Chapter 1

Class 1 Material

1.1 The Laguage of Mathematics

What is Mathematics?

As a graduate student, I had the annoying habbit of asking my teachers and peer students what they think mathematics is. To my surprise, I received many different answers, and to this day, I agree with many of them. In my eyes, mathematics is many things. In mathematics, we will be talking a lot about things being true or being not true. Although this probably happens during every course in every discipline, mathematical truth can be objectively established and agreed upon. To achieve such an objective approach, we have to develop a language that is objectively understood. In this sense, mathematics is also a language.

Mathematics is a collection of *true statements* that are developed, expressed, and interpreted using an objective *language* and rules of *logic*.

To understand mathematics, we need to first agree on an objective language. Reading and writing mathematical notation correctly will be important.

’Then you should say what you mean’, the March Hare went on. ’I do,’ Alice hastily replied; ’at least - at least I mean what I say - that’s the same thing, you know.’ ’Not the same thing a bit!’ said the Hatter. Why, you might just as well say that ’I see what I eat’ is the same thing as ’I eat what I see!’

Lewis Carroll
Alice’s Adventures in Wonderland

Words and Sentences

Mathematical statements are much like English sentences. As English sentences are built from different kind of words, mathematical statements usually contain three types of components: **objects, operations and relations**.

Definition: The concept of an **object** (very much like nouns in English sentences) is usually clearly understood and needs no explanation.

Examples of objects from algebra include $2, 3$, and generally, numbers. Objects from geometry include lines, points, triangles, circles, line segments, etc.

Definition: An **operation** is an action (very much like verbs in English sentences) that can be applied to objects and usually result in new objects.

Examples of operations from algebra include addition, subtraction, multiplication, and division. Operations from geometry include reflection to a line, rotation, or translation that can be performed on points, triangles, circles, line segments, etc.

Definition: A **relation** is something we use to compare two objects.

Relations, unlike operations do not produce new things, we use them to compare already existing objects. For example, the operation addition produces 7 if applied to 2 and 5. Relations in algebra are equal, or less, or greater. Relations from geometry are how geometric objects can be comapred to each other: similar, congruent, parallel, perpendicular.

It is actually pretty difficult to form a meaningful statement without using at least one of each of these three components. In the statement $2 + 5 = 7$ the numbers 2, 5, and 7 are the objects, addition (denoted by $+$) is the operation, and being equal (denoted by $=$) is the relation.

It is a common misconception to think of mathematics as the study of *only* numbers. Numbers are only certain types of objects. As we progress in the study of mathematics, we will find that there are many other types of interesting objects. For example, sets are objects we will soon study. Furthermore, the study of operations and relations is also interesting and fruitful.

So, what kind of are true statements can be established in mathematics? There are three types of true statements in mathematics, **definitions**, **axioms**, and **theorems**.

Definition: A **definition** is a labeling statement in which we agree to use an expression to refer to an object, operation, or relation in mathematics.

Definitions are all true statements, because they simply reflect an agreement in the terms of the language. To be precise, these decisions were made without consultation with us, often decades (if not centuries) before we were even born.

An example for a definition would be if we pointed to a clear sky and said: 'From now on, let's call this color blue'.

Definition: A **theorem** is a statement that we insist on proving before believing that it is true. To **prove** a theorem means to derive it from previously established true statements, using logically correct steps.

If you think about the last definition a little bit, you will see that no theorem can exist unless we agree on accepting a few statements to be true, without proving them. These are our "starting true statements".

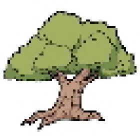
Definition: An **axiom** is a statement we agree to accept to be true without a proof.

Axioms are usually simple and basic statements that are in agreement with our intuition.

For example, the statement '*It is possible to draw a straight line from any point to any other point.*' is an axiom. It has been a constant effort to keep the number of axioms to a minimum. We prove a theorem by deriving its statement from statements already established to be true.

To be precise, when we prove our first theorem, we derive its statement from the axioms. When we prove our second theorem, we derive its statement from the axioms and the first theorem. When proving the third theorem, we can use all the axioms, and the first and second theorems. And so on. For our tenth theorem, we have all the axioms and the first nine theorems at our disposal. At this point, we are buliding a logically sound theory, a unified discipline within mathematics. It is one thing to suspect, to feel, or to have a hunch that something is true. It is entirely different from proving it, with unescapable force of logic.

The ancient Greek mathematician Euclid discussed mathematics in such a manner, i.e stating axioms and building a theory by deriving a sequence of theorems from the axioms. (He called axioms postulates.) Mathematicians immediately accepted and embraced this logical approach to the study of mathematics - and it is how it is done still today. Although Euclid has contributed to several parts of mathematics (including geometry and number theory), he completely axiomatized of what we now call classical geometry or Euclidean geometry. He stated five postulates, accepted them to be true and derived most basic theorems of classical geometry.



Enrichment

1. Look up Euclid's Elements on the internet. (Start at Wikipedia). What is Elements? What is Euclid's contribution to **all** subjects within today's mathematics? What are postulates? List Euclid's five postulates. Explain the significance of these five statements.

2. What is the parallel postulate? Look up the history of Euclid's parallel postulate on the internet. (Start with Wikipedia.) How would we go about proving that an axiom is really a theorem? List statements from geometry that are logically equivalent to the parallel postulate. What exactly could it mean for two axioms to be equivalent to each other?

Euclid (~400 BC - ~300 BC)

1.2 The Real Number System

Definition: The **set of all natural numbers**, denoted by $\mathbb{N}$, is the infinite set

$$\mathbb{N} = \{1, 2, 3, 4, ...\}$$

If we add two natural numbers, the sum is also a natural number. In other words, if x and y are natural numbers, then the sum $x + y$ is also a natural number. When this is true, we say that the set of all natural numbers is **closed under addition**. On the other hand, the set of all natural numbers is not closed under subtraction: while $10 - 3$ is a natural number, $3 - 10$ is not.

Theorem: *The set of all natural numbers is closed under addition and multiplication, but not under subtraction and division.*

Definition: The **set of all integers**, denoted by $\mathbb{Z}$, is the infinite set

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, ...\}$$

Notice that the set of all integers contains all natural numbers. When this happens, we say that the set of all natural numbers is a subset of the set of all integers. Notation: $\mathbb{N} \subseteq \mathbb{Z}$.

Theorem: *The set of all integers is closed under addition, multiplication, and subtraction, but not under division.*

Definition: A number is **rational** if it can be written as a quotient of two integers.

For example, $\dfrac{3}{8}$ is a rational number because both 3 and 8 are integers and so $\dfrac{3}{8}$ is a quotient of two integers.

Definition: The **set of all rational numbers**, denoted by $\mathbb{Q}$, is the infinite set

$$\mathbb{Q} = \left\{ \frac{a}{b} : \ a \text{ and } b \text{ are integers}, \ b \neq 0 \right\}$$

Notice that the set of all rational numbers entirely contains the set of all integers, i.e. $\mathbb{Z} \subseteq \mathbb{Q}$. In fact, the three sets are such that

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$$

For example, the number -5 is an integer and also a rational number, because we can write it as a quotient $\dfrac{-5}{1}$ where both -5 and 1 are integers. The number zero is also a rational number because it can be written as $\dfrac{0}{3}$.

Theorem: *The set of all rational numbers is closed under addition, multiplication, subtraction, and division.*

The closure under the four basic operations is almost perfect: we cannot divide by zero. The first commandment of mathematics is: *"Thou shall not divide by zero. Ever..."*

The expressions $\frac{0}{3}$ and $\frac{3}{0}$ look very similar, and yet they are very different. How can we remember which is which?

The trick is to know that division is defined in terms of multiplication. Consider the easy division $\frac{10}{2}$.

$$\frac{10}{2} = 5 \quad \text{because the multiplication backwards works, i.e.} \quad 10 = 2 \cdot 5$$

We can apply this to $\frac{0}{3}$ and $\frac{3}{0}$.

$$\frac{0}{3} = \square$$

What could we enter into the empty box so that the multiplication backwards will work? Clearly the answer is zero. Indeed,

$$\frac{0}{3} = 0 \quad \text{because} \quad 3 \cdot 0 = 0$$

Consider now

$$\frac{3}{0} = \square$$

What could we enter into the empty box so that the multiplication backwards will work? No matter what number we would write, zero times it will be zero.

$$3 \neq 0 \cdot \square \text{ no matter what value we write in the box}$$

because $0 \cdot \square = 0$. Thus, it is impossible to enter a number into the box to make the multiplication backwards work. On the other hand, the division $\frac{0}{0}$ is also strange, because now every number would work for the multiplication backwards. These two together suggests that division by zero is not something we can or should do.

Any time we are instructed to divide by zero, we need to write the final answer: undefined.

Example: $\dfrac{3-2(-2)}{3-2^2+1} = \dfrac{3-(-4)}{3-4+1} = \dfrac{7}{-1+1} = \dfrac{7}{0} = \text{undefined}$

So this is why b cannot be zero in the definition of the set of all rational numbers:

$$\mathbb{Q} = \left\{ \frac{a}{b} : \ a \text{ and } b \text{ are integers, } b \neq 0 \right\}$$

Mathemaicians have proved that we cannot expand our number system in any way that would make division by zero meaningful. This is the only flaw of the perfect closure; the set of all rational numbers are closed under all four basic operations, addition, subtraction, multiplication and division.

For some strange reason, mathematicians still needed more kind of numbers.

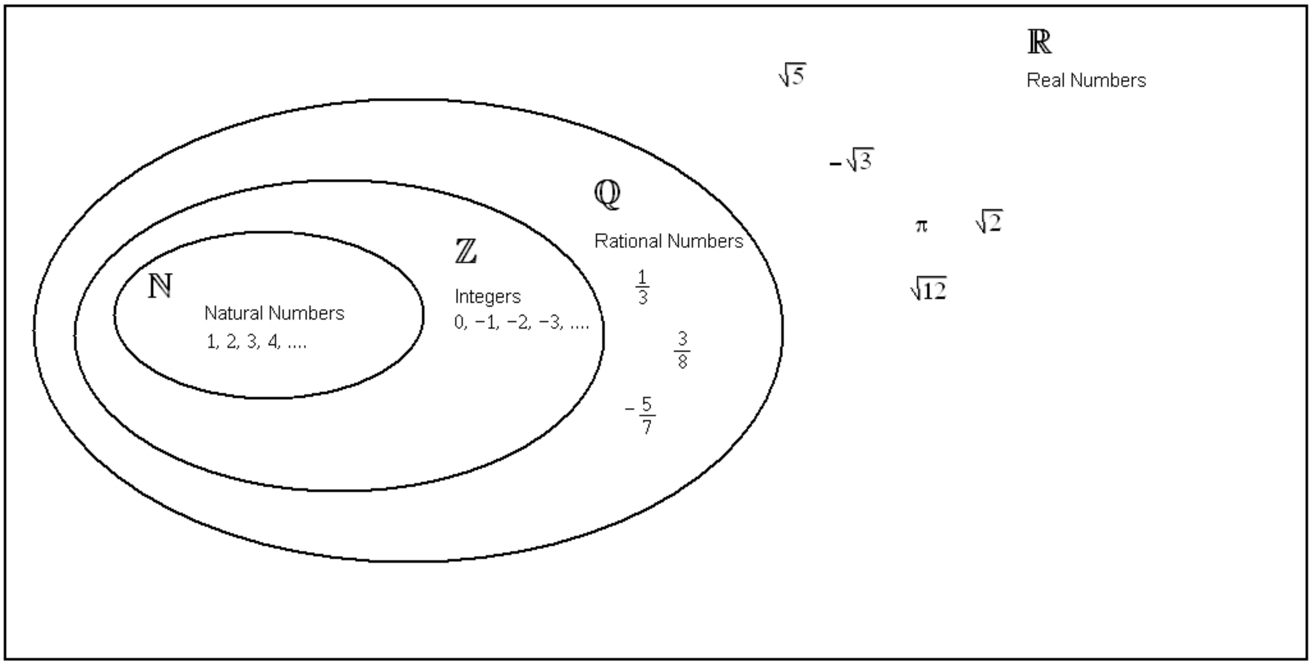
> Definition: A number is **irrational** if it cannot be written as a quotient of two integers.

This is a very strange property because there are so many different integers from which to choose. However, irrational numbers exist. For example, π and $\sqrt{2}$ are irrational numbers. Surprisingly, in a sense, there are many more irrational numbers than rational numbers. (In a fascinating subject within mathematics called set theory, mathematicians have developed language to compare infinite sets. In that comparison, the set of irrational numbers proved to be much greater than the set of rational numbers.)

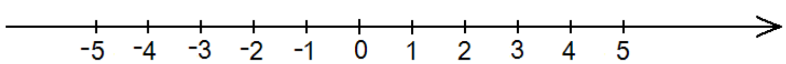
> Definition: The set of all **real numbers**, denoted by $\mathbb{R}$, is the collection of all rational and irrational numbers. The set of all real numbers is also closed under the four basic operations.

The set of all real numbers contain all previous number sets. For example, every rational number is automatically a real number.

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$



Mathematicians proved that there are exactly as many real nubers as many points there are on a straight line. Given a line, for every point on it, we can uniquely assign a real number to that point. Vice versa, for every real number, there is exactly one point on the line. This correspondance is expressed by the concept of the **number line**.



1.3 The Axioms of Real Numbers

The axioms of the real numbers are listed below. Any other statement about the real numbers is a theorem and can be derived from the axioms.

A1. Addition is commutative.
For all $x, y \in \mathbb{R}$, $x + y = y + x$.

A2. Addition is associative.
For all $x, y, z \in \mathbb{R}$, $(x + y) + z = x + (y + z)$.

A3. Additive identity.
There exists a real number d, such that for every real number x, $x + d = x$. (This number is 0).

A4. Additive inverse.
For all $x \in \mathbb{R}$, there exists a real number x^*, such that $x + x^* = 0$. (We denote this number by $-x$).

M1. Multiplication is commutative.
For all $x, y \in \mathbb{R}$, $xy = yx$.

M2. Multiplication is associative.
For all $x, y, z \in \mathbb{R}$, $(xy)z = x(yz)$.

M3. Multiplicative identity.
There exists a real number d, such that for all $x \in \mathbb{R}$, $xd = x$. (This number is 1.)

M4. Multiplicative inverse.
For all $x \in \mathbb{R}$, $x \neq 0$, there exists a real number x^*, such that $xx^* = 1$. (We denote this number by $\frac{1}{x}$).

D1. Distributive Law.
For all $x, y, z \in \mathbb{R}$, $z(x + y) = zx + zy$.

Sets that have these properties are called fields, and are studied in depth in abstract algeba.

The real number system also has the following axioms about ordering.

O1. For all $x, y \in \mathbb{R}$, either $a \leq b$ or $b \leq a$ or both.

O2. If $a \leq b$ and $b \leq a$, then $a = b$.

O3. If $a \leq b$ and $b \leq c$, then $a \leq c$.

O4. If $a \leq b$ then $a + c \leq b + c$.

O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

A set with all these properties is called an ordered field. However, so far these properties hold for both $\mathbb{Q}$ (the set of all rational numbers) and $\mathbb{R}$ (the set of all real numbers.) What distinguishes these two is the completeness property, that is true for $\mathbb{R}$ but not for $\mathbb{Q}$.

Completeness property: Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

Identity and inverse elements

An identity element does 'nothing'. It is a unique elemet of the set that works for every element.

The inverse of an element is another element that when the operation applied, results in the identity. The inverse is different for different numbers.

		Real Numbers with Addition	Real Numbers with Multiplication
	Notation	$\langle \mathbb{R}, + \rangle$	$\langle \mathbb{R}, \cdot \rangle$
Identity	Systematic Name	additive identity	multiplicative identity
	Defining Property	does 'nothing' in addition	does 'nothing' in multiplication
	Value	0	1
Inverse	Systematic Name	additive inverse of a	multiplicative inverse of a
	Non-Systematic Name	opposite of a	reciprocal of a
	Defining Property	$a + (\text{opposite of } a) = 0$ i.e. 'takes' a to the identity	$a \cdot (\text{reciprocal of a}) = 1$ i.e. 'takes' a to the identity
	Notation	$-a$	$\frac{1}{a}$

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