= -4\left(\frac{1}{2}\right)\left(\frac{-3}{2}\right) \\
&= -2\left(\frac{-3}{2}\right) = \frac{-2}{1}\cdot\frac{-3}{2} = \frac{6}{2} = 3
\end{aligned}$$

$$\frac{1}{2} - 2 = \frac{1}{2} - \frac{2}{1} = \frac{1}{2} - \frac{4}{2} = \frac{1-4}{2} = \frac{-3}{2}$$
$$-4\left(\frac{1}{2}\right) = \frac{-4}{1}\cdot\frac{1}{2} = -\frac{4}{2} = -2$$

Since $3 = 3$, the number $\frac{1}{2}$ is a solution.

(d) $x = -3$ $-60 = -60 \implies$ yes

Solution: We evaluate both sides with $x = -3$

$$\begin{aligned}
\text{LHS} &= -x^2 + 2x^3 + 3 = -(\ )^2 + 2(\ )^3 + 3 = -(-3)^2 + 2(-3)^3 + 3 \\
&= -9 + 2(-27) + 3 = -9 - 54 + 3 = -63 + 3 = -60 \\
\text{RHS} &= -4x(x-2) = -4(\ )((\ )-2) = -4(-3)((-3)-2) \\
&= -4(-3)(-5) = 12(-5) = -60
\end{aligned}$$

Since $-60 = -60$, the number -3 is a solution.

5. (Exponential Expressions.) Simplify each of the following.

(a) $(2x^5)(x^4) =$ $2x^9$

Solution: Let us first recall the rules of exponents.

$$\begin{aligned}
&1. & a^n \cdot a^m &= a^{n+m} \\
&2. & \frac{a^n}{a^m} &= a^{n-m} \\
&3. & (a^n)^m &= a^{nm} \\
&4. & (ab)^n &= a^n b^n \\
&5. & \left(\frac{a}{b}\right)^n &= \frac{a^n}{b^n}
\end{aligned}$$

$$(2x^5)(x^4) = 2x^5x^4 = 2x^9 \quad \text{by rule 1}$$

(b) $\left(2x^5\right)^4 = \quad 16x^{20}$
Solution:

$$\begin{aligned} \left(2x^5\right)^4 &= 2^4\left(x^5\right)^4 && \text{by rule 4} \\ &= 16x^{20} && \text{by rule 3} \end{aligned}$$

(c) $(-xy)^2\left(-xy^2\right)^3 = \quad -x^5y^8$
Solution:

$$(-xy)^2\left(-xy^2\right)^3 =$$

$$\begin{aligned} &= (-1xy)^2\left(-1xy^2\right)^3 && \text{the 1's will help with signs} \\ &= (-1)^2x^2y^2(-1)^3x^3\left(y^2\right)^3 && \text{by rule 4} \\ &= 1\cdot x^2y^2(-1)x^3y^6 && \text{by rule 3} \\ &= 1(-1)x^2x^3y^2y^6 && \text{since multiplication is commutative} \\ &= -1x^5y^8 && \text{by rule 1} \end{aligned}$$

(d) $\dfrac{(2ab)^3\left(-3a^2b\right)^2}{-b\left(6ab^2\right)^2} = \quad -2a^5$
Solution:

$$\frac{(2ab)^3\left(-3a^2b\right)^2}{-b\left(6ab^2\right)^2} =$$

$$\begin{aligned} &= \frac{(2ab)^3\left(-3a^2b\right)^2}{-1b\left(6ab^2\right)^2} && \text{the 1 will help with signs} \\ &= \frac{2^3a^3b^3(-3)^2\left(a^2\right)^2b^2}{-1\cdot b\cdot 6^2\cdot a^2\left(b^2\right)^2} && \text{by rule 4} \\ &= \frac{8a^3b^3\cdot 9\cdot a^4b^2}{-1\cdot b\cdot 36\cdot a^2b^4} && \text{by rule 3} \\ &= \frac{8\cdot 9\cdot a^3a^4b^3b^2}{-1\cdot 36\cdot a^2\cdot b\cdot b^4} && \text{since multiplication is commutative} \\ &= \frac{72a^7b^5}{-36a^2b^5} && \text{by rule 1} \\ &= \frac{-2a^7b^5}{a^2b^5} && \text{simplify among numbers: } \frac{72}{-36} = \frac{-72}{36} = \frac{-2}{1} \\ &= \frac{-2a^7}{a^2} && \text{cancel out } b^5 \\ &= \frac{-2a^5}{1} && \text{by rule 2} \\ &= -2a^5 \end{aligned}$$

6. Completely factor each of the following.

(a) $100x - x^2 - 2419 = \; -(x-41)(x-59)$

Solution: We rearrange the polynomial by degree of terms. Since it is quadratic with three terms, it may factor by completing the square. Then we need to factor out -1 to work with a leading coefficient 1 within the parentheses.

$$\begin{aligned}
100x - x^2 - 2419 &= -x^2 + 100x - 2419 \\
&= -\left(x^2 - 100x + 2419\right) \qquad\qquad (x-50)^2 = x^2 - 100x + 2500 \\
&= -\left(\underbrace{x^2 - 100x + 2500} - 2500 + 2419\right) \\
&= -\left((x-50)^2 - 81\right) \\
&= -\left((x-50)^2 - 9^2\right) \\
&= -(x-50+9)(x-50-9) \\
&= -(x-41)(x-59)
\end{aligned}$$

We check by multiplication:

$$\begin{aligned}
-(x-41)(x-59) &= -\left(x^2 - 41x - 59x + 2419\right) \\
&= -\left(x^2 - 100x + 2419\right) \\
&= -x^2 + 100x - 2419
\end{aligned}$$

(b) $2p^4 - 162 = \; 2\left(p^2+9\right)(p+3)(p-3)$

Solution:

$$\begin{aligned}
2p^4 - 162 &= \quad \text{factor out GCF} \\
2\left(p^4 - 81\right) &= \quad \text{re-write both quantities as squares} \\
2\left(\left(p^2\right)^2 - 9^2\right) &= \quad \text{factor via the difference of squares theorem} \\
2\left(p^2+9\right)\left(p^2-9\right) &= \quad \text{second factor will factor again} \\
2\left(p^2+9\right)\left(p^2-3^2\right) &= \quad \text{factor via the difference of squares theorem} \\
&= 2\left(p^2+9\right)(p+3)(p-3)
\end{aligned}$$

We check by multiplication:

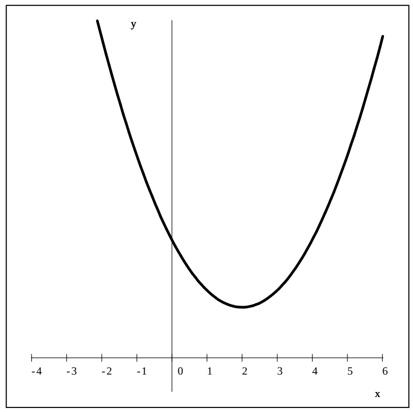
$$\begin{aligned}
2\left(p^2+9\right)\underbrace{(p+3)(p-3)}_{\text{FOIL}} &= 2\left(p^2+9\right)\left(p^2 - 3p + 3p - 9\right) = 2\underbrace{\left(p^2+9\right)\left(p^2-9\right)}_{\text{FOIL}} \\
&= 2\left(p^4 - 9p^2 + 9p^2 - 81\right) = 2\left(p^4 - 81\right) = 2p^4 - 162
\end{aligned}$$

Thus our solution, $2\left(p^2+9\right)(p+3)(p-3)$ is correct.

(c) $x^2 - 4x + 7 =$ does not factor over the real numbers

$$\begin{aligned} x^2 - 4x + 7 &= \\ \underbrace{x^2 - 4x + 4}-4+7 &= (x-2)^2 + 3 \end{aligned} \qquad (x-2)^2 = x^2 - 4x + 4$$

Since the parabola $y = (x-2)^2 + 3$ has its vertex at $(2, 3)$, above the $x-$axis, it does not have any $x-$intercepts. If the expression $(x-2)^2 + 3$ would factor into two linear factors, those two factors would guarantee $x-$intercepts. Thus $(x-2)^2 + 3$ does not factor. In short: the sum of squares does not factor.



(d) $357ab^2 - 30ab^2x - 3ab^2x^2 =$ $-3ab^2(x+17)(x-7)$
Solution: We start with the GCF (greatest common factor) and rearrange the polynomial by degree of terms.

$$\begin{aligned} 357ab^2 - 30ab^2x - 3ab^2x^2 &= 3ab^2\left(119 - x^2 - 10x\right) \\ &= 3ab^2\left(-x^2 - 10x + 119\right) \end{aligned}$$

The expression is quadratic with three terms, and so it may factor by completing the square. Then we need to factor out -1 to make the leading coefficient 1.

$$\begin{aligned} 357ab^2 - 30ab^2x - 3ab^2x^2 &= 3ab^2\left(119 - x^2 - 10x\right) \\ &= -3ab^2\left(x^2 + 10x - 119\right) \qquad (x+5)^2 = x^2 + 10x + 25 \\ &= -3ab^2\left(\underbrace{x^2 + 10x + 25}-25 - 119\right) \\ &= -3ab^2\left((x+5)^2 - 144\right) \\ &= -3ab^2\left((x+5)^2 - 12^2\right) \\ &= -3ab^2(x+5+12)(x+5-12) \\ &= -3ab^2(x+17)(x-7) \end{aligned}$$

(e) $3a^3 - 27ab^2 =$ $3a(a+3b)(a-3b)$
Solution:

$$\begin{aligned} 3a^3 - 27ab^2 &= \quad \text{factor out GCF} \\ 3a\left(a^2 - 9b^2\right) &= \quad \text{re-write } 9b^2 \text{ as } (3b)^2 \\ 3a\left(a^2 - (3b)^2\right) &= \quad \text{factor via the difference of squares theorem} \\ &= 3a(a+3b)(a-3b) \end{aligned}$$

We check by multiplication:

$$\begin{aligned} 3a(a+3b)(a-3b) &= 3a\left(a^2-3ab+3ab-9b^2\right) = 3a\left(a^2-9b^2\right) \\ &= 3a^3-27ab^2 \end{aligned}$$

Thus our solution, $3a(a+3b)(a-3b)$ is correct.

(f) $20x+5x^3 =$ $5x\left(x^2+4\right)$

Solution: We rearrange the terms by degree first and then factor out the GCF.

$$20x+5x^3 = 5x^3+20x = 5x\left(x^2+4\right)$$

Since the sum of squares does not factor, the final answer is $5x\left(x^2+4\right)$. We can easily check the result by mulitplication.

7. (Linear Equations) Solve each of the following equations. Make sure to check your solutions.

(a) $\dfrac{2x-7}{3} = -1$ 2

Solution: We apply all operations to both sides.

$$\begin{aligned} \frac{2x-7}{3} &= -1 && \text{multiply by } 3 \\ 2x-7 &= -3 && \text{add } 7 \\ 2x &= 4 && \text{divide by } 2 \\ x &= 2 \end{aligned}$$

We check:

$$\text{LHS} = \frac{2(2)-7}{3} = \frac{4-7}{3} = \frac{-3}{3} = -1 = \text{RHS}$$

Thus our solution, $x=2$ is correct.

(b) $\dfrac{x+8}{3} = -2$ -14

Solution: We apply all operations to both sides.

$$\begin{aligned} \frac{x+8}{3} &= -2 && \text{multiply by } 3 \\ x+8 &= -6 && \text{subtract } 8 \\ x &= -14 \end{aligned}$$

We check:

$$\text{LHS} = \frac{-14+8}{3} = \frac{-6}{3} = -2 = \text{RHS}$$

Thus our solution, $x=-14$ is correct.

(c) $\frac{x}{3}+8=-2$ -30
Solution: We apply all operations to both sides.

$$\begin{aligned}\frac{x}{3}+8 &= -2 && \text{subtract } 8\\ \frac{x}{3} &= -10 && \text{multiply by } 3\\ x &= -30\end{aligned}$$

We check:

$$\text{LHS}=\frac{-30}{3}+8=-10+8=-2=\text{RHS}$$

Thus our solution, $x=-30$ is correct.

(d) $\frac{1}{5}x-\frac{2}{3}=\frac{26}{15}$ 12
Solution:

$$\begin{aligned}\frac{1}{5}x-\frac{2}{3} &= \frac{26}{15} && \text{add } \frac{2}{3} \text{ to both sides} && \frac{26}{15}+\frac{2}{3}=\frac{26}{15}+\frac{2\cdot 5}{3\cdot 5}=\\ \frac{1}{5}x &= \frac{12}{5} && \text{divide by } \frac{1}{5} && \frac{26}{15}+\frac{10}{15}=\frac{36}{15}=\frac{\not{3}\cdot 12}{\not{3}\cdot 5}=\frac{12}{5}\\ x &= 12 && && \frac{\frac{12}{5}}{\frac{1}{5}}=\frac{12}{5}\cdot\frac{5}{1}=\frac{12\cdot\not{5}}{1\cdot\not{5}}=\frac{12}{1}=12\end{aligned}$$

We check: if $x=12$, then

$$\begin{aligned}\text{LHS} &= \frac{1}{5}\cdot 12-\frac{2}{3}=\frac{1}{5}\cdot\frac{12}{1}-\frac{2}{3}=\frac{12}{5}-\frac{2}{3}=\frac{12\cdot 3}{5\cdot 3}-\frac{2\cdot 5}{3\cdot 5}=\frac{36}{15}-\frac{10}{15}=\frac{26}{15}\\ \text{RHS} &= \frac{26}{15}\end{aligned}$$

Thus our solution, $x=12$ is correct.

(e) $\frac{3}{8}x+\left(1\frac{4}{5}\right)=\frac{3}{10}$ -4
Solution: this is a very simple equation, much like $2x+1=7$, only the numbers are fractions. But the principles and operations regarding equations are the same.

$$\begin{aligned}\frac{3}{8}x+\left(1\frac{4}{5}\right) &= \frac{3}{10} && \text{convert mixed number to improper fraction}\\ \frac{3}{8}x+\frac{9}{5} &= \frac{3}{10} && \text{subtract } \frac{9}{5} \text{ from both sides; } \frac{3}{10}-\frac{9}{5}=\frac{3}{10}-\frac{18}{10}=\frac{3-18}{10}=\frac{-15}{10}=\frac{-3}{2}\\ \frac{3}{8}x &= \frac{-3}{2} && \text{divide both sides by } \frac{3}{8}\\ x &= -4 && \left(\frac{-3}{2}\right)\div\left(\frac{3}{8}\right)=\frac{-3}{2}\cdot\frac{8}{3}=\frac{-24}{6}=-4\end{aligned}$$

We check:

$$\begin{aligned}\text{RHS} &= \frac{3}{8}(-4)+\left(1\frac{4}{5}\right)=\frac{3}{8}\cdot\frac{-4}{1}+\frac{9}{5}=\frac{-12}{8}+\frac{9}{5}=\frac{-3}{2}+\frac{9}{5}=\frac{-15}{10}+\frac{18}{10}=\frac{3}{10}\\ \text{LHS} &= \frac{3}{10}\end{aligned}$$

Thus our solution, -4 is correct.

(f) $3w - 5 = 5(w-2)$ $\frac{5}{2}$
Solution: we first apply the law of distributivity to simplify the right-hand side.

$$\begin{aligned} 3w - 5 &= 5(w-2) \\ 3w - 5 &= 5w - 10 && \text{subtract } 3w \text{ from both sides} \\ -5 &= 2w - 10 && \text{add } 10 \text{ to both sides} \\ 5 &= 2w && \text{divide both sides by 2} \\ \frac{5}{2} &= w \end{aligned}$$

We check: if $w = \frac{5}{2}$, then

$$\begin{aligned} \text{LHS} &= 3\left(\frac{5}{2}\right) - 5 = \frac{3}{1} \cdot \frac{5}{2} - 5 = \frac{15}{2} - 5 = \frac{15}{2} - \frac{5}{1} = \frac{15}{2} - \frac{10}{2} = \frac{5}{2} \\ \text{RHS} &= 5\left(\frac{5}{2} - 2\right) = 5\left(\frac{5}{2} - \frac{2}{1}\right) = 5\left(\frac{5}{2} - \frac{4}{2}\right) = 5\left(\frac{1}{2}\right) = \frac{5}{1} \cdot \frac{1}{2} = \frac{5}{2} \end{aligned}$$

(g) $7(j-5) + 9 = 2(-2j+5) + 5j$ 6
Solution:

$$\begin{aligned} 7(j-5) + 9 &= 2(-2j+5) + 5j && \text{distribute on both sides} \\ 7j - 35 + 9 &= -4j + 10 + 5j && \text{combine like terms} \\ 7j - 26 &= j + 10 && \text{subtract } j \\ 6j - 26 &= 10 && \text{add 26} \\ 6j &= 36 && \text{divide by 6} \\ j &= 6 \end{aligned}$$

We check: if $j = 6$, then

$$\begin{aligned} \text{LHS} &= 7(6-5) + 9 = 7 \cdot 1 + 9 = 7 + 9 = 16 \\ \text{RHS} &= 2(-2 \cdot 6 + 5) + 5 \cdot 6 = 2(-12+5) + 30 = 2(-7) + 30 = -14 + 30 = 16 \end{aligned}$$

Thus our solution is correct.

(h) $3(x-5) - 5(x-1) = -2x + 1$ no solution
Solution:

$$\begin{aligned} 3(x-5) - 5(x-1) &= -2x + 1 && \text{multiply out parentheses} \\ 3x - 15 - 5x + 45 &= -2x + 1 && \text{combine like terms} \\ -2x + 30 &= -2x + 1 && \text{add } 2x \\ 30 &= 1 \end{aligned}$$

Since x disappeared from the equation and we are left with an unconditionally false statement, there is no solution for this equation. This type of an equation is called a contradiction.

8. (Absolutie Value Equations) Solve each of the following equations. Make sure to check your solutions.

(a) $|3x+1|-7=1$ $\quad -3, \frac{7}{3}$

Solution: We first isolate the expression within the absolute value sign.

$$\begin{aligned} |3x+1|-7 &= 1 \quad \text{add } 7 \\ |3x+1| &= 8 \\ 3x+1 &= 8 \quad \text{or} \quad 3x+1=-8 \quad \text{solve for } x \\ 3x &= 7 \quad \text{or} \quad 3x=-9 \\ x &= \frac{7}{3} \quad \text{or} \quad x=-3 \end{aligned}$$

We check. If $x=\frac{7}{3}$, then

$$\begin{aligned} \text{LHS} &= \left|3\left(\frac{7}{3}\right)+1\right|-7=|7+1|-7=|8|-7=8-7=1 \\ \text{RHS} &= 1 \end{aligned}$$

We check. If $x=-3$, then

$$\begin{aligned} \text{LHS} &= |3(-3)+1|-7=|-9+1|-7=|-8|-7=8-7=1 \\ \text{RHS} &= 1 \end{aligned}$$

Thus both solutions, -3 and $\frac{7}{3}$ are correct.

(b) $|3x+1|-1=-11$ no solution

Solution: We first isolate the expression within the absolute value sign.

$$\begin{aligned} |3x+1|-1 &= -11 \quad \text{add } 1 \\ |3x+1| &= -10 \end{aligned}$$

Since absolute values are never negative, there is no solution.

(c) $\left|\frac{1}{2}x-3\right|-2=-23$ no solution

Solution: We first isolate the expression within the absolute value sign.

$$\begin{aligned} \left|\frac{1}{2}x-3\right|-2 &= -23 \quad \text{add } 2 \\ \left|\frac{1}{2}x-3\right| &= -21 \end{aligned}$$

Since absolute values are always non-negative (i.e. positive or zero), this condition will not hold for any x. Thus there is no solution for this equation.

(d) $\left|\frac{1}{2}x-3\right|-2=23$ $\quad -44, 56$

Solution: We first isolate the expression within the absolute value sign.

$$\begin{aligned} \left|\frac{1}{2}x-3\right|-2 &= 23 \quad \text{add } 2 \\ \left|\frac{1}{2}x-3\right| &= 25 \end{aligned}$$

We translate the one equation involving absolute values into two linear equations and solve them separately.

$$\begin{aligned} \frac{1}{2}x - 3 &= -25 \quad \text{or} \quad \frac{1}{2}x - 3 = 25 \quad \text{add } 3 \\ \frac{1}{2}x &= -22 \quad \text{or} \quad \frac{1}{2}x = 28 \quad \text{multiply by } 2 \\ x &= -44 \quad \text{or} \quad x = 56 \end{aligned}$$

The solutions are -44 and 56.
We check: If $x = -44$, then

$$\text{LHS} = \left|\frac{1}{2}(-44) - 3\right| - 2 = |-22 - 3| - 2 = |-25| - 2 = 25 - 2 = 23 = \text{RHS}$$

and if $x = 56$, then

$$\text{LHS} = \left|\frac{1}{2}(56) - 3\right| - 2 = |28 - 3| - 2 = |25| - 2 = 25 - 2 = 23 = \text{RHS}$$

The our solutions, -44 and 56 are correct.

9. (Higher Degree Equations) Solve each of the following equations. Make sure to check your solutions.

(a) $(3x)^2 - (x+3)(5x-3) = (5-2x)^2 - 16$ 0

Solution: There is an algebraically dangerous situation in this problem. We are used to not write parentheses after we FOIL a product of polynomials. In this case, however, we subtract the product. Since we subtract the entire expression and not just its first term, we need to keep the product $(x+3)(5x-3)$ in parentheses. We work out the products first.that require more steps.

$$(x+3)(5x-3) = 5x^2 - 3x + 15x - 9 = 5x^2 + 12x - 9 \quad \text{and}$$
$$(5-2x)^2 = (5-2x)(5-2x) = 25 - 10x - 10x + 4x^2 = 4x^2 - 20x + 25$$

We now are ready to solve the equation.

$$\begin{aligned} (3x)^2 - (x+3)(5x-3) &= (5-2x)^2 - 16 && \text{multiply the polynomials on both sides} \\ 9x^2 - \left(5x^2 + 12x - 9\right) &= 4x^2 - 20x + 25 - 16 && \text{subtract} \\ 9x^2 - 5x^2 - 12x + 9 &= 4x^2 - 20x + 25 - 16 && \text{combine like terms} \\ 4x^2 - 12x + 9 &= 4x^2 - 20x + 9 && \text{subtract } 4x^2 \\ -12x + 9 &= -20x + 9 \end{aligned}$$

At this point, the equation turned out to be linear, since the quadratic terms disappeared.

$$\begin{aligned} -12x + 9 &= -20x + 9 && \text{add } 20x \\ 8x + 9 &= 9 && \text{subtract } 9 \\ 8x &= 0 && \text{divide by } 8 \\ x &= 0 \end{aligned}$$

We check our result:

$$\begin{aligned} \text{LHS} &= (3 \cdot 0)^2 - (0+3)(5 \cdot 0 - 3) = 0^2 - 3(0-3) = 0 - 3(-3) = 0 + 9 = 9 \\ \text{RHS} &= (5 - 2 \cdot 0)^2 - 16 = (5-0)^2 - 16 = 5^2 - 16 = 25 - 16 = 9 \end{aligned}$$

Thus the solution, 0 is correct.

(b) $(x+4)(1-2x) = 3x - 2(x-3)^2$ 1

Solution: We first work out products that require several steps.

$$\begin{aligned} (x-3)^2 &= (x-3)(x-3) = x^2 - 3x - 3x + 9 = x^2 - 6x + 9 \quad \text{and} \\ (x+4)(1-2x) &= x - 2x^2 + 4 - 8x = -2x^2 - 7x + 4 \end{aligned}$$

$$\begin{aligned} (x+4)(1-2x) &= 3x - 2(x-3)^2 && \text{multiply out parentheses} \\ -2x^2 - 7x + 4 &= 3x - 2\left(x^2 - 6x + 9\right) && \text{distribute } -2 \\ -2x^2 - 7x + 4 &= 3x - 2x^2 + 12x - 18 && \text{combine like terms} \\ -2x^2 - 7x + 4 &= -2x^2 + 15x - 18 && \text{add } 2x^2 \\ -7x + 4 &= 15x - 18 \end{aligned}$$

At this point, the quadratic part canceled out, and the equation has turned out to be linear.

$$\begin{aligned} -7x + 4 &= 15x - 18 && \text{add } 7x \\ 4 &= 22x - 18 && \text{add } 18 \\ 22 &= 22x && \text{divide by } 22 \\ 1 &= x \end{aligned}$$

We check:

$$\begin{aligned} \text{LHS} &= (1+4)(1-2(1)) = 5(1-2) = 5(-1) = -5 \\ \text{RHS} &= 3(1) - 2(1-3)^2 = 3(1) - 2(-2)^2 = 3(1) - 2(4) \\ &= 3 - 8 = -5 \end{aligned}$$

Thus our solution, 1 is correct.

(c) $2x^3 = 20x^2 + 1750x$ $35, 0, -25$

Solution: We reduce one side to zero, then factor, and then apply the zero property.

$$\begin{aligned} 2x^3 &= 20x^2 + 1750x \\ 2x^3 - 20x^2 - 1750x &= 0 && \text{factor out GCF} \\ 2x\left(x^2 - 10x - 875\right) &= 0 && \text{divide both sides by 2} \\ x\left(x^2 - 10x - 875\right) &= 0 \end{aligned}$$

We will factor by completing the square. Half of the linear coefficient is -5, and thus we will work with $(x-5)^2 = x^2 - 10x + 25$. We smuggle in 25.

$$\begin{aligned} x\left(x^2 - 10x - 875\right) &= 0 \\ x\left(\underbrace{x^2 - 10x + 25} - 25 - 875\right) &= 0 \\ x\left((x-5)^2 - 900\right) &= 0 \\ x\left((x-5)^2 - 30^2\right) &= 0 \\ x(x - 5 + 30)(x - 5 - 30) &= 0 \\ x(x+25)(x-35) &= 0 \end{aligned}$$

$$x(x+25)(x-35)=0$$

Applying the zero property we obtain 0, -25, and 35 as the solutions. We check (even if it hurts a little...). If $x=0$, then

$$\begin{aligned} \text{LHS} &= 2(0)^3=0 \\ \text{RHS} &= 20(0)^2+1750(0)=0 \end{aligned}$$

If $x=-25$, then

$$\begin{aligned} \text{LHS} &= 2(-25)^3=2(-15\,625)=-31\,250 \\ \text{RHS} &= 20(-25)^2+1750(-25)=20(625)+1750(-25) \\ &= 12\,500-43\,750=-31\,250 \end{aligned}$$

And if $x=35$, then

$$\begin{aligned} \text{LHS} &= 2(35)^3=2(42\,875)=85\,750 \\ \text{RHS} &= 20(35)^2+1750(35)=20(1225)+1750(35) \\ &= 24\,500+61\,250=85\,750 \end{aligned}$$

10. Word Problems.

(a) A TV is priced at \$ 600. How much would it cost if it went on a 15% sale? \$ 510

Solution:

Method 1. The new price is

$$\$\,600-(15\%\ \text{ of }\ \$\,600)$$

We now need to compute 15% of \$ 600. Since 15% is the same as $\frac{15}{100}$,

$$\begin{aligned} &\frac{1}{100}\ \text{ of }\ \$\,600\ \text{ is }\ \$\,6 \\ &\frac{15}{100}\ \text{ of }\ \$\,600\ \text{ is }\ \$\,90 \end{aligned}$$

Thus the sale price is \$ 600 − \$ 90 = \$ 510.

Method 2. Since the new price is

$$\begin{aligned} \$\,600-(15\%\ \text{ of }\ \$\,600) &= \\ (100\%\ \text{ of }\ \$\,600)-(15\%\ \text{ of }\ \$\,600) &= \\ (100\%-15\%)\ \text{ of }\ \$\,600 &= 85\%\ \text{ of }\ \$\,600 \end{aligned}$$

We now need to compute 85% of \$ 600. Since 85% is the same as $\frac{85}{100}$,

$$\begin{aligned} &\frac{1}{100}\ \text{ of }\ \$\,600\ \text{ is }\ \$\,6 \\ &\frac{85}{100}\ \text{ of }\ \$\,600\ \text{ is }\ \$\,510 \end{aligned}$$

Thus the sale price is \$ 510.

(b) We have placed \$ 5000 in a bank account with an annual interest rate of 6%. How much money do we have in the account after one year? \$ 5300

Solution:

Method 1: We will have

$$\$\ 5000 + \text{interest} = \$\ 5000 + (6\% \text{ of } \$\ 5000)$$

We now need to compute 6% of \$ 5000. 6% is the same as $\frac{6}{100}$ of \$ 5000

$$\frac{1}{100} \text{ of } \$\ 5000 \text{ is } \$\ 50$$
$$\frac{6}{100} \text{ of } \$\ 5000 \text{ is } \$\ 300$$

Thus we have $\$\ 5000 + \$\ 300 = \$\ 5300$.

Method 2. We will have the principal and the interest

$$\begin{aligned} \$\ 5000 + (6\% \text{ of } \$\ 5000) &= \\ (100\% \text{ of } \$\ 5000) + (6\% \text{ of } \$\ 5000) &= \\ (100\% + 6\%) \text{ of } \$\ 5000 &= 106\% \text{ of } \$\ 5000 \end{aligned}$$

We now need to compute 106% of \$ 5000. Since 106% is the same as $\frac{106}{100}$,

$$\frac{1}{100} \text{ of } \$\ 5000 \text{ is } \$\ 50$$
$$\frac{106}{100} \text{ of } \$\ 5000 \text{ is } \$\ 5300$$

Thus we have \$ 5300.

(c) Ann took four exams. Her scores on the first three exams were 63, 76, and 68. How many points did she earn on the fourth exam if her average is 71? 77

Solution: Let x denote the score of Ann's fourth exam. The equation will express the average.

$$\begin{aligned} \frac{63+76+68+x}{4} &= 71 && \text{simplify left-hand side by adding the three scores} \\ \frac{x+207}{4} &= 71 && \text{multiply by 4} \\ x+207 &= 284 && \text{subtract 207} \\ x &= 77 \end{aligned}$$

We check: the average of the four exams is

$$\text{Average} = \frac{63+76+68+77}{4} = 71$$

Thus our solution, 77 is correct.

(d) If we multiply a number by -2 and add 7, the result is 25. Find this number. -9
Solution: Let x denote the number. Then the problem translate to

$$\begin{aligned} -2\cdot x+7 &= 25 && \text{solve for } x \\ -2x+7 &= 25 && \text{subtract 7} \\ -2x &= 18 && \text{divide by } -2 \\ x &= -9 \end{aligned}$$

Thus the number is 9. We check: if we multiply our number, -9 by -2, we get 18. Then we add 7, we indeed get 25.

(e) If we subtract 5 from the opposite of a number, we get -1. Find this number. -4
Solution: Let x denote the number. Then the problem translate to

$$\begin{aligned} -x-5 &= -1 && \text{solve for } x, \quad \text{add 5} \\ -x &= 4 && \text{multiply by } -1 \\ x &= -4 \end{aligned}$$

Thus the number is -4. We check: if we subtract 5 from the opposite of -4, we get $-(-4)-5=4-5=-1$. Thus our solution is correct.

(f) Three times a number is 5 more than 16. Find this number. 7
Solution: Let x denote the number. Since 21 is the number that is 5 more than 16, we have

$$\begin{aligned} 3x &= 21 && \text{solve for } x, \quad \text{divide by 3} \\ x &= 7 \end{aligned}$$

Thus the number is 7. We check: three times 7 is 21 which is indeed 5 more than 16. Thus our solution is correct.

(g) One number is 18 less than the other. Find these numbers if their sum is 110. 46 and 64
Solution: Let us denote the smaller number by x. Then the larger number is $x+8$. The equation will express the sum of the numbers.

$$\begin{aligned} x+x+18 &= 110 && \text{combine like terms} \\ 2x+18 &= 110 && \text{subtract 18} \\ 2x &= 92 && \text{divide by 2} \\ x &= 46 \end{aligned}$$

Thus the smaller number is 46 and the smaller number is $46+18=64$.

(h) One number is 18 less than the other. Find these numbers if their product is 1600.
$32, 50$ and $-50, -32$
Solution: Let us denote the smaller number by x. Then the larger number is $x+8$. The equation will express the product of the numbers.

$$\begin{aligned} x(x+18) &= 1600 && \text{multiply} \\ x^2+18x &= 1600 && \text{subtract 1600} \\ x^2+18x-1600 &= 0 && \text{factor by completing the square} \end{aligned}$$

$$\begin{aligned} x^2+18x-1600 &= 0 \\ \underbrace{x^2+18x+81}-81-1600 &= 0 \\ (x+9)^2-1681 &= 0 \\ (x+9)^2-41^2 &= 0 \\ (x+9+41)(x+9-41) &= 0 \\ (x+50)(x-32) &= 0 \end{aligned}$$

$$(x+9)^2 = x^2+18x+81 \qquad \sqrt{1681}=41$$

$$x_1 = -50 \qquad x_2 = 32$$

We did not get one pair! We obtained two candidates for the smaller number in two pairs of numbers. If the smaller number is -50, then the larger one is $-50+18=-32$. If the smaller number is 32, then the larger one is $32+18=50$. It is easy to see that both pairs work:

$$\begin{aligned} -32-(-50) &= 18 \quad \text{and} \quad -32(-50)=1600 \\ 50-32 &= 18 \quad \text{and} \quad 32(50)=1600 \end{aligned}$$

Thus there are two solutions, -50 with -32 and 32 with 50.

(i) The product of 3 and the opposite of a number is -63. Find this number. 21

Solution: Let x denote the number. The problem translates to

$$\begin{aligned} 3(-x) &= -63 && \text{solve for } x \\ -3x &= -63 && \text{divide by } -3 \\ x &= 21 \end{aligned}$$

Thus the number is 21. We check: three times the opposite of 21 is $3(-21)=-63$. Thus our solution is correct.

(j) One side of a rectangle is 4 in shorter than 3 times the other side. Find the sides of the rectangle if its perimeter is 48 in. 7 in and 17 in

Solution: Let us denote the shorter side by x. Then the larger side is $3x-4$. The equation expresses the perimeter.

$$\begin{aligned} 2x+2(3x-4) &= 48 && \text{multiply out parentheses} \\ 2x+6x-8 &= 48 && \text{combine like terms} \\ 8x-8 &= 48 && \text{add } 8 \\ 8x &= 56 && \text{divide by } 8 \\ x &= 7 \end{aligned}$$

Thus the shorter side is 7 in, which makes the longer side $3\cdot7-4=17$ in. Thus the sides are 7 in and 17 in long.

(k) A bank teller has 47 more five-dollar bills than ten-dollar bills. The total value of the money is \$1000. How much of each denomination of bill does he have? 51 ten-dollar bills and 98 five-dollar bills

Solution: Let us denote the number of ten-dollar bills by x. Then we have $x+47$ many five-dollar bills. The equation expresses the value of the bills.

$$\begin{aligned}
\underbrace{10x}_{\text{amount in 10-bills}} + \underbrace{5(x+47)}_{\text{amount in 5-bills}} &= 1000 && \text{distribute}\\
10x + 5x + 235 &= 1000 && \text{combine like terms}\\
15x + 235 &= 1000 && \text{subtract 235}\\
15x &= 765 && \text{divide by 15}\\
x &= 51
\end{aligned}$$

Thus we have 51 tens and $51 + 47 = 98$ fives. We check: $98 - 51 = 47$ and $51(10) + 98(5) = 1000$. Thus our solution; 51 ten-dollar bills and 98 five-dollar bills; is correct.

(l) We throw an object upward from the top of a 1200 ft high building. The height of the object, (measured in feet) t seconds after we threw it is

$$h(t) = -16t^2 + 160t + 1200$$

i. Where is the object 3 seconds after we threw it? 1536 ft

Solution: We need to compute $h(3)$. This means that we substitute 3 into t and evaluate the algebraic expression.

$$\begin{aligned}
h(3) &= -16 \cdot 3^2 + 160 \cdot 3 + 1200 = -16 \cdot 9 + 160 \cdot 3 + 1200\\
&= -144 + 480 + 1200 = 336 + 1200 = 1536
\end{aligned}$$

Thus the object is 1536 ft high after 3 seconds.

ii. How long does it take for the object to hit the ground? 15 seconds.

Solution: we need to solve the equation $t = ?$ so that $h(t) = 0$

$$\begin{aligned}
h(t) &= 0\\
-16t^2 + 160t + 1200 &= 0 && \text{factor out } -16\\
-16\left(t^2 - 10t - 75\right) &= 0
\end{aligned}$$

We will factor $t^2 - 10t + 75$ by completing the square.

$$\begin{aligned}
-16\left(t^2 - 10t - 75\right) &= 0 && (t-5)^2 = t^2 - 10t + 25 \quad \text{smuggle in 25}\\
-16\left(\underbrace{t^2 - 10t + 25} - 25 - 75\right) &= 0\\
-16\left((t-5)^2 - 100\right) &= 0 && \text{re-write 100 as } 10^2\\
-16\left((t-5)^2 - 10^2\right) &= 0 && \text{factor via the difference of squares theorem}\\
-16(t - 5 + 10)(t - 5 - 10) &= 0 && \text{simplify}\\
-16(t+5)(t-15) &= 0 && \text{apply zero property}
\end{aligned}$$

$$\begin{aligned}
t + 5 &= 0 \quad \text{or} \quad t - 15 = 0\\
t &= -5 \quad \text{or} \quad t = 15
\end{aligned}$$

Since the negative solution, $t=-5$ does not make sense in the context of the problem, it is ruled out. We check $t=15$:

$$\begin{aligned} h(3) &= -16\cdot 15^2+160\cdot 15+1200 \\ &= -16\cdot 225+160\cdot 15+1200 \\ &= -3600+2400+1200 \\ &= -1200+1200=0 \end{aligned}$$

Thus the answer is: 15 seconds.

(m) A certain triangle's longest side is one centimeter less than six times the shortest side. The other side is five times the shortest side. The perimeter is thirty-five centimeters. Find the length of the longest side. 17 cm

Solution: Let x denote the shortest side. Then the longest side is $6x-1$, and the other side is $5x$. We obtain the equation by expressing the perimeter of the triangle. Then we solve for x.

$$\begin{aligned} \underbrace{x}_{\text{shortest side}} + \underbrace{6x-1}_{\text{longest side}} + \underbrace{5x}_{\text{other side}} &= 35 && \text{combine like terms} \\ 12x-1 &= 35 && \text{add } 1 \text{ to both sides} \\ 12x &= 36 && \text{divide both sides by } 12 \\ x &= 3 \end{aligned}$$

Now we know that $x=3$. Since the longest side was denoted by $6x-1$, it must be $6(3)-1=18-1=17$ cm long.

(n) Find all numbers that satisfy the following condition: if we square the number, we get back the same number. $0,1$

Solution: Let us denote the number by x. The equation is

$$\begin{aligned} x^2 &= x && \text{reduce one side to zero} \\ x^2-x &= 0 && \text{factor} \\ x(x-1) &= 0 && \text{apply the zero property} \end{aligned}$$

$$\begin{aligned} x &= 0 \quad\text{or}\quad x-1=0 \\ x &= 0 \quad\text{or}\quad x=1 \end{aligned}$$

Thus there are two numbers, 0 and 1, satisfying the property. We check: $0^2=0$ and $1^1=1$. Thus our answer is: 0 and 1

(o) Find all numbers that satisfy the following condition: if we raise the number to the third power, the result is four times the original number. $0,2,-2$

Solution: Let us denote the number by x. The equation is

$$\begin{aligned} x^3 &= 4x && \text{reduce one side to zero} \\ x^3-4x &= 0 && \text{factor out the GCF} \\ x\left(x^2-4\right) &= 0 && \text{factor via the difference of squares theorem} \\ x(x+2)(x-2) &= 0 && \text{apply the zero property} \end{aligned}$$

$$x = 0 \quad \text{or} \quad x+2=0 \quad \text{or} \quad x-2=0$$
$$x = 0 \quad \text{or} \quad x=-2 \quad \text{or} \quad x=2$$

Thus there are three numbers, 0, 2 and -2, satisfying the property. We check: $0^3 = 4 \cdot 0$, $2^3 = 4 \cdot 2$, and $-2^3 = 4(-2)$. Thus our answer is: 0, 2, and -2

11. Consider the equations $2x - y = 2$ and $y = -x + 7$.

(a) Graph these lines in the same coordinate system. Use your graph to find the coordinates where the points intersect.
We first graph the line $2x - y = 2$.

$$\begin{aligned} \text{If } x &= 0,\ y=? && \text{substitute } x=0 \text{ into the equation of the line} \\ 2(0) - y &= 2 && \text{solve for } y \\ 0 - y &= 2 && \\ -y &= 2 && \text{multiply by } -1 \\ y &= -2 && \Longrightarrow \text{ we found } (0,-2) \end{aligned}$$

$$\begin{aligned} \text{If } x &= 1,\ y=? && \text{substitute } x=1 \text{ into the equation of the line} \\ 2(1) - y &= 2 && \text{solve for } y \\ 2 - y &= 2 && \text{subtract } 2 \\ -y &= 0 && \text{multiply by } -1 \\ y &= 0 && \Longrightarrow \text{ we found } (1,0) \end{aligned}$$

$$\begin{aligned} \text{If } x &= 2,\ y=? && \text{substitute } x=2 \text{ into the equation of the line} \\ 2(2) - y &= 2 && \text{solve for } y \\ 4 - y &= 2 && \text{subtract } 4 \\ -y &= -2 && \text{multiply by } -1 \\ y &= 2 && \Longrightarrow \text{ we found } (2,2) \end{aligned}$$

$$\begin{aligned} \text{If } x &= 3,\ y=? && \text{substitute } x=3 \text{ into the equation of the line} \\ 2(3) - y &= 2 && \text{solve for } y \\ 6 - y &= 2 && \text{subtract } 6 \\ -y &= -4 && \text{multiply by } -1 \\ y &= 4 && \Longrightarrow \text{ we found } (3,4) \end{aligned}$$

We graph the points $(0,-2)$, $(1,0)$, $(2,2)$, $(3,4)$, and connect the points. (Green line.)
We now graph the line $y = -x + 7$.

$$\begin{aligned} \text{If } x &= 0,\ y=? && \text{substitute } x=0 \text{ into the equation of the line} \\ y &= -(0) + 7 = 0 + 7 = 7 && \Longrightarrow \text{ we found } (0,7) \end{aligned}$$

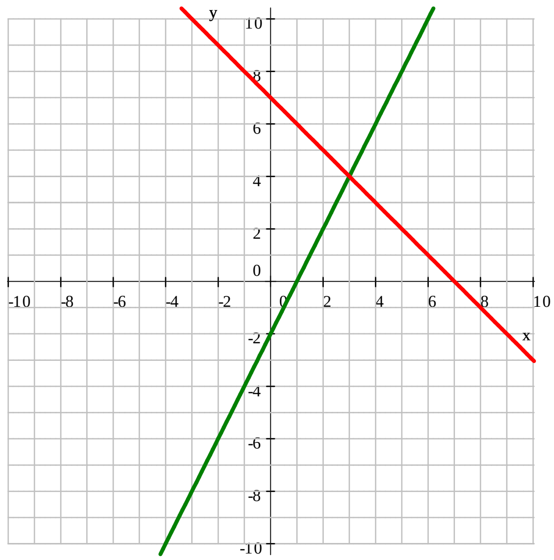
$$\begin{aligned} \text{If } x &= 2,\ y=? && \text{substitute } x=2 \text{ into the equation of the line} \\ y &= -(2) + 7 = 5 && \Longrightarrow \text{ we found } (2,5) \end{aligned}$$

$$\begin{aligned} \text{If } x &= 4,\ y=? && \text{substitute } x=4 \text{ into the equation of the line} \\ y &= -(4)+7=3 && \Longrightarrow \text{ we found } (4,3) \end{aligned}$$

$$\begin{aligned} \text{If } x &= 6,\ y=? && \text{substitute } x=6 \text{ into the equation of the line} \\ y &= -(6)+7=1 && \Longrightarrow \text{ we found } (6,1) \end{aligned}$$

$$\begin{aligned} \text{If } x &= 7,\ y=? && \text{substitute } x=7 \text{ into the equation of the line} \\ y &= -(7)+7=0 && \Longrightarrow \text{ we found } (7,0) \end{aligned}$$

We graph the points $(0,7)$, $(2,5)$, $(4,3)$, $(6,1)$, and $(7,0)$ and connect the points. (Red line.)



We read from the graph that the lines intersect at $(3,4)$.

(b) Use algebraic methods to check your answer for part a).
Solution: Is the point $(3,4)$ on the line $2x-y=2$?

$$\text{LHS} = 2x-y = 2(3)-(4) = 6-4 = 2 = \text{RHS} \quad \Longrightarrow \quad \text{yes}$$

Is the point $(3,4)$ on the line $y=-x+7$?

$$\begin{aligned} \text{LHS} &= y = 4 \\ \text{RHS} &= -x+7 = -3+7 = 4 = \text{LHS} \quad \Longrightarrow \quad \text{yes} \end{aligned}$$

12. Graph the parabola $y = -8x + x^2 + 15$. Clearly label the coordinates of five points on the parabola, including vertex and intercepts.

Solution: We obtain all forms of the equation first.

$$y = x^2 - 8x + 15 \implies \text{polynomial form}$$

Half of the linear coefficient is -4, thus we will work with $(x-4)^2 = x^2 - 8x + 16$

$$\begin{aligned} y &= x^2 - 8x + 15 \\ y &= \underbrace{x^2 - 8x + 16} - 16 + 15 \\ y &= (x-4)^2 - 1 \implies \text{complete square form} \end{aligned}$$

We factor via the difference of squares theorem

$$\begin{aligned} y &= (x-4)^2 - 1^2 \quad \text{since } 1 = 1^2 \\ y &= (x-4+1)(x-4-1) \\ y &= (x-3)(x-5) \implies \text{factored form} \end{aligned}$$

From the polynomial form we obtain the $y-$intecept, $(0, 15)$. From the complete square form, the vertex is $(4, -1)$. Finally, the factored form tells us that there are two $x-$intercepts, $(3, 0)$ and $(5, 0)$. The few missing points, close to the vertex can be found by substituting values for x into any of the three forms of the equations to find y. This time we will work with the polynomial form.

$$\begin{aligned} \text{if } x &= 2, \text{ then } y = (2)^2 - 8(2) + 15 = 4 - 16 + 15 = 3 \\ \text{if } x &= 6, \text{ then } y = (6)^2 - 8(6) + 15 = 36 - 48 + 15 = 3 \end{aligned}$$

We are ready to graph: