1. Simplify each of the following. Show all steps.

(a) $\dfrac{x^3-x}{x+1} = x^2-x$

Solution: We factor the numerator and simplify.

$$\frac{x^3-x}{x+1} = \frac{x\left(x^2-1\right)}{x+1} = \frac{x\left(x+1\right)\left(x-1\right)}{x+1} = x\left(x-1\right) \quad \text{or} \quad x^2-x$$

(b) $\dfrac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}} = \dfrac{13-4\sqrt{10}}{3}$

Solution: We multiply the fraction by 1 as a fraction of whose both numerator and denominator are the conjugate of the denominator.

$$\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}} = \frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}\cdot 1 = \frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}\cdot\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}-\sqrt{5}} = \frac{\left(\sqrt{8}-\sqrt{5}\right)\left(\sqrt{8}-\sqrt{5}\right)}{\left(\sqrt{8}+\sqrt{5}\right)\left(\sqrt{8}-\sqrt{5}\right)}$$

We FOIL out both numerator and denominator

$$= \frac{\sqrt{8}\sqrt{8}-\sqrt{8}\sqrt{5}-\sqrt{5}\sqrt{8}+\sqrt{5}\sqrt{5}}{\sqrt{8}\sqrt{8}-\sqrt{8}\sqrt{5}+\sqrt{5}\sqrt{8}-\sqrt{5}\sqrt{5}} = \frac{8-\sqrt{40}-\sqrt{40}+5}{8-5} = \frac{13-2\sqrt{40}}{3}$$

Note: although this answer is acceptable, the expression can be further simplified.

$$\frac{13-2\sqrt{40}}{3} = \frac{13-2\sqrt{4\cdot 10}}{3} = \frac{13-2\sqrt{4}\cdot\sqrt{10}}{3} = \frac{13-2\cdot 2\cdot\sqrt{10}}{3} = \frac{13-4\sqrt{10}}{3}$$

(c) $\left(-\dfrac{x^3y^0x^{-5}}{y^{-3}}\right)^{-2} = \dfrac{x^4}{y^6}$

Solution: First we simplify the expression within the parentheses.

$$\left(-\frac{x^3y^0x^{-5}}{y^{-3}}\right)^{-2} = \left(-x^{3+(-5)}y^{0-(-3)}\right)^{-2} = \left(-x^{-2}y^3\right)^{-2}$$

One good way of keeping track of the negative sign is to carry it as multiplication by -1:

$$\left(-x^{-2}y^3\right)^{-2} = \left(-1x^{-2}y^3\right)^{-2} = (-1)^{-2}\left(x^{-2}\right)^{-2}\left(y^3\right)^{-2} = (-1)^{-2}x^4y^{-6}$$

We can (in MULTIPLICATION ONLY!) remove the negative exponents by moving these factors to the denominator and re-writing them with the opposite exponent. If we don't have a fraction, we can easily create one by dividing by 1.

$$(-1)^{-2}x^4y^{-6} = \frac{(-1)^{-2}x^4y^{-6}}{1} = \frac{x^4}{(-1)^2y^6} = \frac{x^4}{(1)\,y^6} = \frac{x^4}{y^6} \quad \text{or} \quad x^4y^{-6}$$

(d) $\left(\sqrt{5x}-2\right)\left(\sqrt{5x}+3\right) = \quad 5x+\sqrt{5x}-6$
Solution: We FOIL the expression

$$\left(\sqrt{5x}-2\right)\left(\sqrt{5x}+3\right) = \sqrt{5x}\cdot\sqrt{5x}+\sqrt{5x}\cdot 3-2\cdot\sqrt{5x}-2\cdot(+3) =$$
$$= 5x+3\sqrt{5x}-2\sqrt{5x}-6 = 5x+\sqrt{5}\sqrt{x}-6$$

(e) $\dfrac{x^2-10x+25}{x^2-5x+4}\left(\dfrac{x^2-2x-8}{x^2-6x+5}\div\dfrac{x-5}{x-1}\right) = \quad \dfrac{x+2}{x-1}$
Solution: We factor the polynomials written in the fractions, and re-write division as multiplication by the reciprocal.

$$\frac{x^2-10x+25}{x^2-5x+4}\left(\frac{x^2-2x-8}{x^2-6x+5}\div\frac{x-5}{x-1}\right) =$$

$$= \frac{(x-5)(x-5)}{(x-4)(x-1)}\cdot\left(\frac{(x-4)(x+2)}{(x-5)(x-1)}\cdot\frac{x-1}{x-5}\right)$$
$$= \frac{(x-5)(x-5)(x-4)(x+2)(x-1)}{(x-4)(x-1)(x-5)(x-1)(x-5)} = \frac{x+2}{x-1}$$

(f) $\dfrac{14-13i}{2+i} = \quad 3-8i$
Solution: We multiply the fraction by 1 where 1 is a fraction, both numerator and denominator being the conjugate of $2+i$.

$$\frac{14-13i}{2+i} = \frac{14-13i}{2+i}\cdot\frac{2-i}{2-i} = \frac{(14-13i)(2-i)}{2^2-i^2} = \frac{28-14i-26i+13i^2}{4-(-1)} = \frac{28-40i+13(-1)}{5}$$
$$= \frac{28-13-40i}{5} = \frac{15-40i}{5} = \frac{5(3-8i)}{5} = 3-8i$$

2. Factor completely each of the following expressions.

(a) $3a^4x-48x = \quad 3x\left(a^2+4\right)(a+2)(a-2)$
Solution: We first factor out the greatest common factor.

$$3a^4x-48x = 3x\left(a^4-16\right)$$

Then we factor via the difference of squares theorem.

$$3x\left(a^4-16\right) = 3x\left(\left(a^2\right)^2-4^2\right) = 3x\left(a^2+4\right)\left(a^2-4\right)$$

Now the last factor, a^2-4 factors via the difference of squares theorem, since $a^2-4=a^2-2^2=(a+2)(a-2)$. Thus $3x\left(a^2+4\right)\left(a^2-4\right) = 3x\left(a^2+4\right)(a+2)(a-2)$

(b) $21x^2-18ax^2-3a^2x^2 = \quad -3x^2(a+7)(a-1)$
Solution: Let us factor out the greatest common factor first.

$$21x^2-18ax^2-3a^2x^2 = 3x^2\left(7-a^2-6a\right)$$

Now we rearrange the terms in the second factor.

$$3x^2\left(7-a^2-6a\right)=3x^2\left(-a^2-6a+7\right)$$

It is easier to factor a polynomial if its leading coefficient is positive. To obtain this, we factor out -1

$$3x^2\left(-a^2-6a+7\right)=-3x^2\left(a^2+6a-7\right)$$

Now we factor the second factor. Since $a^2+6a-7=(a+7)(a-1)$. So the answer is $-3x^2(a+7)(a-1)$.

3. Solve each of the following equations. Make sure to check your solution(s).

(a) $\dfrac{3x+17}{2}=x-1+\dfrac{x+19}{2}$ identity, all numbers are solution

Solution:

$$\begin{aligned}
\frac{3x+17}{2} &= x-1+\frac{x+19}{2} && \text{express everything as a fraction}\\
\frac{3x+17}{2} &= \frac{x-1}{1}+\frac{x+19}{2} && \text{bring everything to the common denominator}\\
\frac{3x+17}{2} &= \frac{2(x-1)}{2}+\frac{x+19}{2} && \text{add fractions on right hand side}\\
\frac{3x+17}{2} &= \frac{2(x-1)+x+19}{2} && \text{multiply out parentheses}\\
\frac{3x+17}{2} &= \frac{2x-2+x+19}{2} && \text{combine like terms}\\
\frac{3x+17}{2} &= \frac{3x+17}{2} && \text{multiply by 2}\\
3x+17 &= 3x+17
\end{aligned}$$

Because the left hand side is now identical to the right hand side, this equation is an identity, and all real numbers are solution.

(b) $|3-2x|+2=5$ 0, 3

Solution:

$$\begin{aligned}
|3-2x|+2 &= 5 \quad \text{subtract } 2\\
|3-2x| &= 3
\end{aligned}$$

$$\begin{array}{rclcrcl}
3-2x &=& 3 & \text{or} & 3-2x=-3 & & \text{subtract } 3\\
-2x &=& 0 & \text{or} & -2x=-6 & & \text{divide by } -2\\
x &=& 0 & \text{or} & x=3 & &
\end{array}$$

(c) $\frac{2}{3}(x-7)=\frac{4}{5}(x+1)$ -41
Solution:

$$\begin{aligned}
\frac{2}{3}(x-7) &= \frac{4}{5}(x+1) \\
\frac{2}{3}\cdot\frac{x-7}{1} &= \frac{4}{5}\cdot\frac{x+1}{1} \\
\frac{2(x-7)}{3} &= \frac{4(x+1)}{5} && \text{bring fractions to common denominator} \\
\frac{5\cdot 2(x-7)}{15} &= \frac{3\cdot 4(x+1)}{15} && \text{multiply both sides by 15}
\end{aligned}$$

$$\begin{aligned}
10(x-7) &= 12(x+1) && \text{multiplty out parentheses} \\
10x-70 &= 12x+12 && \text{subtract } 10x \\
-70 &= 2x+12 && \text{subtract } 12 \\
-82 &= 2x && \text{divide by } 2 \\
-41 &= x
\end{aligned}$$

We check:

$$\begin{aligned}
\text{LHS} &= \frac{2}{3}(-41-7) = \frac{2}{3}(-48) = -32 \\
\text{RHS} &= \frac{4}{5}(-41+1) = \frac{4}{5}(-40) = -32
\end{aligned}$$

Thus our solution, -41 is correct.

(d) $7x^2+(x+3)(2x-1)=(3x+1)^2$ -4
Solution:

$$\begin{aligned}
7x^2+(x+3)(2x-1) &= (3x+1)^2 && \text{multiply the polynomials on both sides} \\
7x^2+2x^2+5x-3 &= 9x^2+6x+1 && \text{combine like terms} \\
9x^2+5x-3 &= 9x^2+6x+1 && \text{subtract } 9x^2 \\
5x-3 &= 6x+1 && \text{subtract } 5x \\
-3 &= x+1 && \text{subtract } 1 \\
-4 &= x
\end{aligned}$$

We check our result:

$$\begin{aligned}
\text{LHS} &= 7(-4)^2+((-4)+3)(2(-4)-1) = 7\cdot 16+(-1)(-9) = 112+9 = 121 \\
\text{RHS} &= (3(-4)+1)^2 = (-12+1)^2 = (-11)^2 = 121
\end{aligned}$$

Thus the solution, -4 is correct.

(e) $3x^2 - 30x + 69 = 0$

Solution 1: By Completing the square.

$$\begin{aligned} 3x^2 - 30x + 69 &= 0 \\ 3\left(x^2 - 10x + 23\right) &= 0 \qquad (x-5)^2 = x^2 - 10x + 25 \\ 3\left(\underbrace{x^2 - 10x + 25} - 25 + 23\right) &= 0 \\ 3\left((x-5)^2 - 2\right) &= 0 \\ 3\left((x-5)^2 - \left(\sqrt{2}\right)^2\right) &= 0 \\ 3\left(x - 5 + \sqrt{2}\right)\left(x - 5 - \sqrt{2}\right) &= 0 \end{aligned}$$

$$x_1 = 5 - \sqrt{2} \quad \text{or} \quad x_2 = 5 + \sqrt{2}$$

We check: when $x = 5 - \sqrt{2}$, then

$$\begin{aligned} \text{LHS} &= 3\left(5 - \sqrt{2}\right)^2 - 30\left(5 - \sqrt{2}\right) + 69 = 3\left(27 - 10\sqrt{2}\right) - 30\left(5 - \sqrt{2}\right) + 69 \\ &= 81 - 30\sqrt{2} - 150 + 30\sqrt{2} + 69 = 0 = \text{RHS} \end{aligned}$$

and when when $x = 5 + \sqrt{2}$, then

$$\begin{aligned} \text{LHS} &= 3\left(5 + \sqrt{2}\right)^2 - 30\left(5 + \sqrt{2}\right) + 69 = 3\left(27 + 10\sqrt{2}\right) - 30\left(5 + \sqrt{2}\right) + 69 \\ &= 81 + 30\sqrt{2} - 150 - 30\sqrt{2} + 69 = 0 = \text{RHS} \end{aligned}$$

Solution 2: By the quadratic formula

$$\begin{aligned} 3x^2 - 30x + 69 &= 0 \\ 3\left(x^2 - 10x + 23\right) &= 0 \qquad \text{divide both sides by 3} \\ x^2 - 10x + 23 &= 0 \end{aligned}$$

$$a = 1 \qquad b = -10 \qquad c = 23$$

$$\begin{aligned} x_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(23)}}{2(1)} = \frac{10 \pm \sqrt{100 - 92}}{2} = \frac{10 \pm \sqrt{8}}{2} \\ &= \frac{10 \pm 2\sqrt{2}}{2} = \frac{2\left(5 \pm \sqrt{2}\right)}{2} = 5 \pm \sqrt{2} \end{aligned}$$

We check: when $x = 5 - \sqrt{2}$, then

$$\begin{aligned} \text{LHS} &= 3\left(5 - \sqrt{2}\right)^2 - 30\left(5 - \sqrt{2}\right) + 69 = 3\left(27 - 10\sqrt{2}\right) - 30\left(5 - \sqrt{2}\right) + 69 \\ &= 81 - 30\sqrt{2} - 150 + 30\sqrt{2} + 69 = 0 = \text{RHS} \end{aligned}$$

and when when $x = 5 + \sqrt{2}$, then

$$\begin{aligned} \text{LHS} &= 3\left(5 + \sqrt{2}\right)^2 - 30\left(5 + \sqrt{2}\right) + 69 = 3\left(27 + 10\sqrt{2}\right) - 30\left(5 + \sqrt{2}\right) + 69 \\ &= 81 + 30\sqrt{2} - 150 - 30\sqrt{2} + 69 = 0 = \text{RHS} \end{aligned}$$

4. Graph the straight lines $3x + 5y = -1$ and $y = -x - 1$ in the same coordinate system.

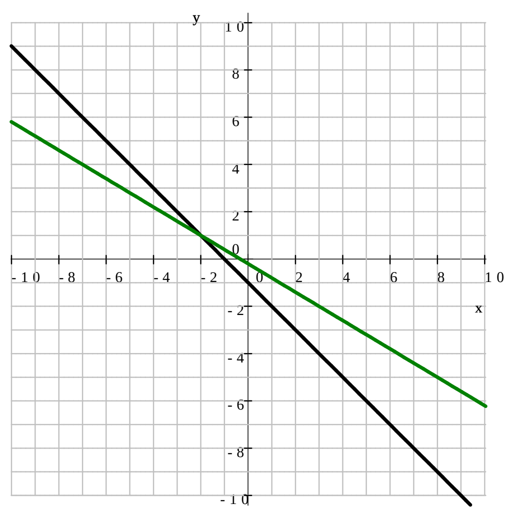
   (a) Use your graph to find the coordinates of the point where the lines intersect. $(-2, 1)$
   Solution: We start with $y = -x - 1$. The $y-$intercept is clearly $(0, -1)$. Since the slope is -1, we graph other points starting from $(0, -1)$ by stepping 1 unit to the right, 1 unit down. Now for the other line. $3x + 5y = -1$. We first bring the equation to its slope-intercept form by solving for y.

$$\begin{aligned} 3x + 5y &= -1 && \text{subtract } 3x \\ 5y &= -3x - 1 && \text{divide by } 5 \\ y &= \frac{-3x - 1}{5} = -\frac{3}{5}x - \frac{1}{5} \end{aligned}$$

   Since the $y-$intercept, $\left(0, -\frac{1}{5}\right)$ is not useful for precise graphing, we look for another convenient point. If $x = 3$, then

$$y = \frac{-3(3) - 1}{5} = \frac{-9 - 1}{5} = \frac{-10}{5} = -2$$

   We thus start at the point $(3, -2)$ and graph other points using the slope. Since the slope is $\frac{-3}{5}$, we step from $(3, -2)$ 5 units to the right, 3 units down.



   The point of intersection is $(-2, 1)$.

   (b) Use algebraic methods of checking your solution.
   Solution: We substitute $x = -2$ and $y = 1$ into both equations.

$$\begin{aligned} 1 &= -(-2) - 1 && \Longrightarrow \quad (-2, 1) \text{ is on the line } y = -x - 1 \\ 3(-2) + 5(1) &= -6 + 5 = -1 && \Longrightarrow \quad (-2, 1) \text{ is on the line } 3x + 5y = -1 \end{aligned}$$

   Thus the point $(-2, 1)$ is on both lines.

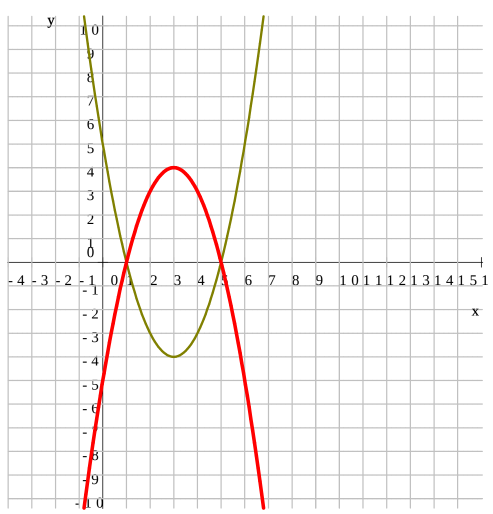
5. Graph the parabola $y = 6x - x^2 - 5$. Clearly label the coordinates of five points on the parabola, including vertex and intercepts.
Solution: After we rearrange the terms, we obtain $y = -x^2 + 6x - 5$. Clearly the y-intecept is $(0, -5)$.
To find the $x-$intercepts and vertex, we complete the square.

$$\begin{aligned} y &= -x^2 + 6x - 5 \\ y &= -\left(x^2 - 6x + 5\right) \qquad\qquad (x-3)^2 = x^2 - 6x + 9 \\ y &= -\left(x^2 - 6x + 9 - 9 + 5\right) \\ y &= -\left((x-3)^2 - 4\right) \\ y &= -\left((x-3)^2 - 2^2\right) \\ y &= -(x - 3 + 2)(x - 3 - 2) \\ y &= -(x-1)(x-5) \end{aligned}$$

Thus the $x-$intercepts are $(1, 0)$ and $(5, 0)$. However, the vertex is NOT $(3, -4)$ as it may appear for first sight.
Recall that multiplying an expression by -1 results in a reflection to the $x-$axis. If we graph the two parabolas $y = x^2 - 6x + 5$ and $y = -\left(x^2 - 6x + 5\right)$ in the same coordinate system, we see that the vertex is $(3, 4)$. This can be seen algebraically as well:

$$\begin{aligned} y &= -1\left((x-3)^2 - 4\right) \qquad \text{distribute } -1 \\ y &= -1(x-3)^2 + 4 \end{aligned}$$

When $x = 3$, y is clearly 4.



6. There is an animal farm where chickens and cows live. All together, there are 53 heads and 174 legs. How many chickens, how many cows? 19 chickens and 34 cows
Solution: Denote the number of chickens by x and the number of cows by y. The first equation will express the number of heads, the second equation will express the number of heads.

$$\begin{aligned} x + y &= 53 \\ 2x + 4y &= 174 \end{aligned}$$

To eliminate x, we multiply the first equation by -1 and divide the second equation by 2.

$$\begin{aligned} -x-y &= -53 \\ x+2y &= 87 \end{aligned}$$

Now we add the two equations.

$$y = 34$$

We use the first equation to find x.

$$\begin{aligned} x+34 &= 53 \quad \text{subtract 34} \\ x &= 19 \end{aligned}$$

Thus we have 19 chickens and 34 cows. We check: the number of heads is $19+34=53$, and the number of legs is $2(19)+4(34)=38+136=174$. So our solution is correct.

7. The area of a rectangle is $1260\ \text{m}^2$. Find the dimensions of the rectangle if we know that one side is 48 m longer than three times the other side. 14 m by 90 m
Solution: Let us denote the shorter side by x. Then the longer side is $3x+48$. We obtain the equation for the area:

$$x(3x+48) = 1260$$

Since this equation is quadratic, we will reduce one side to zero, and factor the other side to solve the equation.

$$\begin{aligned} x(3x+48) &= 1260 \quad \text{distribute} \\ 3x^2+48x &= 1260 \quad \text{subtract 1260} \\ 3x^2+48x-1260 &= 0 \quad \text{factor out the GCF, 3} \\ 3\left(x^2+16x-420\right) &= 0 \quad \text{divide by 3} \\ x^2+16x-420 &= 0 \end{aligned}$$

Factor by completing the square.

$$\begin{aligned} x^2+16x-420 &= 0 \qquad (x+8)^2 = x^2+16x+64 \\ \underbrace{x^2+16x+64}-64-420 &= 0 \\ (x+8)^2-484 &= 0 \\ (x+8)^2-22^2 &= 0 \\ (x+8+22)(x+8-22) &= 0 \\ (x+30)(x-14) &= 0 \\ x_1 &= -30 \quad \text{and} \quad x_2 = 14 \end{aligned}$$

Since distances can not be negative, $x=-30$ is ruled out. If $x=14$ m, then the other side is $3(14\ \text{m})+48\ \text{m}=90\ \text{m}$. We check: $90\ \text{m}=3(14\ \text{m})+48\ \text{m}$ and $14\ \text{m}(90\ \text{m})=1260\ \text{m}^2$. Thus the rectangle's dimensions are indeed 14 m by 90 m.

8. We invested \$10000 into two bank accounts. One account earns 14% per year, the other account earns 8% per year. How much did we invest into each account if the combined interest from the two accounts is \$1238 after the first year? \$ 7300 at 14% and \$ 2700 at 8%
Solution: Let us denote the amount invested at 14% by x and the amount invested at 8% by y. The two equations express that

$$\begin{aligned} x + y &= 10000 && \text{the amounts add up to \$10000} \\ 0.14x + 0.08y &= 1238 && \text{the interests earned add up to \$1238} \end{aligned}$$

We solve the system of equation by elimination. But let us first make the second equation simpler:

$$\begin{aligned} 0.14x + 0.08y &= 1238 && \text{multiply by 100} \\ 14x + 8y &= 123800 && \text{divide by 2} \\ 7x + 4y &= 61900 \end{aligned}$$

We now have

$$\begin{aligned} x + y &= 10000 \\ 7x + 4y &= 61900 \end{aligned}$$

We will multiply the first equation by -4 to eliminate y.

$$\begin{aligned} -4x - 4y &= -40000 \\ 7x + 4y &= 61900 \end{aligned}$$

We add the equations and solve for x.

$$\begin{aligned} 3x &= 21900 && \text{divide by 3} \\ x &= 7300 \end{aligned}$$

Thus we invested \$7300 at 14%. The other amount is then from the first equation:

$$\begin{aligned} 7300 + y &= 10000 \\ y &= 2700 \end{aligned}$$

We invested \$ 7300 at 14% and \$ 2700 at 8%. We check: the amounts add up to $\$7300 + \$2700 = \$10000$. The interest from the accounts are

$$\begin{aligned} 14\% \text{ of } 7300 \text{ is } 0.14\,(7300) &= 1022 \text{ and} \\ 8\% \text{ of } 2700 \text{ is } 0.08\,(2700) &= 216 \end{aligned}$$

Since $1022 + 216 = 1238$, our solution is correct.

9. Ann started to walk southbound in the morning, with a rate of $120\frac{\text{ft}}{\text{min}}$ (feet per minute). Ten minutes later, Betty followed her with a rate of $150\frac{\text{ft}}{\text{min}}$. How long will it take for Betty to catch up with Ann? 40 minutes

Solution: Let x denote the amount of time (in minutes) that Betty spent walking. Then Ann walked for $x+10$ minutes before they met. The distance traveled by Ann and Beety is the same, and so

$$\begin{aligned} 120(x+10) &= 150(x) && \text{solve for } x \\ 120x+1200 &= 150x && \\ 1200 &= 30x && \text{divide by 30} \\ 40 &= x && \end{aligned}$$

Thus it took 40 minutes for Betty to catch up with Ann.