# Part 1

1. Simplify $5-2(4b-5(b-3))$.

   (a) $2b+35$
   (b) $35-18b$
   (c) $2b-25$
   (d) $35-2b$
   Solution:

$$\begin{aligned} 5-2(4b-5(b-3)) &= && \text{distribute} \\ 5-2(4b-5b+15) &= && \text{combine like terms} \\ 5-2(-b+15) &= && \text{distribute} \\ 5+2b-30 &= && \text{combine like terms} \\ &= 2b-25 \end{aligned}$$

   which is C).

2. Simplify the expression $\left(\sqrt{x}-\sqrt{2}\right)^2$

   (a) $x-2\sqrt{2x}+2$
   (b) $x-2$
   (c) $x-2\sqrt{x}+2\sqrt{2}-\sqrt{x}\sqrt{2}$
   (d) $x-4\sqrt{x}+4$
   Solution:

$$\begin{aligned} \left(\sqrt{x}-\sqrt{2}\right)^2 &= \left(\sqrt{x}-\sqrt{2}\right)\left(\sqrt{x}-\sqrt{2}\right) \qquad \text{we FOIL} \\ &= \sqrt{x}\sqrt{x}-\sqrt{x}\sqrt{2}-\sqrt{2}\sqrt{x}+\sqrt{2}\sqrt{2} \\ &= x-2\sqrt{2x}+2 \end{aligned}$$

   which is A).

3. Solve the equation $x^2+29=4x$ over the complex numbers.

   (a) There is no solution
   (b) $2-\sqrt{33}$ and $2+\sqrt{33}$
   (c) $2-5i$ and $2+5i$
   (d) $-25$
   Solution:

$$\begin{aligned} x^2+29 &= 4x && \text{subtract } 4x \\ x^2-4x+29 &= 0 && (x-2)^2 = x^2-4x+4 \\ \underbrace{x^2-4x+4}-4+29 &= 0 && \text{complete the square} \\ (x-2)^2+25 &= 0 \\ (x-2)^2-(-25) &= 0 \\ (x-2)^2-(5i)^2 &= 0 && \text{factor} \\ (x-2+5i)(x-2-5i) &= 0 \end{aligned}$$

$$x_1 = 2 - 5i \qquad \text{and} \qquad x_2 = 2 + 5i$$

which is C).

4. Perform the indicated operations and simplify. $\dfrac{x^2 - 9}{x^2 + 7x + 12} \div \dfrac{x - 3}{x + 5} =$

(a) $\dfrac{x + 5}{x + 4}$

(b) $\dfrac{x^2 - 6x + 9}{9x + x^2 + 20}$

(c) $\dfrac{x - 3}{9x + x^2 + 20}$

(d) $\dfrac{x + 5}{x - 4}$

Solution: We factor everything we can and re-write the division as multiplication by the reciprocal. Finally, we cancel.

$$\frac{x^2 - 9}{x^2 + 7x + 12} \div \frac{x - 3}{x + 5} = \frac{(x - 3)(x + 3)}{(x + 4)(x + 3)} \cdot \frac{x + 5}{x - 3} = \frac{x + 5}{x + 4}$$

which is A).

5. Solve the equation $x^2 = 4x + 1$.

(a) $-\dfrac{1}{2}, \ \sqrt{5} + 1$

(b) $2 - \sqrt{5}, \ 2 + \sqrt{5}$

(c) $2 - \sqrt{10}, \ 2 + \sqrt{10}$

(d) $2 + \sqrt{20}, \quad 2 - \sqrt{20}$

Solution 1: We will apply the quadratic formula. First we reduce one side to zero and then identify the coefficients.

$$x^2 - 4x - 1 = 0 \qquad \Longrightarrow \quad a = 1, \quad b = -4 \quad \text{and} \quad c = -1$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)} = \frac{4 \pm \sqrt{16 + 4}}{2} = \frac{4 \pm \sqrt{20}}{2} =$$

$$= \frac{4 \pm 2\sqrt{5}}{2} = \begin{cases} \dfrac{4 + 2\sqrt{5}}{2} = \dfrac{2\left(2 + \sqrt{5}\right)}{2} = 2 + \sqrt{5} \\ \dfrac{4 - 2\sqrt{5}}{2} = \dfrac{2\left(2 - \sqrt{5}\right)}{2} = 2 - \sqrt{5} \end{cases} \quad \text{or} \quad 2 \pm \sqrt{5}$$

which is B).

Solution 2: Complete the square.

$$\begin{aligned} x^2 &= 4x+1 \\ x^2-4x-1 &= 0 \qquad\qquad (x-2)^2 = x^2-4x+4 \\ \underbrace{x^2-4x+4}-4-1 &= 0 \\ (x-2)^2-5 &= 0 \\ (x-2)^2-\left(\sqrt{5}\right)^2 &= 0 \\ \left(x-2+\sqrt{5}\right)\left(x-2-\sqrt{5}\right) &= 0 \\ x_1 &= 2-\sqrt{5} \quad \text{and} \quad x_2 = 2+\sqrt{5} \end{aligned}$$

which is B).

6. Simplify the expression $\dfrac{1-x^{-2}}{1+x^{-1}}$.

(a) $\dfrac{x-1}{x}$
(b) $\dfrac{1-x}{x^2+1}$
(c) 1
(d) $-\dfrac{1}{x-1}$

Solution:

$$\frac{1-x^{-2}}{1+x^{-1}} = \frac{1-\dfrac{1}{x^2}}{1+\dfrac{1}{x}} = \frac{\dfrac{x^2-1}{x^2}}{\dfrac{x+1}{x}} = \frac{x^2-1}{x^2}\cdot\frac{x}{x+1} = \frac{(x+1)(x-1)}{x^2}\cdot\frac{x}{x+1} = \frac{x-1}{x}$$

which is A).

7. Perform the indicated operations and simplify. $\dfrac{1}{x-y}-\dfrac{1}{x+y}$

(a) 0
(b) $-\dfrac{2}{x+y}$
(c) $\dfrac{-2y}{y^2-x^2}$
(d) $\dfrac{2x}{y^2-x^2}$

Solution:

$$\begin{aligned} \frac{1}{x-y}-\frac{1}{x+y} &= \frac{x+y}{(x-y)(x+y)}-\frac{x-y}{(x-y)(x+y)} \\ &= \frac{(x+y)-(x-y)}{(x-y)(x+y)} = \frac{x+y-x+y}{(x-y)(x+y)} = \frac{2y}{(x-y)(x+y)} \end{aligned}$$

This answer appears to be missing, but it is actually listed:

$$\frac{2y}{(x-y)(x+y)} = \frac{2y}{x^2-y^2} = \frac{2y}{x^2-y^2}\cdot 1 = \frac{2y}{x^2-y^2}\cdot\frac{-1}{-1} = \frac{-2y}{y^2-x^2}$$

which is C).

8. Simplify $\dfrac{2^{1/2}4^{-1/2}}{64^{-2/3}}$.

(a) $\sqrt{2}$

(b) $\dfrac{1}{8}\sqrt{2}$

(c) $-32\sqrt{2}$

(d) $8\sqrt{2}$

Solution 1: First we get rid of the negative exponents using the rule $a^{-n} = \dfrac{1}{a^n}$.

$$\frac{2^{1/2}4^{-1/2}}{64^{-2/3}} = \frac{2^{1/2}64^{2/3}}{4^{1/2}}$$

We then interpret the fractional exponents as roots, using the rule $a^{\frac{n}{m}} = \left(\sqrt[m]{a}\right)^n$.

$$\frac{2^{1/2}64^{2/3}}{4^{1/2}} = \frac{\sqrt{2}\left(\sqrt[3]{64}\right)^2}{\sqrt{4}} = \frac{\sqrt{2}\cdot 4^2}{2} = \frac{\sqrt{2}\cdot 16}{2} = 8\sqrt{2}$$

which is D).

Solution 2: We first we get rid of the negative exponents using the rule $a^{-n} = \dfrac{1}{a^n}$.

$$\frac{2^{1/2}4^{-1/2}}{64^{-2/3}} = \frac{2^{1/2}64^{2/3}}{4^{1/2}}$$

Notice now that $4 = 2^2$ and $64 = 2^6$and then we can easily apply other rules of exponents.

$$\frac{2^{1/2}64^{2/3}}{4^{1/2}} = \frac{2^{1/2}\left(2^6\right)^{2/3}}{\left(2^2\right)^{1/2}} = \frac{2^{1/2}\cdot 2^{6\cdot\left(\frac{2}{3}\right)}}{2^{2\cdot\left(\frac{1}{2}\right)}} = \frac{2^{1/2}\cdot 2^4}{2^1} = \frac{2^{4+\frac{1}{2}}}{2^1} = 2^{4+\frac{1}{2}-1} = 2^{3\frac{1}{2}} = 2^3\cdot 2^{1/2} = 8\sqrt{2}$$

which is D).

9. Find the equation of the perpendicular bisector of the line segment determined by the points $A(-1,-5)$ and $B(5,7)$.

(a) $y = 2x - 3$

(b) $y = \dfrac{1}{2}x - \dfrac{9}{2}$

(c) $4x - y = 13$

(d) $y = -\dfrac{1}{2}x + 2$

Solution: **The perpendicular bisector of the line segment $AB$ is the straight line that is perpendicular to the line determined by $A$ and $B$, and passes through the midpoint of the line segment $AB$.** Consequently, the problem breaks down into three distinct parts.

Part 1. Determine the slope of the perpendicular bisector.

Since the bisector is perpendicular, its slope is the negative reciprocal of the slope of the line determined by $A$ and $B$. Using the slope formula, we determine The slope of the line determined by $A$ and $B$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-(-5)}{5-(-1)} = \frac{12}{6} = 2$$

The slope of the perpendicular bisector is the negative reciprocal of 2, which is $-\dfrac{1}{2}$.

Part 2. Determine a point on the bisector.
The berpendicular bisector passes through the midpont $M$ of the line segment $AB$. We can easily compute the coordinates:

$$x_M = \frac{x_1 + x_2}{2} = \frac{-1+5}{2} = 2 \text{ and } y_M = \frac{y_1 + y_2}{2} = \frac{-5+7}{2} = 1$$

Thus $M(2,1)$ is a point on bisector.
Part 3. Given the point and slope obtained previously, write the equation of the line.
We need to write the equation of the line that has slope $-\frac{1}{2}$ and passes through $(2,1)$. We will find the slope intercept-form of the equation, $y = mx + b$.

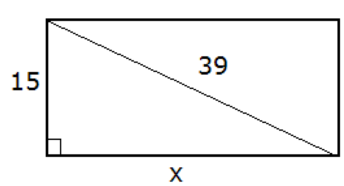
$$\begin{aligned} y &= mx + b && m = -\frac{1}{2} \\ y &= -\frac{1}{2}x + b && (2,1) \text{ is on the line} \\ 1 &= -\frac{1}{2}(2) + b && \text{Solve for } b\text{:} \\ 1 &= -1 + b && \text{add } 1 \\ 2 &= b \end{aligned}$$

Thus the equation is $y = -\frac{1}{2}x + 2$, which is D).

10. Find the area of a rectangle if its diagonal is 39 cm long and one of its sides is 15 cm long.

(a) $292.5\ \text{cm}^2$
(b) $540\ \text{cm}^2$
(c) $585\ \text{cm}^2$
(d) $102\ \text{cm}^2$

Solution:




We have a right triangle where one leg is 15 cm and the hypotenuse is 39 cm. We find the other leg using the Pythagorean Theorem.

$$\begin{aligned} 15^2 + x^2 &= 39^2 \\ 225 + x^2 &= 1521 && \text{subtract } 1521 \\ x^2 - 1296 &= 0 \\ x^2 - 36^2 &= 0 \\ (x+36)(x-36) &= 0 \\ x_{1,2} &= \pm 36 \implies \text{since distances can not be negative, } x = 36 \end{aligned}$$

Thus the rectangle is 15 cm by 36 cm, and thus its area is $A = 15\ \text{cm}\ (36\ \text{cm}) = 540\ \text{cm}^2$, which is B).

# Part 2

1. Simplify each of the following expressions. Show all work.

(a) $2^{-2} - 2^{-3} =$

$$2^{-2} - 2^{-3} = \frac{1}{2^2} - \frac{1}{2^3} = \frac{1}{4} - \frac{1}{8} = \frac{2-1}{8} = \frac{1}{8}$$

(b) $\dfrac{\left(x^{-2}\right)^{-2} y^3 x^0 \left(-2yxy^{-2}x^{-2}\right)^{-3}}{yx^5\left(y^{-2}x\right)^{-3}\left(2x^{-1}yx^3\right)^{-1}} =$

Solution: We will first omit $x^0$ and simplify within each parentheses. Then we apply the rules of exponents as indicated.

$$\begin{aligned}
\frac{\left(x^{-2}\right)^{-2} y^3 x^0 \left(-2yxy^{-2}x^{-2}\right)^{-3}}{yx^5\left(y^{-2}x\right)^{-3}\left(2x^{-1}yx^3\right)^{-1}} &= \frac{\left(x^{-2}\right)^{-2} y^3 \left(-2y^{-1}x^{-1}\right)^{-3}}{yx^5\left(y^{-2}x\right)^{-3}\left(2x^2y\right)^{-1}} && \text{use } (ab)^n = a^n b^n \\
&= \frac{\left(x^{-2}\right)^{-2} y^3 (-2)^{-3}\left(y^{-1}\right)^{-3}\left(x^{-1}\right)^{-3}}{yx^5\left(y^{-2}\right)^{-3} x^{-3} 2^{-1}\left(x^2\right)^{-1} y^{-1}} && \text{use } (a^n)^m = a^{nm} \\
&= \frac{x^4 y^3 (-2)^{-3} y^3 x^3}{yx^5 y^6 x^{-3} 2^{-1} x^{-2} y^{-1}}
\end{aligned}$$

We apply $a^{-n} = \dfrac{1}{a^n}$ and arrange terms: first numbers, then letters, alphabetized.

$$\begin{aligned}
\frac{x^4 y^3 (-2)^{-3} y^3 x^3}{yx^5 y^6 x^{-3} 2^{-1} x^{-2} y^{-1}} &= \frac{2^1 x^4 y^3 y^3 x^3 x^3 x^2 y^1}{(-2)^3 yx^5 y^6} && \text{Now apply } a^n a^m = a^{n+m} \\
&= \frac{2x^{12}y^7}{-8x^5y^7} && \text{apply } \frac{a^n}{a^m} = a^{n-m} \text{ and move } - \text{ sign upstairs} \\
&= \frac{-x^7}{4} \quad \text{or} \quad -\frac{1}{4}x^7.
\end{aligned}$$

(c) $\sqrt{48x^5y^3} =$

$$\sqrt{48x^5y^3} = \sqrt{16x^4y^2 \cdot 3xy} = \sqrt{16x^4y^2}\sqrt{3xy} = 4x^2y\sqrt{3xy}$$

(d) $\sqrt{80a^{11}} - 2\sqrt{180a^{11}} + 3\sqrt{245a^{11}} =$

$$\begin{aligned}
\sqrt{80a^{11}} - 2\sqrt{180a^{11}} + 3\sqrt{245a^{11}} &= \\
\sqrt{16a^{10} \cdot 5a} - 2\sqrt{36a^{10} \cdot 5a} + 3\sqrt{49a^{10} \cdot 5a} &= \\
\sqrt{16a^{10}}\sqrt{5a} - 2\sqrt{36a^{10}}\sqrt{5a} + 3\sqrt{49a^{10}}\sqrt{5a} &= \\
4a^5\sqrt{5a} - 2\left(6a^5\right)\sqrt{5a} + 3\left(7a^5\right)\sqrt{5a} &= \\
4a^5\sqrt{5a} - 12a^5\sqrt{5a} + 21a^5\sqrt{5a} &= \\
(4 - 12 + 21)\, a^5\sqrt{5a} &= 13a^5\sqrt{5a}
\end{aligned}$$

(e) $\sqrt[3]{56} + 4\sqrt[3]{189} - \sqrt[3]{875} =$

$$\begin{aligned}
\sqrt[3]{56} + 4\sqrt[3]{189} - \sqrt[3]{875} &= \sqrt[3]{8 \cdot 7} + 4\sqrt[3]{27 \cdot 7} - \sqrt[3]{125 \cdot 7} \\
&= \sqrt[3]{8}\sqrt[3]{7} + 4\sqrt[3]{27}\sqrt[3]{7} - \sqrt[3]{125}\sqrt[3]{7} \\
&= 2\sqrt[3]{7} + 4(3)\sqrt[3]{7} - 5\sqrt[3]{7} \\
&= 2\sqrt[3]{7} + 12\sqrt[3]{7} - 5\sqrt[3]{7} \\
&= (2 + 12 - 5)\sqrt[3]{7} = 9\sqrt[3]{7}
\end{aligned}$$

(f) $(2-\sqrt{x})(3+2\sqrt{x}) =$

$$\begin{aligned}\left(2-\sqrt{x}\right)\left(3+2\sqrt{x}\right) &= 6+4\sqrt{x}-3\sqrt{x}-2\sqrt{x}\sqrt{x}\\ &= 6+\sqrt{x}-2x\end{aligned}$$

(g) $\dfrac{\sqrt{5}-1}{\sqrt{5}-2} =$

$$\frac{\sqrt{5}-1}{\sqrt{5}-2}\cdot\frac{\sqrt{5}+2}{\sqrt{5}+2} = \frac{\left(\sqrt{5}-1\right)\left(\sqrt{5}+2\right)}{\left(\sqrt{5}-2\right)\left(\sqrt{5}+2\right)} = \frac{5+2\sqrt{5}-\sqrt{5}-2}{5+2\sqrt{5}-2\sqrt{5}-4} = \frac{3+\sqrt{5}}{1} = 3+\sqrt{5}$$

(h) Simplify $\dfrac{px^2-16q-16p+qx^2}{x^2+5x+6}\cdot\dfrac{x^2+6x+9}{4px^2+px^3+4qx^2+qx^3} =$

Solution: $x^2+5x+6=(x+2)(x+3)$ and $x^2+6x+9=(x+3)(x+3)$ are easy. With the other expressions, we start with the greatest common factor.

$$\begin{aligned}px^2-16q-16p+qx^2 &= \underbrace{px^2+qx^2}\underbrace{-16p-16q} && \text{grouping}\\ &= x^2(p+q)-16(p+q) && \text{factor out } p+q\\ &= \left(x^2-16\right)(p+q) && \text{difference of squares}\\ &= (x+4)(x-4)(p+q)\end{aligned}$$

$$\begin{aligned}4px^2+px^3+4qx^2+qx^3 &= x^2\left(\underbrace{4p+4q}\underbrace{+px+qx}\right) && \text{grouping}\\ &= x^2\left(4(p+q)+x(p+q)\right) && \text{factor out } p+q\\ &= x^2(x+4)(p+q)\end{aligned}$$

We have all the pieces now:

$$\begin{aligned}\frac{px^2-16q-16p+qx^2}{x^2+5x+6}\cdot\frac{x^2+6x+9}{4px^2+px^3+4qx^2+qx^3} &=\\ \frac{(x+4)(x-4)(p+q)}{(x+2)(x+3)}\cdot\frac{(x+3)(x+3)}{x^2(x+4)(p+q)} &= \frac{(x+3)(x-4)}{x^2(x+2)} = \frac{x^2-x-12}{2x^2+x^3}\end{aligned}$$

(i) $i^{210} =$

Solution: If we divide 210 by 4, we get $210\div 4 = 52$ R 2. Accordingly, using the fact that $i^4=1$ and $i^2=-1$,

$$i^{210} = i^{4\cdot 52+2} = i^{4\cdot 52}\cdot i^2 = \left(i^4\right)^{52}\cdot i^2 = 1^{52}(-1) = -1$$

(j) $(3-2i)^2 =$

Solution:

$$(3-2i)^2 = (3-2i)(3-2i) = 9-6i-6i+4i^2 = 9-12i+4(-1) = 5-12i$$

(k) $\dfrac{7-4i}{2+i} =$

Solution: We will use the complex conjugate.

$$\begin{aligned}\frac{7-4i}{2+i} &= \frac{7-4i}{2+i}\cdot\frac{2-i}{2-i} = \frac{(7-4i)(2-i)}{2^2-i^2} = \frac{14-7i-8i+4i^2}{4-(-1)} \\ &= \frac{14-15i+4(-1)}{4+1} = \frac{10-15i}{5} = \frac{5(2-3i)}{5} = 2-3i\end{aligned}$$

(l) $\dfrac{(3-i)^3-(2+i)^2}{1-2i} =$

Solution: We will follow the order of operations. Exponents first:

$$\begin{aligned}(3-i)^3 &= (3-i)(3-i)(3-i) \\ &= \left(9-3i-3i+i^2\right)(3-i) \\ &= (8-6i)(3-i) \\ &= 24-8i-18i+6i^2 \\ &= 24-26i+6(-1) \\ &= 18-26i\end{aligned}$$

$$(2+i)^2 = 4+2i+2i+i^2 = 3+4i$$

Now the expression is

$$\frac{(3-i)^3-(2+i)^2}{1-2i} = \frac{(18-26i)-(3+4i)}{1-2i} = \frac{18-26i-3-4i}{1-2i} = \frac{15-30i}{1-2i} = \frac{15(1-2i)}{1-2i} = 15$$

2. Completely factor each of the following.

(a) $357ab^2-30ab^2x-3ab^2x^2 =$

$$\begin{aligned}357ab^2-30ab^2x-3ab^2x^2 &= 3ab^2\left(119-x^2-10x\right) \\ &= -3ab^2\left(x^2+10x-119\right) \qquad (x+5)^2 = x^2+10x+25 \\ &= -3ab^2\left(\underbrace{x^2+10x+25}-25-119\right) \\ &= -3ab^2\left((x+5)^2-144\right) \\ &= -3ab^2\left((x+5)^2-12^2\right) \\ &= -3ab^2(x+5+12)(x+5-12) \\ &= -3ab^2(x+17)(x-7)\end{aligned}$$

(b) $4a^2px^5-2a^2qx-4a^2px+2a^2qx^5 =$

$$\begin{aligned}&4a^2px^5-2a^2qx-4a^2px+2a^2qx^5 \\ =\ &2a^2x\left(2px^4-q-2p+qx^4\right) \\ =\ &2a^2x\left(2px^4+qx^4-2p-q\right) \\ =\ &2a^2x\left(x^4(2p+q)-1(2p+q)\right) \\ =\ &2a^2x\left(x^4-1\right)(2p+q) \qquad x^4-1=\left(x^2\right)^2-1^2 \text{ factors} \\ =\ &2a^2x\left(x^2+1\right)\left(x^2-1\right)(2p+q) \qquad x^2-1 \text{ factors} \\ =\ &2a^2x(x-1)(x+1)\left(x^2+1\right)(2p+q)\end{aligned}$$

3. Factor via completing the square:

(a) $100x - x^2 - 2419 =$

$$\begin{aligned} 100x - x^2 - 2419 &= -\left(x^2 - 100x + 2419\right) \qquad\qquad (x-50)^2 = x^2 - 100x + 2500 \\ &= -\left(\underbrace{x^2 - 100x + 2500}\,-2500 + 2419\right) \\ &= -\left((x-50)^2 - 81\right) \\ &= -\left((x-50)^2 - 9^2\right) \\ &= -(x-50+9)(x-50-9) \\ &= -(x-41)(x-59) \end{aligned}$$

(b) $x^2 - x - 462 =$

$$\begin{aligned} x^2 - x - 462 &= \qquad \left(x - \frac{1}{2}\right)^2 = x^2 - x + \frac{1}{4} \\ &= \underbrace{x^2 - x + \frac{1}{4}}\,-\frac{1}{4} - 462 \\ &= \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{1848}{4} \\ &= \left(x - \frac{1}{2}\right)^2 - \frac{1849}{4} \\ &= \left(x - \frac{1}{2}\right)^2 - \left(\frac{43}{2}\right)^2 \\ &= \left(x - \frac{1}{2} + \frac{43}{2}\right)\left(x - \frac{1}{2} - \frac{43}{2}\right) \\ &= (x+21)(x-22) \end{aligned}$$

(c) $11x + 6x^2 - 10 =$

Solution: we first rearrange the polynomial and then factor out the leading coefficient.

$$11x + 6x^2 - 10 = 6x^2 + 11x - 10 = 6\left(x^2 + \frac{11}{6}x - \frac{5}{3}\right)$$

The magic number is $\frac{11}{6} \div 2 = \frac{11}{6}\left(\frac{1}{2}\right) = \frac{11}{12}$ and so we FOIL $\left(x + \frac{11}{12}\right)^2$

$$\begin{aligned} \left(x + \frac{11}{12}\right)^2 &= \left(x + \frac{11}{12}\right)\left(x + \frac{11}{12}\right) = x^2 + \frac{11}{12}x + \frac{11}{12}x + \left(\frac{11}{12}\right)^2 \\ &= x^2 + \frac{11}{6}x + \frac{121}{144} \qquad\qquad \text{thus we will smuggle in } \frac{121}{144} \end{aligned}$$

$$\begin{aligned}
6\left(x^2+\frac{11}{6}x-\frac{5}{3}\right) &= 6\left(\underbrace{x^2+\frac{11}{6}x+\frac{121}{144}}-\frac{121}{144}-\frac{5}{3}\right)\\
&= 6\left(\left(x+\frac{11}{12}\right)^2-\frac{121}{144}-\frac{240}{144}\right)\\
&= 6\left(\left(x+\frac{11}{12}\right)^2-\frac{361}{144}\right)\\
&= 6\left(\left(x+\frac{11}{12}\right)^2-\left(\frac{19}{12}\right)^2\right)\\
&= 6\left(x+\frac{11}{12}+\frac{19}{12}\right)\left(x+\frac{11}{12}-\frac{19}{12}\right)\\
&= 6\left(x+\frac{30}{12}\right)\left(x-\frac{8}{12}\right)\\
&= 2\left(x+\frac{5}{2}\right)3\left(x-\frac{2}{3}\right) = (2x+5)(3x-2)
\end{aligned}$$

(d) $x^2-8x+13=$

$$\begin{aligned}
x^2-8x+13 &= \qquad (x-4)^2 = x^2-8x+16\\
\underbrace{x^2-8x+16}-16+13 &=\\
(x-4)^2-3 &=\\
(x-4)^2-\left(\sqrt{3}\right)^2 &= \left(x-4+\sqrt{3}\right)\left(x-4-\sqrt{3}\right)
\end{aligned}$$

(e) $x^2-4x+7$

$$\begin{aligned}
x^2-4x+7 &= \qquad (x-2)^2 = x^2-4x+4\\
\underbrace{x^2-4x+4}-4+7 &=\\
(x-2)^2+3 &= \quad \text{does not factor over real number}
\end{aligned}$$

However, the expression factors over the complex number plane.

$$\begin{aligned}
(x-2)^2+3 &= (x-2)^2-(-3) =\\
&= (x-2)^2-\left(\sqrt{-3}\right)^2\\
&= (x-2)^2-\left(\sqrt{3}\sqrt{-1}\right)^2\\
&= (x-2)^2-\left(\sqrt{3}i\right)^2\\
&= \left(x-2+\sqrt{3}i\right)\left(x-2-\sqrt{3}i\right)
\end{aligned}$$

4. Graphing.

(a) Graph the parabola $y=-2x^2+3x+1$. Clearly label the coordinates of at least 5 points, including vertex and intercepts.

Solution: The $y$-intercept is clearly $(0,1)$. The $x$-coordinate of the vertex is always $x_V = \dfrac{-b}{2a}$ where $a=-2,\ \ b=3,\ $ and $\ c=1$

$$x_V = \frac{-b}{2a} = \frac{-3}{2(-2)} = \frac{3}{4}$$

For the $y$-coordinate of the vertex, we plug its $x$-coordinate into the formula:

$$y_V = -2(x_V)^2 + 3(x_V) + 1 = -2\left(\frac{3}{4}\right)^2 + 3\left(\frac{3}{4}\right) + 1 = -2\left(\frac{9}{16}\right) + \frac{9}{4} + 1 = \frac{-9}{8} + \frac{18}{8} + \frac{8}{8} = \frac{17}{8}$$

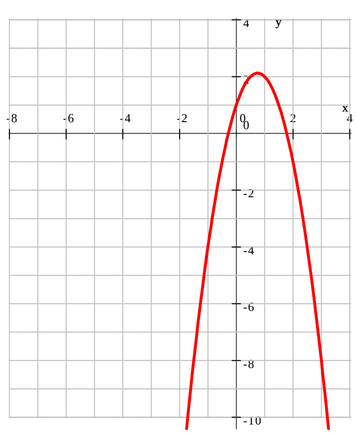
Thus the vertex is $\left(\dfrac{3}{4}, \dfrac{17}{8}\right)$. Now we know to pug in numbers close to $\dfrac{3}{4}$:

$$\begin{aligned}
\text{if } x &= -2, \text{ then } y = -2(-2)^2 + 3(-2) + 1 = -2(4) - 6 + 1 = -13 \\
\text{if } x &= -1, \text{ then } y = -2(-1)^2 + 3(-1) + 1 = -2(1) - 3 + 1 = -4 \\
\text{if } x &= 0, \text{ then } y = -2(0)^2 + 3(0) + 1 = 1 \\
\text{if } x &= 1, \text{ then } y = -2(1)^2 + 3(1) + 1 = -2(1) + 3 + 1 = 2 \\
\text{if } x &= 2, \text{ then } y = -2(2)^2 + 3(2) + 1 = -2(4) + 6 + 1 = -1
\end{aligned}$$

For the $x$-intercepts, we apply the quadratic formula.

$$\begin{aligned}
a &= -2, \quad b = 3 \quad \text{and} \quad c = 1 \\
x_{1,2} &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-3 \pm \sqrt{(-3)^2 - 4(-2)(1)}}{2(-2)} = \frac{-3 \pm \sqrt{9+8}}{-4} = \\
&= \frac{-3 \pm \sqrt{17}}{-4} = \begin{cases} \dfrac{-3+\sqrt{17}}{-4} \cdot \dfrac{-1}{-1} = \dfrac{-1\left(-3+\sqrt{17}\right)}{4} = \dfrac{3-\sqrt{17}}{4} \\ \dfrac{-3-\sqrt{17}}{-4} \cdot \dfrac{-1}{-1} = \dfrac{-1\left(-3-\sqrt{17}\right)}{4} = \dfrac{3+\sqrt{17}}{4} \end{cases} \quad \text{or} \quad \frac{3 \pm \sqrt{17}}{4}
\end{aligned}$$

and so the $x$-intercepts are $\left(\dfrac{3-\sqrt{17}}{4}, 0\right) \simeq (-0.281, 0)$ and $\left(\dfrac{3+\sqrt{17}}{4}, 0\right) \simeq (1.781, 0)$. Now we are ready to graph:



(b) Graph the parabola $y = 5x - 2x^2 + 3$ and the line $y = 5x - 5$ in the same coordinate system. Use your graph to find the coordinates of the points where they intersect.

Solution: We graph the straight line first. The equation given is already in the slope-intercept form, and so it is easy to graph it. We start at the $y-$intercept, $(0, -5)$ and move on the grid one to the right, five up. Now for the parabola: because the leading coefficient is negative, the parabola iopens downward. From the polynomial form, $y = -2x^2 + 5x + 3$, we obtain the $y-$intercept: $(0, 3)$ and the coefficients

$$a = -2,\ b = 5,\ c = 3$$

The vertex is at

$$x_V = \frac{-b}{2a} = \frac{-5}{2(-2)} = \frac{5}{4}$$

Then the $y-$coordinate of the vertex is easy. If $x = \frac{5}{4}$, then

$$y = -2\left(\frac{5}{4}\right)^2 + 5\left(\frac{5}{4}\right) + 3 = -2\left(\frac{25}{16}\right) + \frac{25}{4} + 3 = \frac{-50}{16} + \frac{25}{4} + 3 = \frac{-25}{8} + \frac{25}{4} + 3 = \frac{-25}{8} + \frac{50}{8} + \frac{24}{8} = \frac{49}{8}$$

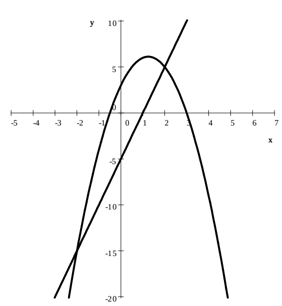
Thus the vertex is $\left(\frac{5}{4}, \frac{49}{8}\right)$. We will conduct the $5-$point dance around 1, the closest integer to $\frac{5}{4}$. Because we no longer work around the vertex, we loose the symmetry.

$$\begin{aligned}
\text{If } x &= -1, \text{ then } y = -2(-1)^2 + 5(-1) + 3 = -2(1) - 5 + 3 = -7 + 3 = -4 \implies (-1, -4) \\
&\phantom{=} \text{We have already found the } y - \text{intercept, } (0, 3) \\
\text{If } x &= 1, \text{ then } y = -2(1)^2 + 5(1) + 3 = -2(1) + 5 + 3 = -2 + 5 + 3 = 6 \implies (1, 6) \\
&\phantom{=} \text{We have already found the vertex, } \left(\frac{5}{4}, \frac{49}{8}\right) \\
\text{If } x &= 2, \text{ then } y = -2(2)^2 + 5(2) + 3 = -2(4) + 10 + 3 = -8 + 10 + 3 = 5 \implies (2, 5)
\end{aligned}$$

The $x-$intercepts are at $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$x_{1,2} = \frac{-5 \pm \sqrt{5^2 - 4(-2)3}}{2(-2)} = \frac{-5 \pm \sqrt{25 + 24}}{-4} = \frac{-5 \pm \sqrt{49}}{-4} = \frac{-5 \pm 7}{-4} = \begin{cases} \dfrac{-5+7}{-4} = \dfrac{2}{-4} = -\dfrac{1}{2} \\ \dfrac{-5-7}{-4} = \dfrac{-12}{-4} = 3 \end{cases}$$

Thus the $x-$intercepts are $\left(-\frac{1}{2}, 0\right)$ and $(3, 0)$. We compute points close to the vertex and where it appears to get closer to the line.




The intersections are $(2, 5)$ and $(-2, -15)$. We can check algebraically.

5. Solve each of the following.

(a) $7-(3+4t)+2t=-5(1-t)+3-t$

$$\begin{aligned} 7-(3+4t)+2t &= -5(1-t)+3-t && \text{distribute} \\ 7-3-4t+2t &= -5+5t+3-t && \text{combine like terms} \\ -2t+4 &= 4t-2 && \text{add } 2t \\ 4 &= 6t-2 && \text{add } 2 \\ 6 &= 6t && \text{divide by } 6 \\ 1 &= t \end{aligned}$$

We check:

$$\begin{aligned} \text{LHS} &= 7-(3+4(1))+2(1)=7-(3+4)+2=7-7+2=2 \\ \text{RHS} &= -5(1-1)+3-1=-5(0)+3-1=2 \end{aligned}$$

(b) $\dfrac{2x-1}{3}-\dfrac{-3-x}{4}=x-1$

$$\begin{aligned} \frac{2x-1}{3}-\frac{-3-x}{4} &= \frac{x-1}{1} && \text{common denominator} \\ \frac{4(2x-1)}{12}-\frac{3(-3-x)}{12} &= \frac{12(x-1)}{12} && \text{multiply by 12} \\ 4(2x-1)-3(-3-x) &= 12(x-1) && \text{distribute} \\ 8x-4+9+3x &= 12x-12 && \text{combine like terms} \\ 11x+5 &= 12x-12 && \text{subtract } 11x \\ 5 &= x-12 && \text{add } 12 \\ 17 &= x \end{aligned}$$

We check:

$$\begin{aligned} \text{LHS} &= \frac{2(17)-1}{3}-\frac{-3-17}{4}=\frac{33}{3}-\frac{-20}{4}=11-(-5)=16 \\ \text{RHS} &= 17-1=16 \end{aligned}$$

(c) $3x^3-x^2=x$

$$\begin{aligned} 3x^3-x^2 &= x \\ 3x^3-x^2-x &= 0 \\ x\left(3x^2-x-1\right) &= 0 \\ x &= 0 \quad \text{or} \quad 3x^2-x-1=0 \end{aligned}$$

We apply the quadratic formula with $a=3$, $b=-1$ and $c=-1$.

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{1\pm\sqrt{(-1)^2-4(3)(-1)}}{2(3)}=\frac{1\pm\sqrt{1+12}}{6}=\frac{1\pm\sqrt{13}}{6}$$

So the solutions are $x=0$ or $x=\dfrac{1+\sqrt{13}}{6}$ or $x=\dfrac{1-\sqrt{13}}{6}$

(d) $5-\sqrt{2x+1}=-2$

$$\begin{aligned}
5-\sqrt{2x+1} &= -2 && \text{add } \sqrt{2x+1} \\
5 &= -2+\sqrt{2x+1} && \text{add } 2 \\
7 &= \sqrt{2x+1} && \text{square} \\
49 &= 2x+1 && \text{subtract } 1 \\
48 &= 2x && \text{divide by } 2 \\
24 &= x
\end{aligned}$$

We check: if $x=24$, then

$$\begin{aligned}
\text{LHS} &= 5-\sqrt{2(24)+1}=5-\sqrt{49}=5-7=-2 \\
\text{RHS} &= -2 \\
\text{Since LHS} &= \text{RHS, } 24 \text{ is a solution.}
\end{aligned}$$

Thus $x=24$ is correct.

6. Word Problems.

(a) One side of a rectangle is 16 cm longer than the other side. The area of the rectangle is 80 $\text{cm}^2$. Find the dimensions of the rectangle. Include units in your answer.

Solution: Let us denote the shorter side by $x$. Then the longer side is $x+16$. We obtain the equation for the area:

$$\begin{aligned}
x(x+16) &= 80 && \text{distribute} \\
x^2+16x &= 80 && \text{subtract } 80 \\
x^2+16x-80 &= 0 && \text{factor} \qquad (x+8)^2=x^2+16x+64 \\
\underbrace{x^2+16x+64}-64-80 &= 0 \\
(x+8)^2-144 &= 0 \\
(x+8)^2-12^2 &= 0 \\
(x+8+12)(x+8-12) &= 0 \\
(x+20)(x-4) &= 0 \\
x_1 &= -20 \quad \text{and} \quad x_2=4
\end{aligned}$$

Since distances are non-negative, $x=-20$ is ruled out as a solution. Thus the shorter side is 4 cm, and the longer side is $4+16=20$ cm. We check: the area is $4(20)=80$ $\text{cm}^2$. Thus the solution is: 4 cm by 20 cm

(b) The sides of a right triangle have lengths (in centimeters) that are consecutive even integers. What are the lengths of the sides?

Solution: Let us denote the length of the shortest side by $x$. Then the other sides are $x+2$ and $x+4$ long. Clearly, $x+4$ must the length of the hypotenuse. The equation expresses the Pythagorean theorem.

$$\begin{aligned}
x^2+(x+2)^2 &= (x+4)^2 && \text{distribute}\\
x^2+x^2+4x+4 &= x^2+8x+16 && \text{combine like terms}\\
2x^2+4x+4 &= x^2+8x+16 && \text{subtract } x^2\\
x^2+4x+4 &= 8x+16 && \text{subtract } 8x\\
x^2-4x+4 &= 16 && \text{subtract } 16\\
x^2-4x-12 &= 0 && \text{factor by completing the square}\\
\underbrace{x^2-4x+4}-4-12 &= 0\\
(x-2)^2-16 &= 0\\
(x-2)^2-4^2 &= 0\\
(x-2-4)(x-2+4) &= 0\\
(x-6)(x+2) &= 0 && \text{apply the zero property}
\end{aligned}$$

$$x_1 = 6 \quad \text{and} \quad x_2 = -2$$

Since distances are non-negative, $x=-2$ is ruled out and so the sides of the triangle are 6 cm, 8 cm, and 10 cm long. We check: 6, 8, and 10 are indeed consecutive even numbers and work with the Pyhtagorean theorem as well, since $6^2+8^2=36+64=100=10^2$ ✓.

(c) Two investments produce an annual interest income of 708. The total amount of money invested is \$8000, and the two interest rates paid are 7% and 11%. How much money is invested at each rate?

Solution 1. Let us denote the amount invested at 11% by $x$. Then the other account must be $8000-x$ since the two accounts add to \$8000. (Remember: one information goes into labeling, the other one gives you the equation.) The equation expresses the combined interest. Since we invested

$$x \text{ at } 11\% \qquad \text{and } 8000-x \text{ at } 7\%$$

the equation is

$$\begin{aligned}
0.11x+0.07(8000-x) &= 708 && \text{multiply both sides by 100}\\
11x+7(8000-x) &= 70800 && \text{distribute}\\
11x+56000-7x &= 70800 && \text{combine like terms}\\
4x+56000 &= 70800 && \text{subtract 56000}\\
4x &= 14800 && \text{divide by 4}\\
x &= 3700
\end{aligned}$$

We invested \$3700 at 11%. The other amount is then $8000-x=8000-3700=4300$. Thus we invested \$3700 at 11% and \$4300 at 7%. We check: the accounts add up to $\$3700+\$4300=\$8000$ ✓. The interest from each account is

$$\begin{aligned}
11\% \text{ of } 3700 \text{ is} \qquad 0.11(3700) &= 407 \text{ and}\\
7\% \text{ of } 4300 \text{ is} \qquad 0.07(4300) &= 301
\end{aligned}$$

Since $407 + 301 = 708$ ✓, our solution is correct.

Solution 2: Let us denote the amount invested at 11% by $x$ and the amount invested at 7% by $y$. The two equations express that

$$\begin{aligned} x + y &= 8000 && \text{the accounts add up to \$8000} \\ 0.11x + 0.07y &= 708 && \text{the combined interest is \$708} \end{aligned}$$

We solve the system of equation by elimination. Before, let us first make the second equation simpler by multiplying by 100. We now have

$$\begin{aligned} x + y &= 8000 \\ 11x + 7y &= 70800 \end{aligned}$$

We will multiply the first equation by $-7$ to eliminate $y$

1.) $-7x - 7y = -56000$
2.) $11x + 7y = 70800$ add the equations
$4x = 14800$ divide by 4
$x = 3700$

Thus we invested \$3700 at 11%. The other amount is then from the first equation:

$$\begin{aligned} 3700 + y &= 8000 && \text{subtract } 3700 \\ y &= 4300 \end{aligned}$$

We invested \$3700 at 11% and \$4300 at 7%. We check: the amounts add up to $\$3700 + \$4300 = \$8000$ ✓. The combined interest from the accounts is

$$11\% \text{ of } 3700 \text{ is } 0.11\,(3700) = 407 \quad \text{and} \quad 7\% \text{ of } 4300 \text{ is } 0.07\,(4300) = 301$$

Since $407 + 301 = 708$, our solution is correct.

(d) A bank teller has 47 more five-dollar bills than ten-dollar bills. The total value of the money is \$1000. How much of each denomination of bill does he have?

Solution: Let us denote the number of ten-dollar bills by $x$. Then we have $x + 47$ many five-dollar bills. The equation expresses the value of the bills.

$$\begin{aligned} 10x + 5\,(x + 47) &= 1000 && \text{distribute} \\ 10x + 5x + 235 &= 1000 && \text{combine like terms} \\ 15x + 235 &= 1000 && \text{subtract 235} \\ 15x &= 765 && \text{divide by 15} \\ x &= 51 \end{aligned}$$

Thus we have 51 tens and $51 + 47 = 98$ fives. We check: $98 - 51 = 47$ and $51\,(10) + 98\,(5) = 1000$. Thus our solution; 51 ten-dollar bills and 98 five-dollar bills; is correct.

(e) Two cars are 400 miles apart. Both start at the same time and travel toward one another. They meet 4 hours later. If the speed of one car is 20 $\frac{\text{mi}}{\text{hr}}$ faster than the other, what is the speed of each car?

Solution: Let us denote the velocity of the slower car by $x$. Then

| | $v\ \left(\frac{\text{mi}}{\text{hr}}\right)$ | $t$ (hr) | $s$ (mi) |
|---|---|---|---|
| slower car | $x$ | 4 | $4x$ |
| faster car | $x+20$ | 4 | $4(x+20)$ |

$$\begin{aligned} 4x + 4(x+20) &= 400 \\ 4x + 4x + 80 &= 400 \\ 8x + 80 &= 400 \\ 8x &= 320 \\ x &= 40 \end{aligned}$$

Thus the speed of the slower car is 40 $\frac{\text{mi}}{\text{hr}}$ and that of the faster one is 60 $\frac{\text{mi}}{\text{hr}}$.