Part I

1. The equation of the line passing through the points $(0,4)$ and $(1,6)$ is

   (a) $y = 2x + 4$

   (b) $y = -\frac{1}{2}x + 4$

   (c) $y = \frac{1}{2}x + 4$

   (d) $y = -2x + 8$

   Solution: We apply the slope formula to find the slope. Labeling the points $(x_1, y_1) = (0,4)$ and $(x_2, y_2) = (1,6)$,

   $$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-4}{1-0} = 2$$

   Thus the slope is 2. Since one of the points given is $(0,4)$, the $y-$intercept, we also immediately have $b = 4$. Thus the equation is

   $$y = 2x + 4$$

   which is choice A.

2. Perform the operation(s) and simplify.

   $$\frac{a^2x^2 - a^2y^2 - 25b^2x^2 + 25b^2y^2}{ax + ay - 5bx - 5by} \div \frac{ax - ay + 5bx - 5by}{3x - y}$$

   (a) $\dfrac{(y-3x)(5b-a)}{a+5b}$

   (b) $\dfrac{(y-3x)(x+y)}{y-x}$

   (c) $5b - a$

   (d) $3x - y$

   Solution: We first factor the first numerator. Since there is no GCF, we proceed to factor by grouping.

   $$\begin{aligned} \underbrace{a^2x^2 - a^2y^2}\underbrace{-25b^2x^2 + 25b^2y^2} &= \\ a^2\left(x^2 - y^2\right) - 25b^2\left(x^2 - y^2\right) &= \left(a^2 - 25b^2\right)\left(x^2 - y^2\right) \end{aligned}$$

   Both factors factor further via the difference of squares theorem.

   $$\begin{aligned} \left(a^2 - 25b^2\right)\left(x^2 - y^2\right) &= \left(a^2 - (5b)^2\right)\left(x^2 - y^2\right) \\ &= (a+5b)(a-5b)(x+y)(x-y) \end{aligned}$$

   We now factor the first denominator by grouping.

   $$\begin{aligned} \underbrace{ax + ay}\underbrace{-5bx - 5by} &= \\ a(x+y) - 5b(x+y) &= (a-5b)(x+y) \end{aligned}$$

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Now the second numerator:

$$\underbrace{ax - ay}\underbrace{+5bx - 5by} = $$
$$a(x-y) + 5b(x-y) = (a+5b)(x-y)$$

We are now ready to simplify the expression. We re-write the division as multiplication by the reciprocal.

$$\begin{aligned}
\frac{a^2x^2 - a^2y^2 - 25b^2x^2 + 25b^2y^2}{ax+ay-5bx-5by} \div \frac{ax-ay+5bx-5by}{3x-y} &= \\
\frac{a^2x^2 - a^2y^2 - 25b^2x^2 + 25b^2y^2}{ax+ay-5bx-5by} \cdot \frac{3x-y}{ax-ay+5bx-5by} &= \\
\frac{(a+5b)(a-5b)(x+y)(x-y)}{(a-5b)(x+y)} \cdot \frac{3x-y}{(a+5b)(x-y)} &= \quad \text{cancellations} \\
&= 3x - y
\end{aligned}$$

which is choice D.

3. If the point (x, y) is the solution of the system

$$\begin{aligned}
5x + 6y &= 15 \\
4x - 3y &= -27
\end{aligned}$$

then x is equal to

(a) 3
(b) 5
(c) -3
(d) -5

Solution: We will solve the system of linear equations by elimination. We can eliminate y if we multiply the second equation by 2.

$$\begin{aligned}
5x + 6y &= 15 \\
8x - 6y &= -54
\end{aligned}$$

We now add the lines:

$$\begin{aligned}
13x &= -39 \quad \text{divide by } 13 \\
x &= -3
\end{aligned}$$

We now solve for y in the first equation.

$$\begin{aligned}
5(-3) + 6y &= 15 \\
-15 + 6y &= 15 \quad \text{add } 15 \\
6y &= 30 \quad \text{divide by } 6 \\
y &= 5
\end{aligned}$$

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We check. If $x=-3$ and $y=5$, then the first equation

$$\text{LHS} = 5(-3)+6(5) = -15+30 = 15 = \text{RHS} \ \checkmark$$

and the second equation

$$\text{LHS} = 8(-3)-6(5) = -24-30 = -54 = \text{RHS} \ \checkmark$$

Thus our solution is correct. The correct answer is C.

4. Simplify $\dfrac{1-\dfrac{1}{x-1}}{1+\dfrac{1}{x-1}}$

(a) $\dfrac{x-2}{x}$

(b) 1

(c) -1

(d) $\dfrac{x^2-x-1}{x^2-x+1}$

Solution: An expression like this is called a complex fraction. Although it contains a variable, we perform the same operations as with regular fractions. We first work out the numerator.

$$1-\frac{1}{x-1} = \frac{1}{1}-\frac{1}{x-1} \quad \text{the common denominator is } x-1$$
$$= \frac{x-1}{x-1}-\frac{1}{x-1}$$
$$= \frac{x-1-1}{x-1} = \frac{x-2}{x-1}$$

We now work out the denominator

$$1+\frac{1}{x-1} = \frac{1}{1}+\frac{1}{x-1} \quad \text{the common denominator is } x-1$$
$$= \frac{x-1}{x-1}+\frac{1}{x-1}$$
$$= \frac{x-1+1}{x-1} = \frac{x}{x-1}$$

We are now ready to perform the division:

$$\frac{1-\dfrac{1}{x-1}}{1+\dfrac{1}{x-1}} = \frac{\dfrac{x-2}{x-1}}{\dfrac{x}{x-1}} = \frac{x-2}{x-1}\cdot\frac{x-1}{x} = \frac{x-2}{x}$$

which is choice A.

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5. Simplify

$$\frac{1}{a-2}-\frac{9-a}{3a+a^2-10}$$

(a) $-\dfrac{4}{a^2+3a-10}$

(b) $\dfrac{2}{a+5}$

(c) $\dfrac{-8-a}{(a-2)(a^2+3a-10)}$

(d) $2(a-2)^2$

Solution: We first factor the second denominator. After we rearrange the terms, we conduct the $pq-$game.

$$3a+a^2-10=a^2+3a-10$$

$$\begin{aligned} pq &= -10 \\ p+q &= 3 \end{aligned}$$

We find 5 and -2 for the values of p and q. Then we factor by grouping

$$\begin{aligned} a^2+3a-10 &= \underbrace{a^2+5a}\underbrace{-2a-10} \\ &= a(a+5)-2(a+5) \\ &= (a-2)(a+5) \end{aligned}$$

$$\frac{1}{a-2}-\frac{9-a}{3a+a^2-10}=\frac{1}{a-2}-\frac{9-a}{(a-2)(a+5)}$$

Now we see that the common denominator is $(a-2)(a+5)$.

$$\frac{1}{a-2}-\frac{9-a}{(a-2)(a+5)}=\frac{a+5}{(a-2)(a+5)}-\frac{9-a}{(a-2)(a+5)}$$

Attention! It is a very frequent mistake to write now $\dfrac{a+5-9-a}{(a-2)(a+5)}$ We have to be careful when writing the subtraction over the common denominator: We are subtracting **the entire** expression $9-a$ and not just its first part!

$\dfrac{a+5}{(a-2)(a+5)}-\dfrac{9-a}{(a-2)(a+5)}$	$=$	re-write with common denominator
$\dfrac{(a+5)-(9-a)}{(a-2)(a+5)}$	$=$	to subtract is to add the opposite
$\dfrac{(a+5)+(-9+a)}{(a-2)(a+5)}$	$=$	drop parentheses, combine like terms
$\dfrac{2a-4}{(a-2)(a+5)}$	$=$	factor top
$\dfrac{2(a-2)}{(a-2)(a+5)}$	$=$	cancel out $a-2$
	$=$ $\dfrac{2}{a+5}$	

which is choice B.

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6. Solve $|2x-3|-7=-2$

(a) -3 and 6
(b) -1 and 4
(c) -4 and 1
(d) No solution

Solution: We first isolate the expression within the absolute value sign.

$$\begin{aligned} |2x-3|-7 &= -2 && \text{add 7} \\ |2x-3| &= 5 \end{aligned}$$

We translate this equation into two linear equations, and solve both for x.

$$\begin{aligned} 2x-3 &= 5 \quad \text{or} \quad 2x-3=-5 && \text{add } 3 \\ 2x &= 8 \quad \text{or} \quad 2x=-2 && \text{divide by } 2 \\ x &= 4 \quad \text{or} \quad x=-1 \end{aligned}$$

We check. If $x=4$, then

$$\text{LHS} = |2\cdot 4-3|-7 = |8-3|-7 = |5|-7 = 5-7 = -2 = \text{RHS} \quad \checkmark$$

If $x=-1$, then

$$\text{LHS} = |2(-1)-3|-7 = |-2-3|-7 = |-5|-7 = 5-7 = -2 = \text{RHS} \quad \checkmark$$

Thus our solution, -1 and 4 is correct. The answer is choice B.

7. Simplify

$$\frac{2abx^2-2ab}{4ax-7a+3ax^2}$$

(a) $\dfrac{2abx}{x-3}$
(b) $\dfrac{2b(x-1)}{3x-7}$
(c) $\dfrac{2b(x+1)}{3x+7}$
(d) $\dfrac{-2ab}{3a+x}$

Solution: We factor both numerator and denominator. The numerator:

$$\begin{aligned} 2abx^2-2ab &= && \text{factor out the GCF} \\ 2ab\left(x^2-1\right) &= \\ 2ab\left(x^2-1^2\right) &= && \text{difference of squares theorem} \\ &= 2ab(x+1)(x-1) \end{aligned}$$

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The denominator:

$$\begin{aligned} 4ax - 7a + 3ax^2 &= && \text{factor out GCF} \\ a\left(4x - 7 + 3x^2\right) &= && \text{rearrange terms} \\ a\left(3x^2 + 4x - 7\right) &= \end{aligned}$$

We will factor by grouping. First we need to take apart the linear term. To find out how, we conduct the $pq-$game.

$$\begin{aligned} pq &= -21 \\ p + q &= 4 \end{aligned}$$

It is easy to see that 7 and -3 will work. We now factor by grouping.

$$\begin{aligned} a\left(3x^2 + 4x - 7\right) &= \\ a\left(\underbrace{3x^2 + 7x}\underbrace{-3x - 7}\right) &= \\ a\left(x\left(3x + 7\right) - 1\left(3x - 7\right)\right) &= a\left(x - 1\right)\left(3x + 7\right) \end{aligned}$$

We are ready to simplify the fraction:

$$\frac{2abx^2 - 2ab}{4ax - 7a + 3ax^2} = \frac{2ab\left(x + 1\right)\left(x - 1\right)}{a\left(x - 1\right)\left(3x + 7\right)} = \frac{2b\left(x + 1\right)}{3x + 7}$$

which is choice C

8. Completely factor

$$8p^3 + 1$$

(a) $(2p - 1)\left(4p^2 + 2p + 1\right)$
(b) $(2p + 1)^3$
(c) $(2p - 1)(2p + 1)(2p - 1)$
(d) $(2p + 1)\left(4p^2 - 2p + 1\right)$

Solution: Since $8p^3 + 1 = (2p)^3 + 1^3$, we simply apply the sum of cubes formula:

$$a^3 + b^3 = (a + b)\left(a^2 + ab + b^2\right)$$

We will substitute $2p$ into a and 1 into b.

$$\begin{aligned} 8p^3 + 1 &= ((2p) + 1)\left((2p)^2 - (2p)1 + 1^2\right) \\ &= (2p + 1)\left(4p^2 - 2p + 1\right) \end{aligned}$$

which is choice D

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9. Solve the formula $\dfrac{3c}{b-2} = \dfrac{2}{a} - \dfrac{c}{b}$ for a.

(a) $\dfrac{2b(b-2)}{c(2b+1)}$

(b) $\dfrac{b(b-2)}{c(2b-1)}$

(c) $\dfrac{2b^2-4b+6c^2}{c(b-2)}$

(d) $\dfrac{b}{c(2b+1)}$

$$
\begin{aligned}
\frac{3c}{b-2} &= \frac{2}{a} - \frac{c}{b} && \text{bring fractions to common denominator} \\
\frac{3abc}{ab(b-2)} &= \frac{2b(b-2)}{ab(b-2)} - \frac{ac(b-2)}{ab(b-2)} && \text{multiply by } ab(b-2) \\
3abc &= 2b(b-2) - ac(b-2) && \text{add } ac(b-2) \\
3abc + ac(b-2) &= 2b(b-2) && \text{distribute } ac \text{ in righ-hand side} \\
3abc + abc - 2ac &= 2b(b-2) && \text{combine like terms } (3abc+abc) \\
4abc - 2ac &= 2b(b-2) && \text{factor out } a \\
a(4bc-2c) &= 2b(b-2) && \text{divide by } 4bc-2c \\
a &= \frac{2b(b-2)}{4bc-2c} && \text{factor out the GCF in denominator} \\
a &= \frac{2b(b-2)}{2c(2b-1)} && \text{cancel } 2 \\
a &= \frac{b(b-2)}{c(2b-1)}
\end{aligned}
$$

which is choice B.

10. Find the domain of the function $f(x) = \dfrac{x-2}{5x^2-14x-3}$

(a) all real numbers, except $x \neq 3$ and $x \neq -\dfrac{1}{5}$

(b) all real numbers

(c) all real numbers, except $x \neq 3$ and $x \neq -\dfrac{1}{5}$ and $x \neq 2$

(d) all real numbers, except $x \neq -3$ and $x \neq \dfrac{1}{5}$

Solution: the domain can be any real number, except those that would make the denominator zero. To find out which numbers need to be ruled out, we solve the quadratic equation $5x^2 - 14x - 3$.

$$5x^2 - 14x - 3 = 0$$

We will factor by grouping. We will take apart the linear term first. To know how, we conduct the $pq-$game.

$$
\begin{aligned}
pq &= -15 \\
p+q &= -14
\end{aligned}
$$

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We easily find -15 and 1 for the values of p and q.

$$\begin{aligned} 5x^2-14x-3 &= 0 \\ \underbrace{5x^2+x}\underbrace{-15x-3} &= 0 \\ x(5x+1)-3(5x+1) &= 0 \\ (x-3)(5x+1) &= 0 \\ x_1 &= 3 \qquad x_2=-\frac{1}{5} \end{aligned}$$

Thus the domain is every real number, except for 3 and $-\frac{1}{5}$, which is choice A.

Part II

1. Simplify each of the following expressions.

(a) $\dfrac{ax^2-18b-9a+2bx^2}{ax-6b-3a+2bx}=$

Solution: We factor both numerator and denominator by grouping, and then simplify. The numerator:

$$\begin{aligned} ax^2-18b-9a+2bx^2 &= \qquad \text{rearrange} \\ \underbrace{ax^2+2bx^2}\underbrace{-9a-18b} &= \\ x^2(a+2b)-9(a+2b) &= \\ \left(x^2-9\right)(a+2b) &= \\ \left(x^2-3^2\right)(a+2b) &= \qquad \text{difference of squares theorem} \\ &= (x+3)(x-3)(a+2b) \end{aligned}$$

The denominator:

$$\begin{aligned} ax-6b-3a+2bx &= \qquad \text{rearrange} \\ \underbrace{ax+2bx}\underbrace{-3a-6b} &= \\ x(a+2b)-3(a+2b) &= (x-3)(a+2b) \end{aligned}$$

We are now ready to simplify the fraction.

$$\frac{ax^2-18b-9a+2bx^2}{ax-6b-3a+2bx}=\frac{(x+3)(x-3)(a+2b)}{(x-3)(a+2b)}=x+3$$

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(b) $\dfrac{2-\dfrac{3}{x+1}}{3-\dfrac{2x}{x+1}} =$

Solution: We first work out the numerator

$$\begin{aligned} 2-\frac{3}{x+1} &= \frac{2}{1}-\frac{3}{x+1} \qquad \text{common denominator is } x+1 \\ &= \frac{2(x+1)}{x+1}-\frac{3}{x+1} \\ &= \frac{2(x+1)-3}{x+1} = \frac{2x+2-3}{x+1} \\ &= \frac{2x-1}{x+1} \end{aligned}$$

Now for the denominator

$$\begin{aligned} 3-\frac{2x}{x+1} &= \frac{3}{1}-\frac{2x}{x+1} \qquad \text{common denominator is } x+1 \\ &= \frac{3(x+1)}{x+1}-\frac{2x}{x+1} = \\ &= \frac{3(x+1)-2x}{x+1} = \frac{3x+3-2x}{x+1} \\ &= \frac{x+3}{x+1} \end{aligned}$$

Now the division:

$$\begin{aligned} \frac{2-\dfrac{3}{x+1}}{3-\dfrac{2x}{x+1}} &= \frac{\dfrac{2x-1}{x+1}}{\dfrac{x+3}{x+1}} = \qquad \text{to divide is to multiply by the reciprocal} \\ &= \frac{2x-1}{x+1}\cdot\frac{x+1}{x+3} = \frac{2x-1}{x+3} \end{aligned}$$

(c) $\dfrac{x}{x-5}-\dfrac{x-2}{5-x} =$

Solution: The trick is to realize that $x-5$ and $5-x$ are opposites, since

$$-1(5-x) = -5+x = x-5$$

We will transform the second fraction by multiplying numerator and denominator by -1.

$$\begin{aligned} \frac{x}{x-5}-\frac{x-2}{5-x} &= \frac{x}{x-5}-\frac{-1(x-2)}{-1(5-x)} = \frac{x}{x-5}-\frac{-x+2}{x-5} \\ &= \frac{x-(-x+2)}{x-5} = \frac{x+(x-2)}{x-5} = \\ &= \frac{2x-2}{x-5} = \frac{2(x-1)}{x-5} \end{aligned}$$

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(d) $\dfrac{64x^3-1}{19x+4x^2-5} \div \dfrac{4x+16x^2+1}{x^2-6x-55} =$
Solution: We first factor whatever we can. The first numerator factors via the difference of cubes theorem.

$$\begin{aligned} 64x^3-1 &= (4x)^3-1^3 \\ &= (4x-1)\left((4x)^2+(4x)1+1^2\right) \\ &= (4x-1)\left(16x^2+4x+1\right) \end{aligned}$$

We will factor the first numerator by grouping. After we rearrange the terms, we have $4x^2+19x-5$, thus

$$\begin{aligned} pq &= -20 \\ p+q &= 19 \end{aligned}$$

Clearly p and q are 20 and -1. Then

$$\begin{aligned} \underbrace{4x^2+20x}\underbrace{-x-5} &= \\ 4x(x+5)-1(x+5) &= (4x-1)(x+5) \end{aligned}$$

The second numerator does not factor, but it will be cancelled out by the longer factor of the difference of cubes. Finally, we facotr the second denominator by grouping. Clearly p and q are -11 and 5.

$$\begin{aligned} x^2-6x-55 &= \\ \underbrace{x^2+5x}\underbrace{-11x-55} &= \\ x(x+5)-11(x+5) &= (x-11)(x+5) \end{aligned}$$

Now we are ready the perform the operation indicated.

$$\begin{aligned} \frac{64x^3-1}{19x+4x^2-5} \div \frac{4x+16x^2+1}{x^2-6x-55} &= \frac{64x^3-1}{4x^2+19x-5} \cdot \frac{x^2-6x-55}{16x^2+4x+1} \\ &= \frac{(4x-1)\left(16x^2+4x+1\right)}{(4x-1)(x+5)} \cdot \frac{(x-11)(x+5)}{16x^2+4x+1} \\ &= \frac{(4x-1)\left(16x^2+4x+1\right)(x+5)(x-11)}{(4x-1)\left(16x^2+4x+1\right)(x+5)} = x-11 \end{aligned}$$

(e) $\dfrac{10b+10}{2b+b^2-3} - \dfrac{2-b}{b+3} =$
Solution: We factor the first denominator first.

$$b^2+2b-3 = (b+3)(b-1)$$

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The common denominator is $(b+3)(b-1)$

$$\begin{aligned}\frac{10b+10}{2b+b^2-3}-\frac{2-b}{b+3} &= \frac{10b+10}{(b+3)(b-1)}-\frac{2-b}{b+3}\\ &= \frac{10b+10}{(b+3)(b-1)}-\frac{(2-b)(b-1)}{(b+3)(b-1)}=\frac{10b+10-(2-b)(b-1)}{(b+3)(b-1)}\\ &= \frac{10b+10-\left(2b-2-b^2+b\right)}{(b+3)(b-1)}=\frac{10b+10-\left(-b^2+3b-2\right)}{(b+3)(b-1)}\\ &= \frac{10b+10+\left(b^2-3b+2\right)}{(b+3)(b-1)}=\frac{b^2+7b+12}{(b+3)(b-1)}\\ &= \frac{b^2+7b+12}{(b+3)(b-1)}=\frac{(b+3)(b+4)}{(b+3)(b-1)}=\frac{b+4}{b-1}\end{aligned}$$

2. Completely factor each of the following.

(a) $ax^2-4ay^2+2bx^2-8by^2=$

Solution: Since there is no GCF, we proceed to factor by grouping.

$$\begin{aligned}\underbrace{ax^2-4ay^2}\underbrace{+2bx^2-8by^2} &=\\ a\left(x^2-4y^2\right)+2b\left(x^2-4y^2\right) &=\\ (a+2b)\left(x^2-4y^2\right) &=\\ (a+2b)\left(x^2-(2y)^2\right) &= (a+2b)(x+2y)(x-2y)\end{aligned}$$

(b) $30a^3b-6ab^3-3a^2b^2=$

Solution: We start with the GCF

$$\begin{aligned}30a^3b-6ab^3-3a^2b^2 &= 3ab\left(10a^2-2b^2-ab\right)\\ &= 3ab\left(10a^2-ab-2b^2\right)\end{aligned}$$

This is a trinomial similar to $10x^2-x-2$. a will play the role of x, and b will play the role of numbers. Thus, factoring by grouping is appropriate.

$$\begin{aligned}pq &= -20\\ p+q &= -1\end{aligned}$$

We easily find -5 and 4 for p and q.

$$\begin{aligned}3ab\left(10a^2-ab-2b^2\right) &=\\ 3ab\left(\underbrace{10a^2+4ab}\underbrace{-5ab-2b^2}\right) &=\\ 3ab\left(2a(5a+2b)-b(5a-2b)\right) &= 3ab(2a-b)(5a+2b)\end{aligned}$$

(c) $16q^3t^2-2t^2=$

Solution: We start with the GCF

$$\begin{aligned}16q^3t^2-2t^2 &= 2t^2\left(8q^3-1\right)\\ &= 2t^2\left((2q)^3-1^3\right) \qquad \text{difference of cubes}\\ &= 2t^2(2q-1)\left((2q)^2+(2q)1+1^2\right)\\ &= 2t^2(2q-1)\left(4q^2+2q+1\right)\end{aligned}$$

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3. Solve each of the following equations. Make sure to check your solution.

(a) $\dfrac{2x-4}{7}-\dfrac{3-x}{2}=2x+4$

Solution:

$$\begin{aligned}
\frac{2x-4}{7}-\frac{3-x}{2} &= 2x+4 && \text{make everything a fraction}\\
\frac{2x-4}{7}-\frac{3-x}{2} &= \frac{2x+4}{1} && \text{common denominator}\\
\frac{2(2x-4)}{14}-\frac{7(3-x)}{14} &= \frac{14(2x+4)}{14} && \text{multiply by 14}\\
2(2x-4)-7(3-x) &= 14(2x+4) && \text{distribute}\\
4x-8-21+7x &= 28x+56 && \text{combine like terms}\\
11x-29 &= 28x+56 && \text{subtract } 11x\\
-29 &= 17x+56 && \text{subtract 56}\\
-85 &= 17x && \text{divide by 17}\\
-5 &= x
\end{aligned}$$

We check: if $x=-5$, then

$$\begin{aligned}
\text{LHS} &= \frac{2(-5)-4}{7}-\frac{3-(-5)}{2}=\frac{-10-4}{7}-\frac{3+5}{2}=\frac{-14}{7}-\frac{8}{2}=-2-4=-6\\
\text{RHS} &= 2(-5)+4=-10+4=-6
\end{aligned}$$

Thus our solution, $x=-5$ is correct.

(b) $\dfrac{8}{x-3}-\dfrac{10}{x+3}=1$

Solution:

$$\begin{aligned}
\frac{8}{x-3}-\frac{10}{x+3} &= 1 && \text{make everything a fraction}\\
\frac{8}{x-3}-\frac{10}{x+3} &= \frac{1}{1} && \text{common denominator is } (x+3)(x-3)\\
\frac{8(x+3)}{(x-3)(x+3)}-\frac{10(x-3)}{(x+3)(x-3)} &= \frac{(x+3)(x-3)}{(x+3)(x-3)} && \text{multiply by } (x+3)(x-3)\\
8(x+3)-10(x-3) &= (x+3)(x-3) && \text{distribute}\\
8x+24-10x+30 &= x^2-9 && \text{combine like terms}\\
-2x+54 &= x^2-9 && \text{add } 2x\\
54 &= x^2+2x-9 && \text{subtract 54}\\
0 &= x^2+2x-63 && \text{factor}\\
0 &= (x+9)(x-7)
\end{aligned}$$

$$x_1=-9 \text{ and } x_2=7$$

We check: if $x=-9$, then

$$\text{LHS}=\frac{8}{-9-3}-\frac{10}{-9+3}=\frac{8}{-12}-\frac{10}{-6}=\frac{-2}{3}+\frac{5}{3}=\frac{3}{3}=1=\text{RHS} \quad \checkmark$$

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If $x=7$, then

$$\text{LHS}=\frac{8}{7-3}-\frac{10}{7+3}=\frac{8}{4}-\frac{10}{10}=2-1=1=\text{RHS} \quad \checkmark$$

Thus our solutions $-9,7$ are correct.

(c) $1-\dfrac{1}{x-3}=\dfrac{x-4}{x+2}$

Solution:

$$\begin{aligned}
1-\frac{1}{x-3} &= \frac{x-4}{x+2} && \text{make everything a fraction}\\
\frac{1}{1}-\frac{1}{x-3} &= \frac{x-4}{x+2} && \text{common denominator is } (x-3)(x+2)\\
\frac{(x-3)(x+2)}{(x-3)(x+2)}-\frac{x+2}{(x-3)(x+2)} &= \frac{(x-4)(x-3)}{(x-3)(x+2)} && \text{multiply by } (x-3)(x+2)\\
(x-3)(x+2)-(x+2) &= (x-4)(x-3) && \text{distribute}\\
x^2+2x-3x-6-x-2 &= x^2-3x-4x+12 && \text{combine like terms}\\
x^2-2x-8 &= x^2-7x+12 && \text{subtract } x^2\\
-2x-8 &= -7x+12 && \text{add } 7x\\
5x-8 &= 12 && \text{add } 8\\
5x &= 20\\
x &= 4
\end{aligned}$$

We check: if $x=4$, then

$$\begin{aligned}
\text{LHS} &= 1-\frac{1}{4-3}=1-\frac{1}{1}=1-1=0\\
\text{RHS} &= \frac{4-4}{4+2}=\frac{0}{6}=0
\end{aligned}$$

Thus our solution, 4 is correct.

(d) $15x^2+6x^3=9x$

Solution:

$$\begin{aligned}
15x^2+6x^3 &= 9x && \text{subtract } 9x \text{ and rearrange}\\
6x^3+15x^2-9x &= 0 && \text{factor out GCF}\\
3x\left(2x^2+5x-3\right) &= 0
\end{aligned}$$

We will factor by grouping. The $pq-$game gives us

$$\begin{aligned}
pq &= -6\\
p+q &= 5
\end{aligned}$$

$p=6$ and $q=-1$. We now factor by grouping:

$$\begin{aligned}
3x\left(2x^2+5x-3\right) &= 0\\
3x\left(\underbrace{2x^2+6x}\underbrace{-x-3}\right) &= 0\\
3x\left(2x\left(x+3\right)-1\left(x+3\right)\right) &= 0\\
3x\left(2x-1\right)\left(x+3\right) &= 0
\end{aligned}$$

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$$x_1 = 0 \quad \text{and} \quad x_2 = \frac{1}{2} \quad \text{and} \quad x_3 = -3$$

We check: if $x = 0$, then

$$\begin{aligned} \text{LHS} &= 15(0)^2 + 6(0)^3 = 0 \\ \text{RHS} &= 9(0) = 0 \end{aligned}$$

if $x = \frac{1}{2}$, then

$$\begin{aligned} \text{LHS} &= 15\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right)^3 = 15\left(\frac{1}{4}\right) + 6\left(\frac{1}{8}\right) = \frac{15}{4} + \frac{3}{4} = \frac{18}{4} = \frac{9}{2} \\ \text{RHS} &= 9\left(\frac{1}{2}\right) = \frac{9}{2} \quad \checkmark \end{aligned}$$

if $x = -3$, then

$$\begin{aligned} \text{LHS} &= 15(-3)^2 + 6(-3)^3 = 15(9) + 6(-27) = 135 - 162 = -27 \\ \text{RHS} &= 9(-3) = -27 \quad \checkmark \end{aligned}$$

Thus all three values of x, $\frac{1}{2}, 0, -3$ are correct.

(e) $\left|\frac{3}{4}x - 2\right| - 1 = 7$

Solution: We first isolate the expression within the absolute value sign.

$$\begin{aligned} \left|\frac{3}{4}x - 2\right| - 1 &= 7 \qquad \text{add } 1 \\ \left|\frac{3}{4}x - 2\right| &= 8 \end{aligned}$$

We now translate the absolute value equation into a pair of linear equation, and solve each for x.

$$\begin{aligned} \frac{3}{4}x - 2 &= 8 \quad \text{or} \quad \frac{3}{4}x - 2 = -8 \qquad \text{add } 2 \\ \frac{3}{4}x &= 10 \quad \text{or} \quad \frac{3}{4}x = -6 \qquad \text{divide by } \frac{3}{4} \\ x &= \frac{40}{3} \quad \text{or} \quad x = -8 \end{aligned}$$

We check: if $x = \frac{40}{3}$, then

$$\text{LHS} = \left|\frac{3}{4} \cdot \frac{40}{3} - 2\right| - 1 = |10 - 2| - 1 = 8 - 1 = 7 = \text{RHS} \quad \checkmark$$

and if $x = -8$, then

$$\text{LHS} = \left|\frac{3}{4}(-8) - 2\right| - 1 = |-6 - 2| - 1 = |-8| - 1 = 8 - 1 = 7 = \text{RHS} \quad \checkmark$$

Thus both values, -8 and $\frac{40}{3}$ are correct.

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(f) $|2x-1|-5=30$
Solution: We first isolate the expression within the absolute value sign.

$$\begin{aligned} |2x-1|-5 &= 30 \qquad \text{add } 5 \\ |2x-1| &= 35 \end{aligned}$$

We now translate the absolute value equation into a pair of linear equation, and solve each for x.

$$\begin{aligned} 2x-1 &= 35 \quad \text{or} \quad 2x-1=-35 \qquad \text{add } 1 \\ 2x &= 36 \quad \text{or} \quad 2x=-34 \qquad \text{divide by } 2 \\ x &= 18 \quad \text{or} \quad x=-17 \end{aligned}$$

We check: if $x=18$, then

$$\text{LHS} = |2(18)-1|-5 = |36-1|-5 = |35|-5 = 35-5 = 30 = \text{RHS} \quad \checkmark$$

and if $x=-17$, then

$$\text{LHS} = |2(-17)-1|-5 = |-34-1|-5 = |-35|-5 = 35-5 = 30 = \text{RHS} \quad \checkmark$$

Thus both values, -17 and 18 are correct.

(g) $|2x-1|-5=-30$
Solution: We first isolate the expression within the absolute value sign.

$$\begin{aligned} |2x-1|-5 &= -30 \qquad \text{add } 5 \\ |2x-1| &= -25 \end{aligned}$$

There is no number that has a negative absolute value, this equation has no solution.

4. Solve each of the following formulas.

(a) $y=\dfrac{x-2}{3x+4}$ for x.
Solution:

$$\begin{aligned} y &= \frac{x-2}{3x+4} && \text{multiply by } 3x+4 \\ y(3x+4) &= x-2 && \text{distribute} \\ 3xy+4y &= x-2 && \text{subtract } 3xy \\ 4y &= -3xy+x-2 && \text{add } 2 \\ 4y+2 &= -3xy+x && \text{factor out } x \\ 4y+2 &= x(-3y+1) && \text{divide by } -3y+1 \\ \frac{4y+2}{-3y+1} &= x \end{aligned}$$

The answer is $x=\dfrac{4y+2}{-3y+1}$.

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(b) $\dfrac{1}{a}=\dfrac{1}{b-1}+\dfrac{1}{b+1}$ for a.
Solution:

$$\begin{aligned}
\frac{1}{a} &= \frac{1}{b-1}+\frac{1}{b+1} && \text{common denominator is } a(b-1)(b+1)\\
\frac{(b-1)(b+1)}{a(b-1)(b+1)} &= \frac{a(b+1)}{a(b-1)(b+1)}+\frac{a(b-1)}{a(b-1)(b+1)} && \text{multiply by } a(b-1)(b+1)\\
(b-1)(b+1) &= a(b+1)+a(b-1) && \text{distribute}\\
b^2-1 &= ab+a+ab-a && \text{combine like terms}\\
b^2-1 &= 2ab && \text{divide by } 2b\\
\frac{b^2-1}{2b} &= a
\end{aligned}$$

The answer is $a=\dfrac{b^2-1}{2b}$.

(c) $C=\dfrac{5F-160}{9}$ for F.
Solution:

$$\begin{aligned}
C &= \frac{5F-160}{9} && \text{multiply by } 9\\
9C &= 5F-160 && \text{add } 160\\
9C+160 &= 5F && \text{divide by } 5\\
\frac{9C+160}{5} &= F \quad \text{or} \quad F=\frac{9}{5}C+\frac{160}{5}=\frac{9}{5}C+32
\end{aligned}$$

The answer is $F=\dfrac{9C+160}{5}=\dfrac{9}{5}C+32$.

5. Solve each of the following inequalities

(a) $-8\le 2x+12\le 14$
Solution:

$$\begin{aligned}
-8 &\le 2x+12\le 14 && \text{subtract } 12\\
-20 &\le 2x\le 2 && \text{divide by } 2\\
-10 &\le x\le 1
\end{aligned}$$

The answer is $-10\le x\le 1$ or, in interval notation, $[-10,1]$.

(b) $-8\le -2x+12<14$
Solution: When multiplying or dividing an inequality by a negative number, the inequality sign must be reversed.

$$\begin{aligned}
-8 &\le -2x+12<14 && \text{subtract } 12\\
-20 &\le -2x<2 && \text{divide by } -2\\
10 &\ge x>-1
\end{aligned}$$

The answer is $-1<x\le 10$, or in interval notation: $(-1,10]$

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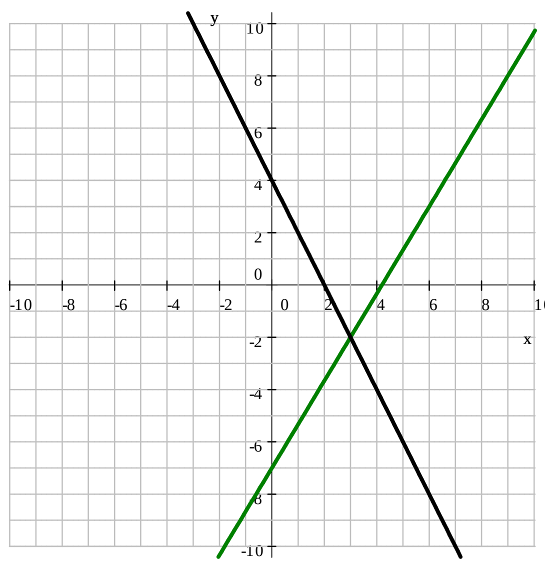
6. Graph the straight lines $\mathbf{5}x - 3y = 21$ and $2x + y = 4$ in the same coordinate system.

Solution: We start with $5x - 3y = 21$. To obtain the slope-intercept form, we solve for y

$$\begin{aligned} 5x - 3y &= 21 && \text{add } 3y \\ 5x &= 3y + 21 && \text{subtract } 21 \\ 5x - 21 &= 3y && \text{divide by } 3 \\ \frac{5x-21}{3} &= y \\ y &= \frac{5x}{3} - \frac{21}{3} \\ y &= \frac{5}{3}x - 7 \end{aligned}$$

We can easily graph the line now. The $y-$intercept is $(0, -7)$. Since the slope is $\frac{5}{3}$, from this point, we step 3 to the right, 5 up. (Green line)

We now graph the line $2x + y = 4$. We solve for y : $y = -2x + 4$. The $y-$intercept is $(0, 4)$. Since the slope is 2, from the $y-$intercept we step 1 to the right, 2 up.



(a) Use your graph to find the coordinates of the point where the lines intersect. $(3, -2)$

(b) Use algebraic methods to check your solution.

The point $(3, -2)$ is on the line $\mathbf{5}x - 3y = 21$ since when substituting its coordinates, the equal sign is true.

$$\text{LHS} = 5(3) - 3(-2) = 15 + 6 = 21 = \text{RHS}$$

The point $(3, -2)$ is on the line $2x + y = 4$ since when substituting its coordinates, the equal sign is true.

$$\text{LHS} = 2(3) + (-2) = 6 - 2 = 4 = \text{RHS}$$

Since it is on both line, this point must be the point of intersection.

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7. Solve the system of linear equations. Make sure to check your solution.

$$\begin{aligned} 3x + 5y &= 26 \\ 2x + 3y &= 15 \end{aligned}$$

Solution: We will use elimination. If we multiply the first equation by 2 and the second equation by -3, we will eliminate x when adding the equations.

$$\begin{aligned} -6x - 10y &= -52 \\ 6x + 9y &= 45 \end{aligned}$$

We add the two equations:

$$\begin{aligned} -y &= -7 \quad \text{multiply by } -1 \\ y &= 7 \end{aligned}$$

We will use the second equation to find the value of x.

$$\begin{aligned} 2x + 3(7) &= 15 \\ 2x + 21 &= 15 \quad \text{subtract 15} \\ 2x &= -6 \quad \text{divide by 2} \\ x &= -3 \end{aligned}$$

Thus $x = -3$ and $y = 7$. We check:

$$\begin{aligned} \text{LHS} &= 3(-3) + 5(7) = -9 + 35 = 26 = \text{RHS} \checkmark \\ \text{LHS} &= 2(-3) + 3(7) = -6 + 21 = 15 = \text{RHS} \checkmark \end{aligned}$$

Thus our solution, $(-3, 7)$ is correct.

8. Word problems.

(a) We have invested \$ 4000 into two bank accounts. One account earns 7% interest per year, the other account earns 5% interest per year. How much did we invest at 5% if the combined interest for the accounts was \$ 244?

Solution: Let x denote the amount we invested at 7% and y denote the amount we invested at 5%. Then

	amount in bank	interest earned
first account	x	7% of $x = 0.07x$
second account	y	5% of $y = 0.05y$

The equations we set up then

$$\begin{aligned} x + y &= 4000 \quad \text{amount in bank accounts} \\ 0.07x + 0.05y &= 244 \quad \text{combined interest} \end{aligned}$$

We can get rid of the decimals if we multiply the second equation by 100

$$\begin{aligned} x + y &= 4000 \\ 7x + 5y &= 24\,400 \end{aligned}$$

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We will use elimination. We will multiply the first equation by -5 to eliminate y when adding the two equations.

$$\begin{aligned} -5x - 5y &= -20\,000 \\ 7x + 5y &= 24\,400 \end{aligned}$$

We add the equations

$$\begin{aligned} 2x &= 4\,400 && \text{divide by 2} \\ x &= 2\,200 \end{aligned}$$

We use the first equation to find y.

$$\begin{aligned} 2\,200 + y &= 4000 && \text{subtract } 2\,200 \\ y &= 1\,800 \end{aligned}$$

Thus we invested \$ 2 200 at 7% and \$ 1 800 at 5%. We check:

$$\begin{aligned} 2200 + 1800 &= 4000 && \text{amount works } \checkmark \\ 0.07\,(2200) + 0.05\,(1800) &= 154 + 90 = 244 && \text{interest works } \checkmark \end{aligned}$$

Thus we invested \$ 1800 at 5%.

(b) One number is one less than twice the other. The product of these numbers is 120. Find these numbers.

Solution: Let x denote the smaller number. Then the larger number is denoted by $2x - 1$. The equation will express the product. Since the equation is quadratic, we will reduce one side to zero, factor the other side, and then apply the zero property.

$$\begin{aligned} x\,(2x-1) &= 120 && \text{distribute } x \\ 2x^2 - x &= 120 && \text{subtract } 120 \\ 2x^2 - x - 120 &= 0 \end{aligned}$$

We factor by grouping. We conduct the $pq-$game.

$$\begin{aligned} pq &= -240 \\ p + q &= -1 \end{aligned}$$

We find -16 and 15. We factor by grouping.

$$\begin{aligned} 2x^2 - x - 120 &= 0 \\ \underbrace{2x^2 - 16x}\underbrace{+15x - 120} &= 0 \\ 2x\,(x-8) + 15\,(x-8) &= 0 \\ (2x+15)\,(x-8) &= 0 \end{aligned}$$

$$x_1 = -\frac{15}{2} \quad \text{and} \quad x_2 = 8$$

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If $x=-\frac{15}{2}$, then the other number is $2\left(-\frac{15}{2}\right)-1=-16$, thus the two numbers are $-\frac{15}{2}$ and -16. These clearly work since

$$\begin{aligned} -16 &= 2\left(\frac{-15}{2}\right)-1 \quad \checkmark \quad \text{and} \\ -16\left(-\frac{15}{2}\right) &= 120 \quad \checkmark \end{aligned}$$

If $x=8$, then the other number is $2(8)-1=15$, thus the two numbers are 8 and 15. These work since

$$\begin{aligned} 15 &= 2(8)-1 \quad \checkmark \quad \text{and} \\ 8(15) &= 120 \quad \checkmark \end{aligned}$$

Thus there are two solutions, $-\frac{15}{2}, -16$ and $8, 15$.

(c) One side of a rectangle is 3 in longer than twice the other side. The numerical value of the area is 29 larger than the numerical value of the perimeter. Find the sides.
Solution: Let x denote the shorter side. Then the longer side is $2x+3$. The equation is then

$$\begin{aligned} A &= P+29 \\ x(2x+3) &= 2x+2(2x+3)+29 \\ 2x^2+3x &= 2x+4x+6+29 \\ 2x^2+3x &= 6x+35 && \text{subtract } 6x \\ 2x^2-3x &= 35 && \text{subtract } 35 \\ 2x^2-3x-35 &= 0 \end{aligned}$$

We will factor by grouping. For p and q,

$$\begin{aligned} pq &= -70 \\ p+q &= -3 \end{aligned}$$

we find -10 and 7.

$$\begin{aligned} 2x^2-3x-35 &= 0 \\ \underbrace{2x^2+7x}\underbrace{-10x-35} &= 0 \\ x(2x+7)-5(2x+7) &= 0 \\ (x-5)(2x+7) &= 0 \end{aligned}$$

$$x_1=5 \quad \text{and} \quad x_2=-\frac{7}{2}$$

The negative solution is ruled out since it can not represent a distance. If $x=5$, then the longer side is $2(5)+3=13$. Thus the sides of the rectangle are 5 in and 13 in long. We check:

$$\begin{aligned} 13 \text{ in} &= 2(5 \text{ in})+3 \text{ in} \quad \checkmark \\ A &= 5 \text{ in}(13 \text{ in}) = 65 \text{ in}^2 \\ P &= 2(5 \text{ in})+2(13 \text{ in}) = 36 \text{ in} \quad \text{and} \\ A-P &= 65-36=29 \quad \checkmark \end{aligned}$$

Thus our solution, 5 in by 13 in, is correct.

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(d) We have 42 coins, all dimes and quarters. The total value of these coins is \$ 7.65. How many quarters, how many dimes?

Solution: Let x denote the number of dimes, and y denote the number of quarters. Then the equations we can write are

$$\begin{aligned} x+y &= 42 && \text{the total number of coins} \\ 10x+25y &= 765 && \text{the value of coins, expressed in cents} \end{aligned}$$

We will solve this system by elimination. Notice first that the second line can be simplifies by division by 5. Then we will eliminate x if we multiply the first equation by -2.

$$\begin{aligned} -2x-2y &= -84 \\ 2x+5y &= 153 \end{aligned}$$

We now add the equations

$$\begin{aligned} 3y &= 69 && \text{divide by 3} \\ y &= 23 \end{aligned}$$

We use the first equation to find x.

$$\begin{aligned} x+23 &= 42 && \text{subtract } 23 \\ x &= 19 \end{aligned}$$

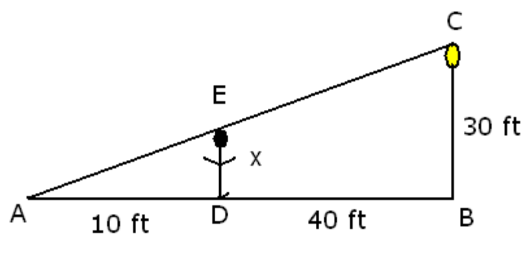
Thus we have 19 dimes and 23 quarters. We check:

$$\begin{aligned} 19+23 &= 42 && \text{we have 42 coins } \checkmark \\ 19\,(0.10)+23\,(0.25) &= 1.90+5.75=765 && \text{in the value of \$ 7.65 } \checkmark \end{aligned}$$

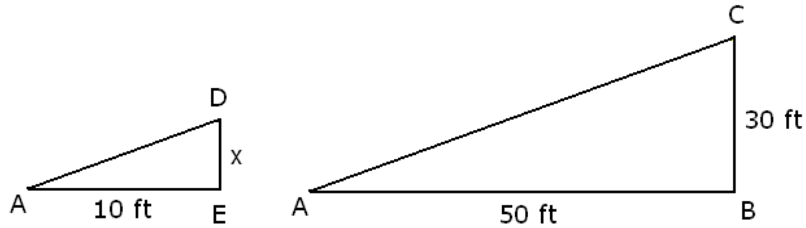
Thus our solution. 19 dimes and 23 quarters are correct.

(e) A person is standing 40 ft away from a street light tha-t is 30 ft tall. How tall is he if his shadow is 10 ft long?

Solution: We draw the following picture.



The triangles ABC and ADE are similar, because two of their angles are equal. The angles ADE and ABC are both right angles, and angle EAD is shared by both triangles. We draw a new picture now:



We now ready to set up and solve a ratio problem to find the value of x.

	horizontal	vertical
small triangle	10	x
large triangle	50	30

The equation:

$$\begin{aligned} 50x &= 10 \cdot 30 \qquad \text{divide by } 50 \\ x &= \frac{300}{50} = 6 \end{aligned}$$

Thus the person is 6 feet tall.

(f) Steve took four exams. His scores on the first three exams were 83, 71, and 93. What was the score given for his fourth exam if the average score of the four exams was 84?

Solution: Let x denote the score earned on the fourth exam. The equation will express the average of the four exam scores.

$$\begin{aligned} \frac{83+71+93+x}{4} &= 84 \\ \frac{x+247}{4} &= 84 \qquad \text{multiply by } 4 \\ x+247 &= 336 \qquad \text{subtract } 247 \\ x &= 89 \end{aligned}$$

Thus the fourth exam score was 89. We check: $\frac{83+71+93+89}{4} = 84$, thus our solution, 89 is correct.

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