Beginning Algebra with Geometry – Weeks 9 – 16

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Contents

Chapter 13

13.1 Interval Notation

Part 1 - Definitions

Some sets are small enough for us to list their elements. With larger sets, we develop notation that helps in defining the set without having to write too much. Consider the following two sets.

$$A = \{x \in \mathbb{Z} : x > 2 \text{ and } x < 7\} \qquad \text{and} \qquad B = \{x \in \mathbb{R} : x > 2 \text{ and } x < 7\}$$

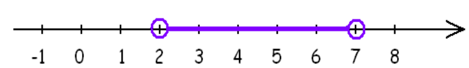
Although the definitions appear to be similar, set A is much smaller than set B. Set A contains all integers greater than 2 and less than 7. That is, A is simply the set $\{3,4,5,6\}$ containing only four elements.

Set B has many more elements, because it is the set of all *real numbers* greater than 2 and less than 7. That means that B contains numbers such as $2,5$, 2.01, 3.1, 6.999999998. If we think of the numbers 2.1, 2.01, 2.001, 2.0001, and so on, these are already infnitely many numbers in B.

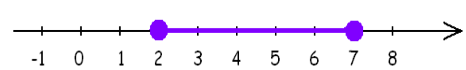
How could we describe set B with less writing? One option is to write the compound inequality $x > 2$ and $x < 7$ in a more effective form, as $2 < x < 7$. Still, we should be able to do better than $B = \{x \in \mathbb{R} : 2 < x < 7\}$.

The set of all real numbers x with $2 < x < 7$ can be also expressed using interval notation.

Definition: The set $\{x \in \mathbb{R} : 2 < x < 7\}$ is also called an **interval**, and is denoted by $(2,7)$. The endpoints of the interval, 2 and 7 are not elements of the set. Such an interval is called an **open interval**.

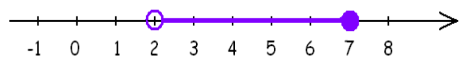


Definition: The set $\{x \in \mathbb{R} : 2 \leq x \leq 7\}$ is denoted by $[2,7]$. The endpoints of the interval, 2 and 7 belong to the set. Such an interval is called a **closed interval**.

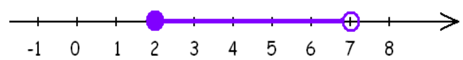


We will see in future courses that open and closed intervals have very different properties. For example, the closed interval $[2,7]$ has a smallest and greatest element. At the same time, the open interval $(2,7)$ has no smallest and greatest element. With respect to the endpoints, there are two other possibilities, as follows.

Definition: The set $\{x \in \mathbb{R} : 2 < x \leq 7\}$ is denoted by $(2,7]$.



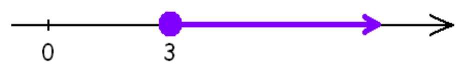
Definition: The set $\{x \in \mathbb{R} : 2 \leq x < 7\}$ is denoted by $[2,7)$.



We also often use interval notation when presenting solution sets of inequalities. Interval notation can also be applied to express simple sets that can be obtained from inequalities such as $x < 3$, $x \leq 3$, $x > 3$, or $x \geq 3$.

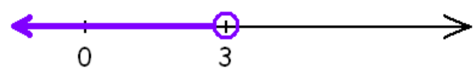
Definition: The set $\{x \in \mathbb{R} : x > 3\}$ in interval notation is denoted by $(3, \infty)$.

Definition: The set $\{x \in \mathbb{R} : x \geq 3\}$ in interval notation is denoted by $[3, \infty)$.

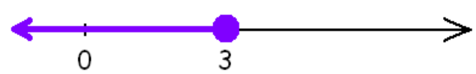


Notice that in case of both $x > 3$ and $x \geq 3$, the closing parentheses indicate that ∞ does not belong to the set. We can also consider the set $(3, \infty)$ to be an open interval. For example, this set has neither smallest nor greatest element.

Definition: The set $\{x \in \mathbb{R} : x < 3\}$ in interval notation is denoted by $(-\infty, 3)$.



Definition: The set $\{x \in \mathbb{R} : x \leq 3\}$ in interval notation is denoted by $(-\infty, 3]$.



Part 2 - Operations

Intervals are sets, and so we can perform set operations on them.

Example 1. Perform each of the following set operations on the intervals.

a) $(2,8) \cap (5,10)$ c) $[1,4] \cap (2,5)$ e) $[-1,2) \cap (3,6]$ g) $(2,8) \cap [4,7]$

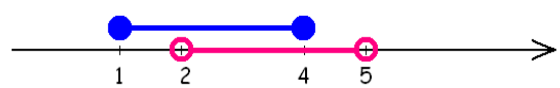
b) $(2,8) \cup (5,10)$ d) $[1,4] \cup (2,5)$ f) $[-1,2) \cup (3,6]$ h) $(2,8) \cup [4,7]$

Solution: a) Plotting the two intervals on the same number line is extremely helpful. We will use the same picture for taking unions and intersections.

The intersection of the intervals $(2,8)$ and $(5,10)$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. That is the line segment between 5 and 8. We consider the endpoints: 5 is not in both sets because 5 is not in $(5,10)$. Similarly, 8 is not in both sets because it is not in $(2,8)$. Consequently, the intersection of the two intervals is $\boxed{(5,8)}$.

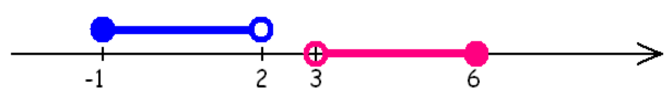
b) The union of the intervals $(2,8)$ and $(5,10)$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. That is the line segment between 2 and 10. We consider the endpoints: 2 is not in either set, so it is not in the union. Similarly, 10 is not in either set, so it is not in the union. Consequently, the union of the two intervals is $\boxed{(2,10)}$.

c) The intersection of the intervals $[1,4]$ and $(2,5)$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. That is the line segment between 2 and 4. We consider the endpoints: 2 is not in both sets because 2 is not in $(2,5)$. However, 4 is in both sets. Therefore, the intersection of the two intervals is $\boxed{(2,4]}$.



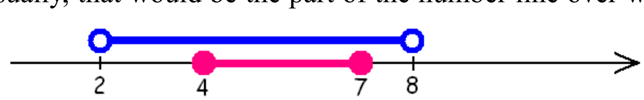
d) The union of the intervals $[1,4]$ and $(2,5)$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. That is the line segment between 1 and 5. We consider the endpoints: 1 is in $[1,4]$, so it is in the union. On the other hand, 5 is not in either set, so it is not in the union. Consequently, the union of the two intervals is $\boxed{[1,5)}$.

e) The intersection of the intervals $[-1,2)$ and $(3,6]$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. In this case, there is no number in both sets, and so the intersection is the empty set, $\boxed{\varnothing}$.



f) The union of the intervals $[-1,2)$ and $(3,6]$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. In this case, the union fails to form a single interval, and so we cannot simplify the expression either. So the answer is $\boxed{[-1,2)\cup(3,6]}$.

g) After we plotted the picture, we might notice that one interval is a subset of the other. In light of that, the answers will not be surprising. (Recall that if A and B are sets and $A \subseteq B$, then $A \cup B = B$ and $A \cap B = A$). The intersection of the intervals $(2,8)$ and $[4,7]$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. The



answer is $\boxed{[4,7]}$.

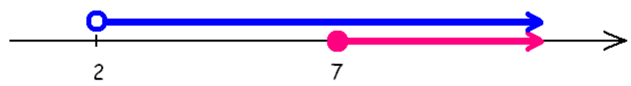
h) The union of the intervals $(2,8)$ and $[4,7]$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. In this case, the union is $\boxed{(2,8)}$.

Example 2. Perform each of the following set operations on the intervals.

a) $(2,\infty)\cap[7,\infty)$ c) $[0,\infty)\cap(-\infty,1)$ e) $(-\infty,4]\cap[9,\infty)$

b) $(2,\infty)\cup[7,\infty)$ d) $[0,\infty)\cup(-\infty,1)$ f) $(-\infty,4]\cup[9,\infty)$

Solution: a) Notice again that one interval is a subset of the other. The intersection of the intervals $(2,\infty)$ and $[7,\infty)$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. In this case, the intersection is $\boxed{[7,\infty)}$.

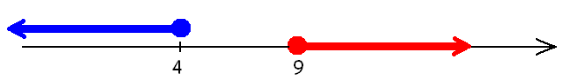


b) The union of the intervals $(2,\infty)$ and $[7,\infty)$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. That is $\boxed{(2,\infty)}$.

c) The intersection of the intervals $[0,\infty)$ and $(-\infty,1)$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. That is the line segment between 0 and 1. We consider the endpoints: 0 is in both sets but 1 is not, since 1 is not in $(-\infty,1)$.Therefore, the intersection of the two intervals is $\boxed{[0,1)}$.

d) The union of the intervals $[0,\infty)$ and $(-\infty,1)$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. In this case, this is the entire number line. Consequently, the union of the two intervals is $\mathbb{R}$, the set of all real numbers. This can be also expressed using interval notation, as $\boxed{(-\infty,\infty)}$.

e) The intersection of the intervals $(-\infty,4]$ and $[9,\infty)$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. In this case, there is no number in both sets, and so the intersection is the empty set, $\boxed{\varnothing}$.



f) The union of the intervals $(-\infty,4]$ and $[9,\infty)$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. In this case, the union fails to form a single interval, and so we cannot simplify the expression either. So the answer is $\boxed{(-\infty,4]\cup[9,\infty)}$.

Practice Problems

1. Re-write each of the given sets using interval notation.

   a) $\{x \in \mathbb{R} : x \geq 3\}$
   b) $\{m \in \mathbb{R} : -2 < m \leq 9\}$
   c) $\mathbb{R}$
   d) $\{y \in \mathbb{R} : y < -5 \text{ or } y > 4\}$
   e) $\{x \in \mathbb{R} : x < 12\}$
   f) $\{r \in \mathbb{R} : r < 10 \text{ and } r \geq 6\}$
   g) $\{x \in \mathbb{R} : x < -2 \text{ or } x > 2\}$
   h) $\{t \in \mathbb{R} : t > 0 \text{ and } t < 7\}$
   i) $\{a \in \mathbb{R} : 3 \leq a \leq 4\}$

2. Perform each of the set operations on the intervals.

   a) $(1,5) \cup (2,7)$
   b) $(1,5) \cap (2,7)$
   c) $[1,5] \cup [2,7]$
   d) $[1,5] \cap [2,7]$
   e) $[1,5] \cup (2,7)$
   f) $[1,5] \cap (2,7)$
   g) $(3,8) \cup (-1,10)$
   h) $(3,8) \cap (-1,10)$
   i) $[3,8] \cup [-1,10]$
   j) $[3,8] \cap [-1,10]$
   k) $[3,8] \cup (-1,10)$
   l) $[3,8] \cap (-1,10)$
   m) $[-2,2] \cup (4,7)$
   n) $[-2,2] \cap (4,7)$
   o) $[-2,1) \cup (0,8]$
   p) $[-2,1) \cap (0,8]$
   q) $[5,10) \cup [7,11)$
   r) $[5,10) \cap [7,11)$

3. Perform each of the set operations on the intervals.

   a) $(-\infty,4) \cup (-\infty,8)$
   b) $(-\infty,4) \cap (-\infty,8)$
   c) $(-\infty,4] \cup (-\infty,8]$
   d) $(-\infty,4] \cap (-\infty,8]$
   e) $(-\infty,4] \cup (-\infty,8)$
   f) $(-\infty,4] \cap (-\infty,8)$
   g) $(-\infty,5) \cup (3,\infty)$
   h) $(-\infty,5) \cap (3,\infty)$
   i) $(-\infty,5] \cup [3,\infty)$
   j) $(-\infty,5] \cap [3,\infty)$
   k) $(-\infty,5] \cup (3,\infty)$
   l) $(-\infty,5] \cap (3,\infty)$
   m) $(-\infty,-2) \cup (1,\infty)$
   n) $(-\infty,-2) \cap (1,\infty)$
   o) $(-\infty,-2] \cup [1,\infty)$
   p) $(-\infty,-2] \cap [1,\infty)$
   q) $(-\infty,-2] \cup (1,\infty)$
   r) $(-\infty,-2] \cap (1,\infty)$

13.2 Graph of an Equation

As we have seen more and more algebraic statements, the solution sets became increasingly more complex. A linear equation usually has a single number solution. In case of linear inequalities, we often have infinitely many solutions. To express those solution sets, we developed interval notation.

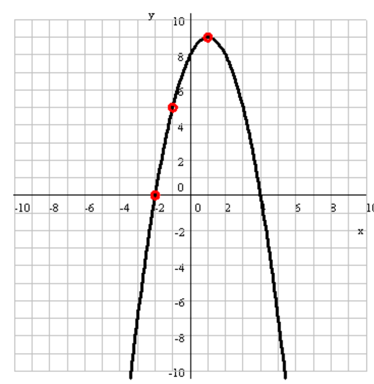
Suppose we have an equation in two variables, x and y. The equations $y = 2x - 3$ or $x^2 - y^2 = 5$ or $xy = -2$ are examples for such equations. A solution for such equations is a set of ordered pairs of numbers, (x, y). For example, $(5, 7)$ is short for $x = 5$ and $y = 7$ and this ordered pair is a solution of the equation $y = 2x - 3$. The ordered pair $(3, -2)$ is a solution of $x^2 - y^2 = 5$, and the ordered pair $(2, -1)$ is a solution of $xy = -2$.

Equations in two variables often have infintely many solutions, where that can no longer meaningfully represented on a number line. We step out into two dimensions, and use a coordinate system to depict solution sets. On a coordinate system, each ordered pair (x, y) can be represented as a point.

Definition: The **graph** of an equation in x, in y, or both in x and y is the set of all points $P(x, y)$ whose coordinates are solution of the equation.

In short, the graph of an equation is a solution set of an equation in x and y. The shape of graphs depends on the type of equation. Before we started to graph equations, it is useful to know that we can do quite a lot just using the definition of graphs.

Example 1. Consider the graph shown. Three points on the graph are marked. These are $A(-2, 0)$, $B(-1, 5)$, and $C(1, 9)$. Use these points to determine, which of the given equations is the one whose graph is the shape we see.



The possible equations offered are:

$$y = 3x + 6$$
$$(x-4)^2 + (y-5)^2 = 25$$
$$y = -x^2 + 2x + 8$$

Solution: Let us consider first the equation $y = 3x + 6$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of A, B, and C. Let's check.

Point $A(-2, 0)$ is on the graph if and only if its coordinates are a solution of $y = 3x + 6$.

Check $y = 3x + 6$ with $x = -2$ and $y = 0$.
The left-hand side is: $\text{LHS} = 0$
and the right-hand side is: $\text{RHS} = 3(-2) + 6 = 0$. $\text{RHS} = \text{LHS}$ ✓

Point A is on the graph of $y = 3x + 6$. This does not mean that $y = 3x + 6$ is the right equation. It only means that we didn't rule it out based on point A alone.

Let's see about point $B(-1, 5)$. Is this pont on the graph of $y = 3x + 6$?

Check $y = 3x + 6$ with $x = -1$ and $y = 5$.
$\text{LHS} = 5$ and $\text{RHS} = 3(-1) + 6 = -3 + 6 = 3$ $\text{RHS} \neq \text{LHS}$

At this point, we can conclude that the graph shown is not of the equation of $y = 3x + 6$, because point B is on the graph but its coordinates are not a solution of this equation. So, we can move on to the next equation.

Consider now the equation $(x-4)^2+(y-5)^2=25$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of A, B, and C. Let's check.

Point $A(-2,0)$ is on the graph if and only if its coordinates are a solution of $(x-4)^2+(y-5)^2=25$.

Check $(x-4)^2+(y-5)^2=25$ with $x=-2$ and $y=0$.

LHS $=(-2-4)^2+(0-5)^2=(-6)^2+(-5)^2=36+25=61$

RHS $=25$. RHS $\neq$ LHS

We can conclude that the graph shown is not of the equation of $(x-4)^2+(y-5)^2=25$, because point A is on the graph but its coordinates are not a solution of this equation. So, we can move on to the next equation.

Consider now the equation $y=-x^2+2x+8$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of A, B, and C. Let's check.

Point $A(-2,0)$ is on the graph if and only if its coordinates are a solution of $y=-x^2+2x+8$.

Check $y=-x^2+2x+8$ with $x=-2$ and $y=0$.

LHS $=0$ and RHS $=-(-2)^2+2(-2)+8=-4-4+8=0$. RHS $=$ LHS ✓

This does not mean that $y=-x^2+2x+8$ is the right equation. It only means that we didn't rule it out based on point A alone. Let's see point B.

Point $B(-1,5)$ is on the graph if and only if its coordinates are a solution of $y=-x^2+2x+8$.

Check $y=-x^2+2x+8$ with $x=-1$ and $y=5$.

LHS $=5$ and RHS $=-(-1)^2+2(-1)+8=-1-2+8=5$. RHS $=$ LHS ✓

This does not mean that $y=-x^2+2x+8$ is the right equation. It only means that we didn't rule it out based on points A and B. Let's see point C.

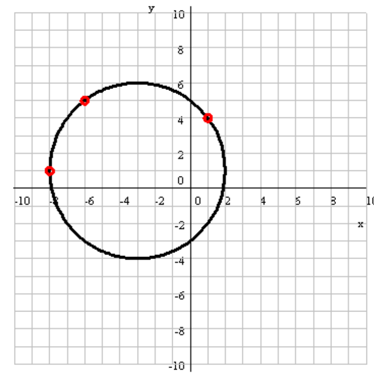
Point $C(1,9)$ is on the graph if and only if its coordinates are a solution of $y=-x^2+2x+8$.

Check $y=-x^2+2x+8$ with $x=1$ and $y=9$.

LHS $=9$ and RHS $=-1^2+2\cdot 1+8=-1+2+8=9$. RHS $=$ LHS ✓

We found that all three points are on the graph of this equation. This still does not mean that $y=-x^2+2x+8$ is the right equation. Given that we were given three equations with the assumption that the correct equation is among them, it can only be this one. So, our answer is that the graph shown is of the equation $y=-x^2+2x+8$. We can find additional nice points on the graph (for example, $(4,0)$ or $(2,8)$) and test them against the equation. Soon we will learn how to graph such shapes.

Example 2. Consider the graph shown. Three points on the graph are marked. These are $A(-8,1)$, $B(-6,5)$, and $C(1,4)$. Use these points to determine, which of the given equations is the one whose graph is the shape we see.



The possible equations offered are:

$$3y = x + 11$$
$$3y + x^2 = -8x + 3$$
$$(x+3)^2 + (y-1)^2 = 25$$

Solution: Let us consider first the equation $3y = x + 11$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of A, B, and C. Let's check.

Point $A(-8,1)$ is on the graph if and only if its coordinates are a solution of $3y = x + 11$.

Check $3y = x + 11$ with $x = -8$ and $y = 1$.

The left-hand side is: $\text{LHS} = 3 \cdot 1 = 3$

and the right-hand side is: $\text{RHS} = -8 + 11 = 3$. $\text{RHS} = \text{LHS}$ ✓

Point A is on the graph of $3y = x + 11$. This does not mean that $3y = x + 11$ is the right equation. It only means that we didn't rule it out based on point A alone. Let's see point B.

Point $B(-6,5)$ is on the graph if and only if its coordinates are a solution of $3y = x + 11$.

Check $3y = x + 11$ with $x = -6$ and $y = 5$.

$\text{LHS} = 3(-6) = -18$ and $\text{RHS} = -6 + 11 = 5$. $\text{RHS} \neq \text{LHS}$

We can conclude that the graph shown is not of the equation of $3y = x + 11$, because point B is on the graph but its coordinates are not a solution of this equation. So, we can move on to the next equation.

Consider now the equation $3y + x^2 = -8x + 3$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of A, B, and C. Let's check.

Point $A(-8,1)$ is on the graph if and only if its coordinates are a solution of $3y + x^2 = -8x + 3$.

Check $3y + x^2 = -8x + 3$ with $x = -8$ and $y = 1$.

$\text{LHS} = 3 \cdot 1 + (-8)^2 = 3 + 64 = 67$ and $\text{RHS} = -8(-8) + 3 = 64 + 3 = 67$. $\text{RHS} = \text{LHS}$ ✓

This does not mean that $3y + x^2 = -8x + 3$ is the right equation. It only means that we didn't rule it out based on point A alone. Let's see point B.

Point $B(-6,5)$ is on the graph if and only if its coordinates are a solution of $3y + x^2 = -8x + 3$.

Check $3y + x^2 = -8x + 3$ with $x = -6$ and $y = 5$.

$\text{LHS} = 3 \cdot 5 + (-6)^2 = 15 + 36 = 51$ and $\text{RHS} = -8(-6) + 3 = 48 + 3 = 51$. $\text{RHS} = \text{LHS}$ ✓

This does not mean that $3y + x^2 = -8x + 3$ is the right equation. It only means that we didn't rule it out based on points A and B. Let's see point C.

Point $C(1,4)$ is on the graph if and only if its coordinates are a solution of $3y + x^2 = -8x + 3$.

Check $3y + x^2 = -8x + 3$ with $x = 1$ and $y = 4$.

$\text{LHS} = 3 \cdot 4 + 4^2 = 12 + 16 = 28$ and $\text{RHS} = -8 \cdot 1 + 3 = -5$. $\text{RHS} \neq \text{LHS}$

We can conclude that the graph shown is not of the equation of $3y+x^2=-8x+3$, because point C is on the graph but its coordinates are not a solution of this equation. So, we can move on to the next equation.

Consider now the equation $(x+3)^2+(y-1)^2=25$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of A, B, and C. Let's check.

Point $A(-8,1)$ is on the graph if and only if its coordinates are a solution of $(x+3)^2+(y-1)^2=25$.

Check $(x+3)^2+(y-1)^2=25$ with $x=-8$ and $y=1$.
LHS $=(-8+3)^2+(1-1)^2=(-5)^2+0^2=25$ and RHS $=25$. RHS $=$ LHS ✓

This does not mean that $(x+3)^2+(y-1)^2=25$ is the right equation. It only means that we didn't rule it out based on point A alone. Let's see point B.

Point $B(-6,5)$ is on the graph if and only if its coordinates are a solution of $(x+3)^2+(y-1)^2=25$.

Check $(x+3)^2+(y-1)^2=25$ with $x=-6$ and $y=5$.
LHS $=(-6+3)^2+(5-1)^2=(-3)^2+4^2=9+16=25$ and RHS $=25$ RHS $=$ LHS ✓

This does not mean that $(x+3)^2+(y-1)^2=25$ is the right equation. It only means that we didn't rule it out based on points A and B. Let's see point C.

Point $C(1,4)$ is on the graph if and only if its coordinates are a solution of $(x+3)^2+(y-1)^2=25$.

Check $(x+3)^2+(y-1)^2=25$ with $x=1$ and $y=4$.
LHS $=(1+3)^2+(4-1)^2=4^2+3^2=16+9=25$ and RHS $=25$ RHS $=$ LHS ✓

We found that all three points are on the graph of this equation. This still does not mean that $(x+3)^2+(y-1)^2=25$ is the right equation. Given that we were given three equations with the assumption that the correct equation is among them, it can only be this one. So, our answer is that the graph shown is of the equation $(x+3)^2+(y-1)^2=25$. We can find additional nice points on the graph (for example, $(2,1)$ or $(-3,-4)$) and test them against the equation.

Practice Problems

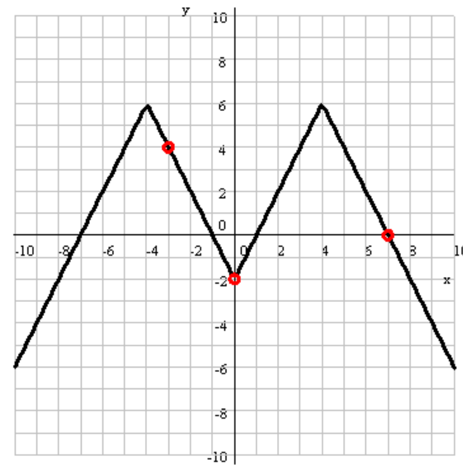
1. Consider the graph shown. Three points on the graph are marked. These are $A(-3,4)$, $B(0,-2)$, and $C(7,0)$. Use these points to determine, which of the given equations is the one whose graph is the shape we see.

   The possible equations offered are:

   $y+2=|2x|$

   $6-y=|8-|2x||$

   $x^2+y^2=1+4(x+y+5)$



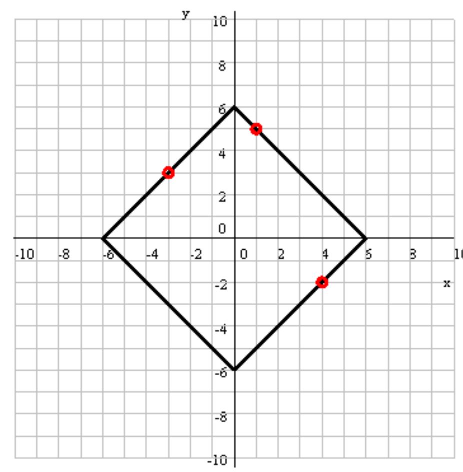
2. Consider the graph shown. Three points on the graph are marked. These are $A(-3,3)$, $B(1,5)$, and $C(4,-2)$. Use these points to determine, which of the given equations is the one whose graph is the shape we see.

   The possible equations offered are:

   $2y=x+9$

   $|x|+|y|=6$

   $y+3=9-|x|$



13.3 The Zero Product Rule

What does factoring mean and why do we do it?

Definition: To factor something means to re-write it as a product.

We factor things for several reasons. For example, reducing a fraction to lowest terms involves factoring both numerator and denominator and then cancelling out all common factors. Another, very important reason for factoring is the Zero Product Rule. It is our only method to solve equations of degree 2, 3, 4, and so on.

Theorem: (The Zero Product Rule) Suppose that we multiply some numbers and the result is zero. Then:

1.) One of the factors must be zero, and
2.) the values of all other factors are irrelevant.

Example 1. Solve the equation $(x-3)(x-7)=0$.

Solution: If we were to expand the left-hand side, we would get a quadratic expression, and so this equation is quadratic. We will solve this equation by applying the Zero Product Rule.

We are multiplying only two factors, $x-3$ and $x-7$, and the result is zero. The only way this is possible if one of the two factors is zero.

Either $x-3$ is zero (and then we can comfortably ignore the other factor, $x-7$) and solve the linear equation $x-3=0$ for x. Or, the other factor, $x-7$ is zero (and now we don't need to worry about $x-3$). Again, we solve the linear equation for x.

$$(x-3)(x-7)=0$$

$$\text{Either } x-3=0 \quad \text{or} \quad x-7=0$$

$$x=3 \quad \text{or} \quad x=7$$

Thus this equation has two solutions, $\boxed{x_1=3 \text{ and } x_2=7}$.

Example 2. Solve the equation $(x-1)(3x+1)(2x-5)=0$

Solution: We multiplied three quantities, and the result was zero. There are only three ways that can happen:

$$\text{Either} \quad x-1=0 \quad \text{or} \quad 3x+1=0 \quad \text{or} \quad 2x-5=0$$

We solve each of the linear equations for x and obtain:

$$\begin{array}{ccccc} x=1 & \text{or} & 3x+1=0 & \text{or} & 2x-5=0 \\ & & 3x=-1 & & 2x=5 \\ x=1 & & x=-\dfrac{1}{3} & & x=\dfrac{5}{2} \end{array}$$

So this equation has three solutions: $\boxed{x_1=1,\, x_2=-\dfrac{1}{3}, \text{ and } x_3=\dfrac{5}{2}}$.

We will leave checking to the reader. It is clear that when we substitute each solution into the original equation, a different factor will be zero, making the product zero.

Example 3. Solve the equation $5(x-2)(x+8)=0$.

Solution: When we apply the Zero Product Rule for the three factors, the first factor, 5 will never be zero, no matter what the value of x. So, even though we have three factors, there are only two solutions of this equation, $x=2$ and $x=-8$.

Example 4. Solve the equation $x^2(x+1)(x-3)=0$.

Solution: We can re-write the product on the left-hand side without exponents:

$$x\cdot x\cdot(x+1)(x-3)=0$$

When we apply the Zero Product Rule, the four factors will give us four solutions: 0, 0, -1, and 3. It is clear that these four factors will produce only three solutions since the first and second solutions are identical. The solutions of this equation are -1, 0, and 3.

Example 5. Write an equation with 3 and 8 as its solutions.

Solution: Consider the computation in Example 1, but backward. Then $x=3$ implies that $x-3=0$. SImilarly, $x=8$ implies $x-8=0$. To have both these numbers as solutions, we will use the zero product rule.

$$x=3 \quad \text{or} \quad x=8$$
$$x-3=0 \quad \text{or} \quad x-8=0$$
$$(x-3)(x-8)=0$$

Notice that our equation is not the only one possible. If $x=8$, then $x-8=0$ and $8-x=0$ both work. We can also include factors that don't yield solutions, for example $-2(x-3)(8-x)=0$ is also perfectly acceptable.

Practice Problems

Solve each of the following equations.

1. $(x+2)(x-5)=0$
2. $x(x-3)(x+1)=0$
3. $x(x+7)^2(x-10)^3=0$
4. $5(x+2)(x-4)=0$
5. $x(x+1)(x-1)(x+6)=0$
6. $4x^2(2x-1)(3x+7)^2(x+8)=0$
7. Write an equation (it can be in a factored form) with solutions 3 and -6.
8. Write an equation (it can be in a factored form) with solutions 0, 8 and -4.
9. Is it possible to have a seven degree equation with just one solution? Find an example.

Problem Set 13

1. Perform each of the given operations.

   a) $(-\infty,5)\cup(-\infty,10]$ c) $(3,8)\cap[4,11]$ e) $(-\infty,9)\cup[2,\infty)$ g) $(-\infty,1]\cap(3,\infty)$

   b) $(-\infty,5)\cap(-\infty,10]$ d) $(3,8)\cup[4,11]$ f) $(-\infty,9)\cap[2,\infty)$ h) $(-\infty,1]\cup(3,\infty)$

2. Simplify each of the following expressions.

   a) $(-2)^4$ b) -2^4 c) a^3a^4 d) $\left(a^3\right)^4$ e) $\left(2a^3\right)(2a)^2$ f) $\dfrac{8a^2}{16a^3}$ g) $\dfrac{(8a)^2}{16a^3}$ h) $(-2x)^2\left(-2x^2\right)$ i) $\dfrac{-3x^4}{(-3x)^2}$ j) $2^{30}\cdot 2^{30}$ k*) $2^{30}+2^{30}$

3. Organize the following quantities from smallest to greatest. You do not need to compute the values to compare them.

   $A=3\cdot 2^{20}$ $B=2\cdot 3^{20}$ $C=6^{20}$ $D=\left(2^3\right)^4$

4. Simplify each of the given expressions.

   a) $\dfrac{6^{2x+1}}{4^{x-1}\cdot 3^{2x+1}}$ b) $\dfrac{\left(-2x^4y\right)^3\left(-xy^2\right)}{\left(-2xy^2x^3\right)^2}$ c) $-(-x)\left(-x^2\right)(-x)^3$

5. Simplify each of the given expressions.

   a) $(5x-2)(-x+8)$ b) $(3x-1)^2$ c) $(3x-1)^3$ d) $\left(2x^4+1\right)^2-\left(2x^4-1\right)^2$ e) $(-2x+5)^2-(x-3)(3x+1)$ f) $\left(x^2-2\right)\left(x^2+1\right)\left(x^2-5\right)$ g) $\left(3x^5+2\right)\left(3x^5-2\right)$

6. Solve each of the following equations.

   a) $5x-3=-38$

   b) $\dfrac{2}{3}x-\dfrac{3}{4}=-\dfrac{5}{12}$

   c) $\dfrac{x-3}{7}=-2$

   d) $\dfrac{x-\frac{5}{6}}{-\frac{3}{8}}=\dfrac{4}{9}$

   e) $3(x+8)=-15$

   f) $\dfrac{3}{8}\left(x-\dfrac{2}{5}\right)=-\dfrac{3}{2}$

   g) $3(2x-5)-2(5x+3)=3x$

   h) $\dfrac{1}{2}\left(6x-\dfrac{2}{3}\right)-\dfrac{5}{6}\left(12x+\dfrac{1}{2}\right)=-\dfrac{31}{4}$

   i) $\dfrac{3}{4}x-\dfrac{1}{2}\left(\dfrac{2}{3}x-\dfrac{3}{5}\right)=\dfrac{1}{20}$

   j) $(2x-3)^2-2x(x-5)=3-(x-1)(-2x+3)$

   k) $(x+3)^2-(x-1)^2=8(x-1)$

   l) $\dfrac{\frac{2x-5}{3}+1}{-2}-1=5$

7. Solve each of the following inequalities. Present your answer using interval notation.

   a) $3(2x-5)-4(5x-3)\geq 3(2x-1)$

   b) $\dfrac{3x+2}{5}-\dfrac{5x-2}{4}>3-x$

   c) $(2x+1)^2-3x(x-2)<x^2+8x+9$

   d) $-\dfrac{2}{3}x+\dfrac{4}{5}\leq\dfrac{7}{15}$

8. Three vertices of a rectangle are given as $A(-3,2)$, $B(5,2)$, and $C(5,6)$.

   a) What are the coordinates of the fourth vertex?

   b) Find both coordinates of the point in which diagonal AC intersects diagonal BD.

9. If we increase the side of a square by 2 units, its area will increase by 20 unit^2. How long are the sides of this square?

10. In our freshman class, 24 students are studying mathematics and 20 students are studying physics, and 10 students are studying neither of these subjects.

    a) What is the smallest possible number possible for the number of students in the frashman class?

    b) What is the greatest possible number possible for the number of students in the frashman class?

    c) How many students are in this class if we also know that 15 students take both mathematics and physics?

11. We had 8000 dollars invested in stocks. First the value of our investment increased by 30%. Later the stocks lost 10% of their value.

    a) How much was our stock worth after the 10% loss?

    b) Express the two changes as a single change. What percentage of a change is this?

12. We spent the entire weekend planting flowers, all red and yellow tulips. After Saturday's work, 84% of the flowers were red. On Sunday, we only planted red tulips. By Sunday night, we have planted 800 flowers, of which 85% were red. How many flowers did we plant on Sunday?

13. Small rolls cost 2 dollars and large bagels cost 5 dollars. We decided to buy rolls and bagels, so that the number of rolls is three less than four times the number of bagels. How many of each did we buy if we paid a total of 85 dollars?

14. Compute each of the following.

    a) $-100+(-99)+\ldots+99+100+101+102$

    b) $-100\cdot(-99)\cdot\ldots\cdot 99\cdot 100\cdot 101\cdot 102$

15. Where is the error in the following computation? Suppose that $a=b=4$

$$\begin{aligned}
a &= b && \text{multiply by } a\\
a^2 &= ab && \text{subtract } b^2\\
a^2-b^2 &= ab-b^2 && \text{factor}\\
(a+b)(a-b) &= b(a-b) && \text{divide by } a-b\\
a+b &= b && \text{subtract } b\\
a &= 0 &&\\
4 &= 0 &&
\end{aligned}$$

16. *Suppose that a, b, and c are numbers such that $a^2+b^2+c^2=14$ and $ab+bc+ca=1$. Find the value of $(a+b+c)^2$

Chapter 14

14.1 Graphing a Line

Equations that are in x, or in y, or in x and y can be graphed. The graph of an equation is our way of representing a large solution set.

> **Definition:** The **graph of an equation** in x and y is the set of all points $P(x,y)$ for which the coordinates x and y form a solution of the equation.

The word linear means: "of degree one". In case of linear equations, the graph is a straight line. There are several forms of a line's equation. Two of them are as follows. There are more, and we will study them later.

$$\begin{aligned} y &= mx+b && \text{slope-intercept form} \\ Ax+By &= C && \text{general form} \end{aligned}$$

There are several methods of graphing a line. We will start with the simplest one, by finding a few points and connecting the dots.

Example 1. Graph the line $y=-2x+3$

Solution: We will find points on this line and connect the dots. Since the graph is a straight line, theoretically it doesn't matter which of its many points we will find. To safeguard against computational errors and to guarantee precision, at least four or five points should be plotted. Here is how we can find a point.

Step 1. Let us freely choose any value for x. We will go with $x=4$. We will look for a point on this line with $x-$coordinate 4.

Step 2. To find the $y-$coordinate of this point, we will use the equation of the line. Once we have a value for x, we can solve for y using the equation of the line.

$$\begin{aligned} y &= \text{ ? if } x=4 \\ x &= 4 \text{ and } y=-2x+3 \implies y=-2\cdot 4+3=-8+3=-5 \end{aligned}$$

So if $x=4$ and the point to be on the line, then y must be -5. Thus we found the point $(4,-5)$ that is on this line. We repeat the process with other values for x to find other points on the line.

Let $x=0$. We will compute the value of y.

$$\begin{aligned} y &= ? \text{ if } x=0 \\ x &= 0 \text{ and } y=-2x+3 \implies y=-2\cdot 0+3=0+3=3 \implies (0,3) \end{aligned}$$

If $x=0$, then $y=3$. Thus we found the point $(0,3)$ that is on this line.

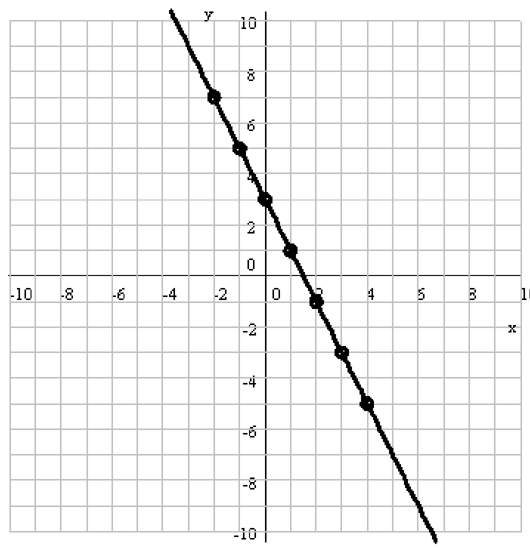
Let $x=-2$. We will compute the value of y.

$$\begin{aligned} y &= ? \text{ if } x=-2 \\ x &= -2 \text{ and } y=-2x+3 \implies y=-2(-2)+3=4+3=7 \implies (-2,7) \end{aligned}$$

If $x=-2$, then $y=7$. Thus we found the point $(-2,7)$ that is on this line.

We continue to find additional points in this manner. We plot these points and connect the dots. We organize the results in a table:

x	y	$\implies$	$P(x,y)$
-2	7		$(-2,7)$
-1	5		$(-1,5)$
0	3		$(0,3)$
1	1		$(1,1)$
2	-1		$(2,-1)$
3	-3		$(3,-3)$
4	-5		$(4,-5)$



Definition: The point where the graph intersects the $x-$axis is called **the** $x-$**intercept**. The point where the graph intersects the $y-$axis is called the $y-$**intercept**.

This line's $y-$intercept is $(0,3)$.

Example 2. Graph the line $y=\frac{1}{2}x-1$

Solution: We freely select any value for x. We find the $y-$coordinate of the point using the equation of the line.

Let $x=-4$. We will compute the value of y.

$$\begin{aligned} y &= ? \text{ if } x=-4 \\ x &= 4 \text{ and } y=\frac{1}{2}x-1 \implies y=\frac{1}{2}(-4)-1=-2-1=-3 \end{aligned}$$

If $x=-4$, then $y=-3$. Thus we found the point $(-4,-3)$ on this line. We repeat the process with other values for x to find other points on the line.

Let $x=-3$. We will compute the value of y.

$$\begin{aligned} y &= ?\ \text{if}\ x=-3 \\ x &= -3 \text{ and } y=\frac{1}{2}x-1 \implies y=\frac{1}{2}(-3)-1=-\frac{3}{2}-1=-\frac{5}{2} \end{aligned}$$

If $x=-3$, then $y=-\frac{5}{2}$. Thus we found the point $\left(-3,-\frac{5}{2}\right)$ on this line. Although the point we found is correct, its $y-$coordinate is not an integer. This makes the plotting of this point more difficult and also inaccurate. Whenever possible, we should rely on graphing lattice. points. **A lattice point is a point of whose both coordinates are integers.**

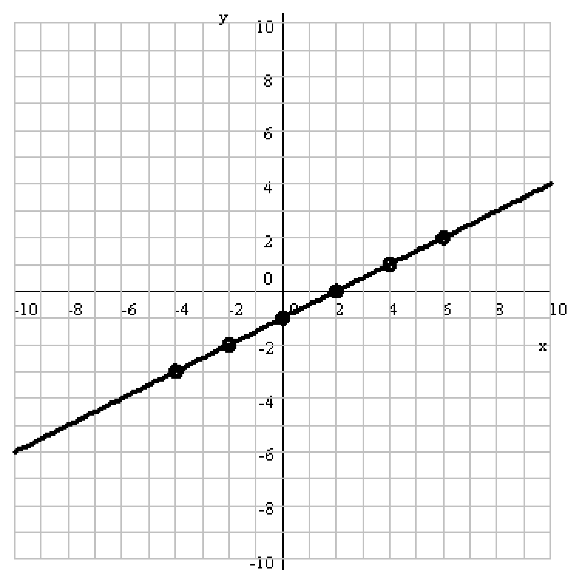
Let $x=-2$. We will compute the value of y.

$$\begin{aligned} y &= ?\ \text{if}\ x=-2 \\ x &= -2 \text{ and } y=\frac{1}{2}x-1 \implies y=\frac{1}{2}(-2)-1=-1-1=-2 \end{aligned}$$

If $x=-2$, then $y=-2$. Thus we found the point $(-2,-2)$ on this line.

We continue to find additional points in this manner. Notice that we get lattice points if we use even numbers for x. We organize the results in a table, and then plot these points and connect the dots.

x	y	$\implies$	$P(x,y)$
-4	-3		$(-4,-3)$
-3	$-\frac{5}{2}$		$\left(-3,-\frac{5}{2}\right)$
-2	-2		$(-2,-2)$
-1	$-\frac{3}{2}$		$\left(-1,-\frac{3}{2}\right)$
0	-1		$(0,-1)$
1	$-\frac{1}{2}$		$\left(1,-\frac{1}{2}\right)$
2	0		$(2,0)$
4	1		$(4,1)$
6	2		$(6,2)$



We can see from our table that this line's $x-$intercept is $(2,0)$ and $y-$intercept is $(0,-1)$.

Example 3. Graph the line $2x+3y=-12$.

Solution: We freely select any value for x. We find the $y-$coordinate of the point using the equation of the line. We substitute the value for x and solve the equation for y.

Let $x=0$. We will compute the value of y.

$$y=?\ \text{if}\ x=0$$

$$\begin{aligned} x=0 \text{ and } 2x+3y=-12 \implies 2\cdot 0+3y &= -12 && \text{solve for } y \\ 3y &= -12 && \text{divide by 3} \\ y &= -4 \end{aligned}$$

If $x=0$, then $y=-4$. Thus we found the point $(0,-4)$ on this line. We repeat the process with other values for x to find other points on the line.

Let $x=2$. We will compute the value of y.

$$y = ? \text{ if } x=2$$

$$x=2 \text{ and } 2x+3y=-12 \implies 2(2)+3y=-12 \quad \text{Solve for } y.$$

$$3y+4=-12 \quad \text{subtract 4}$$

$$y=-\frac{16}{3}$$

If $x=2$, then $y=-\frac{16}{3}$. Thus we found the point $\left(2,-\frac{16}{3}\right)$ on this line. Although the point we found is correct, its $y-$coordinate is not an integer. This makes the plotting of this point more difficult and also inaccurate. Whenever possible, we should rely on graphing lattice points. **A lattice point is a point of whose both coordinates are integers.**

Let $x=3$. We will compute the value of y.

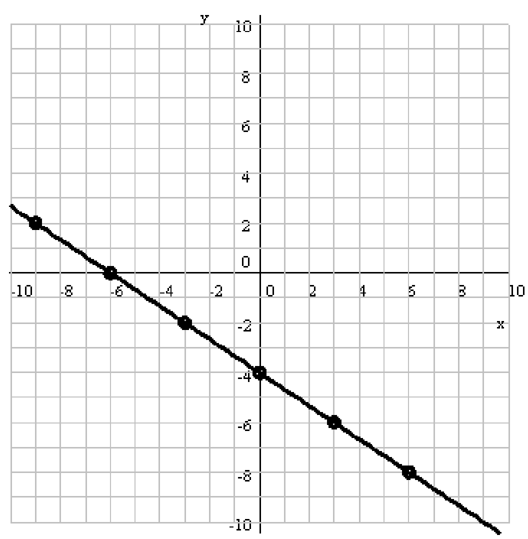
$$\begin{aligned} y &= ? \text{ if } x=3 \\ x &= \text{ and } 2x+3y=-12 \implies 2(3)+3y=-12 \end{aligned}$$

$$\begin{aligned} 2(3)+3y &= -12 && \text{solve for } y \\ 3y+6 &= -12 && \text{subtract 6} \\ 3y &= -18 && \text{divide by 3} \\ y &= -6 \end{aligned}$$

If $x=3$, then $y=-6$. Thus we found the point $(3,-6)$ on this line.

We continue to find additional points in this manner. Notice that we get lattice points if we use numbers for x that are divisible by 3. We organize the results in a table. We plot the lattice points and connect the dots.

x	y	$\implies$	$P(x,y)$
-6	0		$(-6,0)$
-3	-2		$(-3,-2)$
0	-4		$(0,-4)$
3	-6		$(3,-6)$
-9	2		$(-9,2)$
-12	4		$(-12,4)$



This line's $x-$intercept is $(-6,0)$ and $y-$intercept is $(0,-4)$.

Example 4. Graph the line $y = -2$

Solution: We freely select any value for x. We find the $y-$coordinate of the point using the equation of the line. We substitute the value for x and solve the equation for y.

$$\begin{aligned} y &= \ ? \text{ if } x = 0 \\ x &= 0 \text{ and } y = -2 \implies P(0,-2) \end{aligned}$$

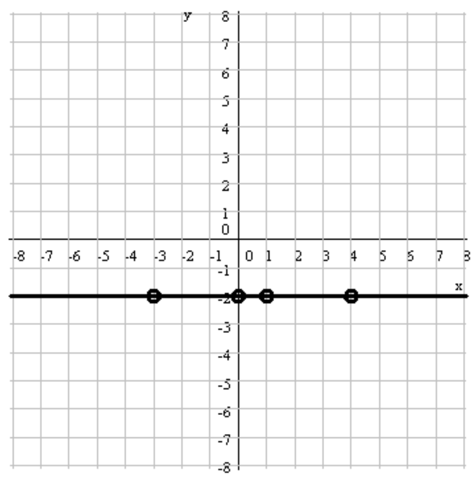
If $x = 0$, then $y = -2$. Thus we found the point $(0,-2)$ on this line. We repeat the process with other values for x to find other points on the line.

$$\begin{aligned} y &= \ ? \text{ if } x = -3 \\ x &= -3 \text{ and } y = -2 \implies P(-3,-2) \end{aligned}$$

If $x = -3$, then $y = -2$. Thus we found the point $(-3,-2)$ on this line.

We continue to find additional points in this manner. It is clear that no matter what the value of x is, y will always be -2. We organize the results in a table. We plot the lattice points and connect the dots.

x	y	$\implies$	$P(x,y)$
-3	-2		$(-3,-2)$
0	-2		$(0,-2)$
1	-2		$(1,-2)$
4	-2		$(4,-2)$



This line's $y-$intercept is $(0,-2)$ and it does not have an $x-$intercept.

Practice Problems

Graph each of the following lines.

1. $3x + 2y = 6$
2. $x = -4$
3. $y = \frac{2}{5}x - 3$
4. $2x - 3y = 10$
5. $y = 1$
6. $y = 3x + 6$
7. $3x + 5y = -30$
8. $2x - y = 7$
9. $y = \frac{1}{3}x$

14.2 Factoring out the GCF and -1

Definition: To **factor** something means to re-write it as a product.

Factoring will be a very important step in solving many types of problems. Most importantly, factoring is key in solving equations of degree 2 (also called *quadratic*), degree 3 (also called *cubic*), degree 4, and so on. This is because of the zero product rule. Let us recall this rule first.

Theorem: Suppose that we multiply some numbers and the result is zero.
Then:
1.) One of the factors must be zero, and
2.) the values of all other factors are irrelevant.

This property is only true for zero. Suppose that the product of two numbers is 100. The value of the two factors depend on each other. Let's say we start with $1 \cdot 100$. If we increase the first factor, the second factor must decrease, as in $2 \cdot 50$ or $5 \cdot 20$. It is a balancing act. Only zero has the very special property that allows us to focus on only one factor while ignoring all other factors.

For example, the zero product rule can be used to solve the equation $(x+3)(x-1)=0$. If two factors multiply to zero, one of the factors must be zero. So, there are only two possibilities: either $x+3=0$ (and we don't need to worry about the second factor), or $x-1=0$ (and we don't need to worry about the value of the first factor.) The zero product rule allowed us to trade in one quadratic (of degree 2) equation for two linear equations: $x+3=0$ and $x-1=0$. We solve these equations and obtain -3 and 1 as solution.

Equations with degree 2, 3, 4, 5, and beyond can be solved by the zero product rule. So, if an equation is of a degree higher than 1, we will reduce one side to zero, factor the other side and apply the zero product rule. For this reason, factoring algebraic expressions is a very important task.

There are many factoring techniques, and we will learn many of them. Different techniques work on different expressions. The process of factoring starts with inspecting the expression to decide which techiques would work. There is one exception to this: in all cases, our first step must be **factoring out the greatest common factor**. We will see later examples in which the additional techniques can not even be applied unless we factor out the greatest common factor or GCF first.

Recall the distributive law:

Axiom (The Distributive Law): For all real numbers a, b, and c,

$$a(b+c) = ab+ac$$

Consider the expression $2(5x-9)$. We can apply the distributive law to expand this expression:

$$2(5x-9) = 10x-18$$

Factoring out the greatest common factor is the reversal of this process.

Example 1. Factor out the greatest common factor in $12x-18$.

Solution: The first step is to identify the greatest common factor or GCF. Both $12x$ and -18 are divisible by 6.

We write $6(\qquad)$ and the rest is a few division problems.

We ask: 6 times what will give us $12x$? The answer is $2x$ because $6\cdot 2x=12x$. Similarly, 6 times what will give us -18? The answer is -3. We can now write:

$$12x-18=\boxed{6(2x-3)}$$

After we wrote down what we think the answer is, we need to ask two questions. Does the multiplication backward work? Did we get all common divisors out? We distribute 6 in $6(2x-3)$ and see that we get the correct product. If we inspect $2x-3$, we see that the two terms do not share any divisors, and so we did factor out the greatest common factor.

Example 2. Factor out the greatest common factor in $10a^3b^2-5ab+30ab^3$.

Solution: We first identify the greatest common factor between the three terms in $10a^3b^2-5ab+30ab^3$. The numbers multiplying the variables, also called coefficients are $10, -5$, and 30. Their greatest common factor is 5. Then we look for $a-$powers. The first term is divisible by a^3, the second term by a, and the third term by a. The greatest common factor between them is a. Similarly, the greatest common factor of b^2, b, and b^3 is b. Therefore, the greatest common factor is $5ab$. So we write $5ab(\qquad)$ and the rest is three division problems.

$$10a^3b^2-5ab+30ab^3=5ab(\qquad)$$

We will need to write three terms into the parentheses. In case of all factoring, we usually ask: does the multiplication backward work? $5ab$ must be multplied by what, so that the product is $10a^3b^2$. The answer is $2a^2b$. So now we have:

$$10a^3b^2-5ab+30ab^3=5ab\left(2a^2b\qquad\right)$$

Once we wrote down the first term, we can check whether the multiplication backwards work. For the second term, $-5ab$, nearly everything was factored out. If this happens, we are left with 1. In this case, we are left with -1.

$$10a^3b^2-5ab+30ab^3=5ab\left(2a^2b-1\qquad\right)$$

For the third term, we ask: $5ab$ times what is $30ab^3$? The answer is $6b^2$, and so we have

$$10a^3b^2-5ab+30ab^3=\boxed{5ab\left(2a^2b-1+6b^2\right)}$$

We ask the two questions. *Does the multiplication backward work?* and *Did we get all the common factors out?* Applying the distributive law, we see that the multiplication backward does work. Inspecting the three terms inside the parentheses, we see that they do not share any divisors. This is especially easy, given that the second term is -1. Thus our solution is correct.

Sometimes we will need to factor out -1 from an expression. This step is usually needed when the coefficient of the highest degree term is -1.

Example 3. Factor out -1 from $8x^5-x^6+3x-2$.

Solution: It is always a good idea to rearrange the terms by degree. Then we write $-1(\qquad)$. Inside the parentheses, we write the opposite of our expression, i.e. change all signs.

$$8x^5-x^6+3x-2=-x^6+8x^5+3x-2=\boxed{-1\left(x^6-8x^5-3x+2\right)}$$

We often omit the 1 and write only $-\left(x^6-8x^5-3x+2\right)$.

Sometimes the greatest common factor is more complicated.

Example 4. Factor out the GCF from $12a^3(a-2)-6a^2(a-2)+24(a-2)$.

Solution: In this case, $a-2$ is part of the GCF. We factor it out:

$$12a^3(a-2)-6a^2(a-2)+24(a-2)=(a-2)\left(12a^3-6a^2+24\right)$$

If we look at the expression in the second pair of parentheses, we see that there is a common factor of 6. Thus the final answer is

$$(a-2)\,6\left(2a^3-a^2+4\right)=\boxed{6(a-2)\left(2a^3-a^2+4\right)}$$

Factoring out the GCF must always be the first step in factoring. In case of the next example, this is all we need.

Example 5. Solve the equation $x^2=6x$

Solution: We realize that this is a quadratic equation. Therefore, we need to reduce one side to zero, factor, and apply the zero product rule. The number multiplying the variables in the highest degree term is called **the leading coefficient**. When reducing one side to zero, we should try to avoid creating negative leading coefficients. In this case, we should subtract $6x$ from both sides.

$$\begin{aligned} x^2 &= 6x && \text{subtract } 6x \\ x^2-6x &= 0 && \text{factor out the GCF} \\ x(x-6) &= 0 \end{aligned}$$

We apply the zero product rule to the two factors:

$$x=0 \quad \text{or} \quad x-6=0$$
$$x=6$$

Therefore, there are two solutions, $\boxed{0 \text{ and } 6}$. We check: if $x=0$, then both sides are zero. If $x=6$, then both sides are 36. Thus our solution is correct.

Example 6. Find all numbers with the following property. The number raised to the third power is five times the number we get if we double the number and then square the result.

Solution: We label this number by x. Then the number raised to the third power is x^3. If we double the number, we get $2x$. We write the equation comparing the square of $2x$ and x^3.

$$\begin{aligned} 5\left((2x)^2\right) &= x^3 \\ 5\left(4x^2\right) &= x^3 \\ 20x^2 &= x^3 && \text{subtract } 20x^2 \\ 0 &= x^3-20x^2 && \text{factor out the GCF} \\ 0 &= x^2(x-20) && \text{apply the zero product rule} \end{aligned}$$

$$x=0 \ \text{ or } \ x=20$$

So there are two such numbers: $\boxed{0 \text{ and } 20}$. We check: 0 clearly works. If the number is 20, it raised to the third power is $20^3=8000$. If we double 20, we get 40. The square of 40 is $40^2=1600$, and indeed 8000 is five times 1600, thus our solution is correct.

Sample Problems

1. Factor out the greatest common factor in each of the given expressions.

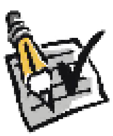
   a) $3x-12$
   b) $16a^2b+20a^3b-12a^2b^2$
   c) $3a^2-12$
   d) $3a^3-12a^2$
   e) $20x+5x^3$
   f) $3x(x-2)+8x^3(x-2)-11(x-2)$

2. Factor out -1 from $-5x^3+2x^2-x-8$.

3. Solve each of the following equations. Make sure to check your solution.

   a) $(x-2)(x+3)(2x+1)=0$ b) $m(m+7)=0$ c) $x^2=9x$ d) $8x^3=50x^2$

4. Find all numbers that satisfy the following condition: if we square the number, we get back the same number.



Practice Problems

1. Factor out the greatest common factor from each of the following.

   a) $10a^2b^2-15ab^3+25a^2b^3c$
   b) $6x^3-3x^2-15x^4$
   c) $a^2-a^3+a^4$
   d) $6a^2b+12a^3b-30a^3b^2$
   e) $x^5-2x^4+4x^3$
   f) $3xy(a-3)+8t(a-3)-200x^5(a-3)$

2. Factor out -1 from each of the following.

   a) x^3-x^5+2 b) $-x^2+3x-1$ c) $-x^2+3x-5$

3. Solve each of the following equations. Make sure to check your solutions.

   a) $(w+5)(w-1)=0$
   b) $x(x-2)(x+3)=0$
   c) $2(x-2)(x+3)=0$
   d) $x^2=4x$
   e) $x^2+6x=0$
   f) $3x^3=75x^2$

4. A number has the following property: if we square it, we obtain the opposite of the number. Find all such numbers.

Problem Set 14

1. Let $U = \{1,2,3,4,5,6,7,8,9,10,11,12\}$, $A = \{2,5,6,7,9,11\}$, $B = \{1,5,6,8,9,10,12\}$, and $C = \{2,4,5,7,9,12\}$
   a) Draw a Venn diagramm depicting these sets.
   b) Find each of the following.
   i) $A \cap (B \cup C)$ ii) $(A \cap B) \cup (A \cap C)$ iii) $A \cup (B \cap C)$ iv) $(A \cap B) \cap C$

2. Simplify each of the following.
   a) $(-\infty,5) \cap [-2,\infty)$ c) $[3,10] \cup (7,12)$ e) $(-\infty,7] \cap (9,\infty)$ g) $(2,\infty) \cup [5,\infty]$
   b) $(-\infty,5) \cup [-2,\infty)$ d) $[3,10] \cap (7,12)$ f) $(-\infty,7] \cup (9,\infty)$ h) $(2,\infty) \cap [5,\infty]$

3. Label each of the following statements as true or false.
   a) If n is a positive integer with $n \geq 2$, then the prime factorization of n^2 will have only even exponents.
   b) If n is a positive integer with $n \geq 2$, then all exponents in the prime factorization of n^3 are divisible by 3.
   c) For any pairs of numbers a and b, if x is the greatest common factor of a and b, and y is the least common multiple of a and b, then x is a factor of y.
   d) Suppose that a and b are integers such that both a and b are divisible by 5. Then $a+b$ is also divisible by 5.
   e) Suppose that a and b are integers such that $a+b$ is divisible by 5. Then both a and b are divisible by 5.

4. a) List all factors of 150.
   b) List all prime numbers between 50 and 70.
   c) Which of the following numbers is not a prime? $29, 79, 89, 109, 119$

5. a) Find the prime factorization of each of the following. i) 180 ii) 1575 iii) 80^{100}
   b) Compute the greatest common factor and least common multiple of 180 and 1575.
   c*) The greatest common factor of 48 and x is 6. The least common multiple of 48 and x is 720. What values are possible for x?
   d*) Is it possible for two numbers to have their greatest common factor be equal to their least common multiple?

6. Perform the operations as indicated.
   a) $\dfrac{\frac{3}{4} + \frac{8}{15} \div \left(-\frac{2}{5}\right)}{2\frac{1}{3}}$
   b) $4 - \left|\left|2^2 - (-2)^4\right| - 5^2\right|$
   c) $\dfrac{2\left((-2)^2 - (3^2 - 3)\right)}{-2^2} - \dfrac{(-3^2+2)\,3}{3-(-4)}$
   d) $\dfrac{\left|\left|-2^2+2\right|-5\right|-1}{-2^2 - \left((-3)^3 + 5^2\right)}$
   e) $\dfrac{(-2)^3 + (-2)^4 + (-2)^5 + (-2)^6}{3^2 - (-2)^2}$
   f) $\dfrac{3}{10} + \left(3 + \dfrac{3}{5}\right) \div \left(1 + \dfrac{1}{3}\right)$

7. Suppose that $A = 32\,000\,000$ and $B = 1250\,000$. Compute each of the following. Present your answer using scientific notation.
   a) A^2 b) AB c) $\dfrac{A}{B}$

8. Evaluate the expression $\dfrac{11a-2a^2-15}{2a-5}$ if

   a) $a=0$ b) $a=2$ c) $a=\dfrac{1}{3}$ d) $a=2\dfrac{1}{2}$ e) $a=-\dfrac{1}{2}$ f) $a=3$ g) $a=1\dfrac{1}{2}$

9. Simplify each of the following. (Re-write it as a simpler exponential expression.)

   a) $2^{100}+2^{103}$ b) $5\cdot 3^{100}-3^{101}$ c) $3^{100}+3^{100}+3^{100}$

10. Simplify each of the following.

    a) $-2a^3(-2a)^2(-2a)^3$

    b) $\left(-x(-2x)^2\left(-\dfrac{1}{2}x\right)\right)^3$

    c) $\left(-2xy^3\right)^2 x(-y)^5 x^2$

    d) $\dfrac{\left(3ba^4b^2\right)^5\left(-2a^3b\right)^3}{\left(-2b^4\right)^2\left(6a^7b^2\right)^4}$

    e) $\dfrac{-\left(x^3(-2x)^2\right)^4}{(-4x)\left(-2x^3\right)^5}$

    f) $\left(\dfrac{2a^2b^7}{-3ab^5}\right)^2\left(\dfrac{-6a^3b^{10}}{4a^2b^8}\right)^4$

    g) $(5x+2)(x-3)$

    h) $(5a-1)^2$

    i) $(a-b)\left(a^3+a^2b+ab^2+b^3\right)$

    j) $\left(3a^5-1\right)\left(3a^5+1\right)$

    k) $(2x-3)^3$

11. Factor out -1 in each of the given expressions.

    a) x^3-2x^4+5x-1 b) $-5y-1$ c) $-3m+5$ d) $a-b$

12. Factor out the greatest common factor in each of the following.

    a) $10a^2b^3-5ab^4c+5ab^3$

    b) $2x^5-12x^4+6x^3$

    c) $6a^2bc-2ab^3c+12a^4b^3c^2$

    d) $4p^2q-6pq+9pq^2$

    e) $x^4-5x^3+x^2$

    f) $3x(a-3)-6x^2(a-3)+12(a-3)$

    g) $8x(5x-2)-(5x-2)$

13. Which one is greater? (Hint: factor out the greatest common factor!)

    a) 6^{100} or $2^{99}\cdot 3^{101}$ b) $2^{100}\cdot 5^{100}$ or 3^{200} c) 3^{120} or 2^{200}

14. Solve each of the following equations. Make sure to check your solutions.

    a) $\dfrac{2}{3}x+\dfrac{3}{5}=-\dfrac{1}{15}$

    b) $2x-3(x-1)=7-x$

    c) $x^2+x=0$

    d) $\dfrac{2x+1}{5}-\dfrac{5-x}{2}=x-1$

    e) $-3(2x-1)=2(x+1)-(x-1)$

    f) $a^2+5a=0$

    g) $36m^4=4m^3$

    h) $\dfrac{x-1}{2}-3+x=\dfrac{3x-7}{2}$

    i) $2-(3-x)(2x+5)=(x-1)(2x-1)$

    j) $(2x-1)^2+x(x-6)=(x+1)^2$

    k) $\dfrac{\dfrac{3x-1}{5}-1}{2}=3$

15. Graph each of the following.

    a) $y=-\dfrac{1}{2}x+3$ b) $3x+2y=-12$ c) $x+y=4$

16. Consider the equations $2x+y=-1$ and $x-y=-5$.

    a) Graph these lines in the same coordinate system. Use your graph to find both coordinates of the point where the lines intersect each other.

    b) Use algebraic methods to check your answer for part a).

17. A TV went on an 15% off sale. The sale price was 340 dollars. What was the original price?

18. The difference between two numbers is 7, their sum is 37. Find these numbers.

19. A total of \$20 000 is to be invested in bonds and stocks. If the amount invested in bonds is to be \$4500 more than the amount invested in stocks, how much money is invested in each category?

20. The tickets for the field trip were purchased yesterday for both students and instructors. Children tickets cost \$5, adult tickets cost \$12. The number of children ticket purchased was three less than four times the number of adults tickets purchased. How many of each were purchased if all of the tickets cost a total of \$209 dollars?

21. Ann and Betty are roommates. The monthly rent is \$950. The amount paid by Ann is \$310 less than twice the amount paid by Betty. How much do they each pay for rent?

22. One side of a rectangle is 4 ft shorter than three times the other side. Find the sides if the perimeter is 64 ft.

23. If we square a number, we get nine times the original number. Find all numbers with this property.

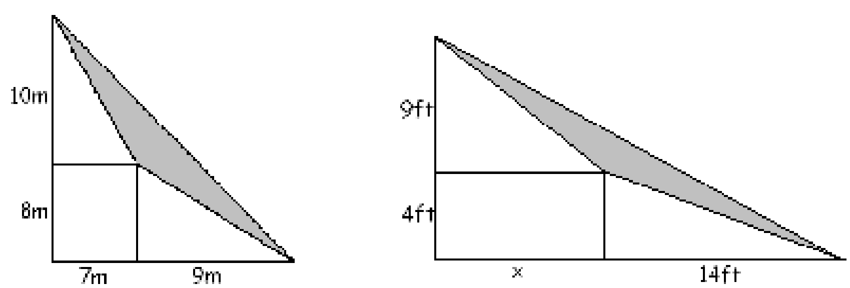
24. Shawn is thinking of a number. If we add one to the number and then square, the result is five less than the square of the number. Of what number is Shawn thinking?

25. A number is four less than the sum of -8 and twice the opposite of the number. Find this number.

26. Find the area of the triangle determined by the points $A(-5,-2)$, $B(7,-2)$, and $C(3,6)$.

27. a) Compute the area of the shaded region shown on the picture. Angles that look like right angles are right angles.

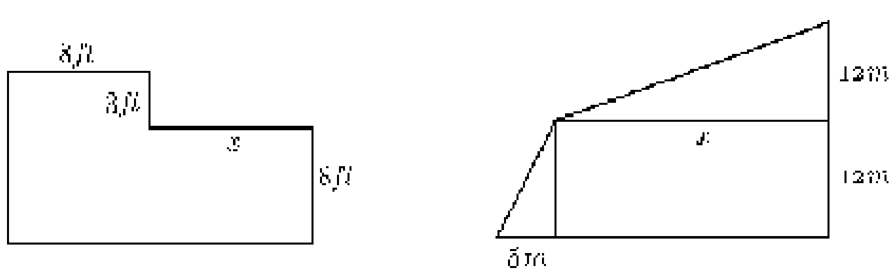
    b) Find the value of x if we know that the shaded region shown on the picture has area 43 ft^2. Angles that look like right angles are right angles.



a) b)

28. a) Find the value of x if the area of the object shown on the picture below is 168 ft^2.

    b) Find the value of x if the area of the object shown on the picture below is 534 m^2.



a) b)

Chapter 15

15.1 The Difference of Squares Theorem

Recall the zero product rule.

> **Theorem:** Suppose we multiply several numbers, and the product is zero. Then
> 1. One of the factors must be zero; and
> 2. The value of all other factors is irrelevant.

This property is very special – only a product of zero allows us to fixate on one factor while ignoring the other factors. No other number has this property. Because of this property, factoring is an essential tool for solving equations of degree 2, 3, 4, and on.

This is one of the reasons why we find factoring so useful. So far, we have learned to factor out the greatest common factor (or GCF) from an expression. We will now further progress. However, factoring the greatest common factor remains important and it must be the first thing we consider when we factor n expression. Often times, we will make several steps. If factoring out the GCF is needed, it should always be the first thing we do, otherwise we might not apply the other techniques. Now we will see one new factoring technique, the difference of squares theorem.

Consider the expression $2a-1$. When asked to find the opposite of this expression, a common error is to think that the opposite of $2a-1$ is $2a+1$. This is of course incorrect. To get to the opposite, we must multiply by -1 and then apply the distributive law. The opposite of $2a-1$ is $-2a+1$.

However, the relationship between $2a-1$ and $2a+1$ is important enough for it to have a name.

> **Definition:** Suppose we are given an algebraic expression that is a sum (or a difference). A **conjugate** of the expression is obtained by changing the sign in front of just one term in the expression.

For example, a conjugate of $x+3$ is $x-3$. Another possible conjugate of $x+3$ is $-x+3$. As long as the expression is organized so that the variable is first, we will be in the habit of changing the second sign. This is not a strict rule however, just a habit.

Conjugates are very useful in algebra. Because of the nearly identical terms and alternating signs, working with conjugates results in cancellations.

Example 1. Consider the expressions $x+3$ and $x-3$. Add, subtract, and multiply these expressions. Simplify your answer.

Solution: a) Let us add the conjugates. We drop the parentheses and combine like terms.

$$(x+3)+(x-3)=2x$$

In this case, the first term doubles up, and the second term is canceled out.

b) Now we will add the conjugates. To subtract is to add the opposite.

$$(x+3)-(x-3)=x+3-x+3=6$$

Now the first term is cancelled out and the second term doubles.

c) Now we multiply the conjugates. We apply FOIL (first, outer, inner last) and then combine like terms, usually O and I from FOIL.

$$(x+3)(x-3)=x^2-3x+3x-9=x^2-9$$

Because of the symmetries, when we combine the like terms from O and I, we have complete cancellation. Only conjugates do this.

When we multiply conjugates, O and I in FOIL completely cancel out each other. We are left with the first and last terms only. To express this in more general terms, $(a+b)(a-b)=a^2-ab+ab-b^2=a^2-b^2$. This is a fairly reasonable statement. If we start out with conjugates, the symmetries and differences result in such a cancellation. We get a much more suprising statement if we swap the two sides.

Theorem: (The difference of squares theorem) For all quantities a and b,

$$a^2-b^2=(a+b)(a-b)$$

In words, the difference of two squares can always be factored into a pair of conjugates.

Example 2. Completely factor each of the following expressions.

a) $3a^2-12$ b) $2x^2-50y^2$ c) x^2-1 d) x^2+1 e) $-x^6+49$

Solution: a) Consider the expression $3a^2-12$. We start with the greatest common factor (or GCF). In this case, the GCF is 3.

$$3a^2-12=3\left(a^2-4\right)$$

What is in the parentheses, a^2-4, can be further factored via the difference of squares theorem.

$$3\left(a^2-4\right)=3\left(a^2-2^2\right)=3(a+2)(a-2)$$

The expressions in neither parentheses can be further factored and so we are done. We check our work by multiplication:

$$3(a+2)(a-2)=3\left(a^2-2a+2a-4\right)=3\left(a^2-4\right)=3a^2-12$$

and so our answer, $\boxed{3(a+2)(a-2)}$ is correct.

Notice that $3a^2$ and 12 are not squares. Therefore, if we did not start with the greatest common factor, then we couldn't apply the difference of squares theorem. It is essential that we always start with the greatest common factor.

b) Consider the expression $2x^2-50y^2$. We start with the greatest common factor (or GCF). In this case, the GCF is 2. We factor it out:

$$2x^2-50y^2=2\left(x^2-25y^2\right)$$

Now x^2-25y^2 can be factored further via the difference of squares theorem.

$$2\left(x^2-25y^2\right)=2\left(x^2-(5y)^2\right)=2(x+5y)(x-5y)$$

The expressions in neither parentheses can be further factored and so we are done. We check our work by multiplication:

$$2(x+5y)(x-5y)=2\left(x^2-5xy+5xy-25\right)=2\left(x^2-25y^2\right)=2x^2-50y^2$$

and so our answer, $\boxed{2(x+5y)(x-5y)}$ is correct.

c) Consider the expression x^2-1. This is the simplest and possibly the most famous difference of two squares. We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. However, x^2-1 can be factored via the difference of squares theorem.

$$x^2-1=x^2-1^2=(x+1)(x-1)$$

The expressions in neither parentheses can not be further factored and so we are done. We check our work by multiplication:

$$(x+1)(x-1)=x^2-x+x-1=x^2-1$$

and so our answer, $\boxed{(x+1)(x-1)}$ is correct.

It is a common mistake to cofuse x^2-1 with $(x-1)^2$. The expression $(x-1)^2$ is a complete square in which O and I from FOIL are identical, so they double up. Only conjugates cause cancellation of the $x-$terms.

$$\begin{aligned}(x-1)^2 &= (x-1)(x-1)=x^2-x-x+1=x^2-2x+1\\ x^2-1 &= (x+1)(x-1)\end{aligned}$$

d) Consider the expression x^2+1. We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. In addition, x^2+1 can NOT be factored via the difference of squares theorem. **The sum of two squares can never be factored.** So, there is nothing that can be done here, and the final answer is $\boxed{x^2+1}$.

e) Conside the expression $-x^6+49$. We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. Before we proceed any further, we rearrange the terms so that the difference of squares becomes easier to observe.

$$-x^6+49=49-x^6$$

This factors via the difference of sqaures theorem. It is x^3 that we need to square to obtain x^6.

$$49-x^6=7^2-\left(x^3\right)^2=\left(7+x^3\right)\left(7-x^3\right)$$

What is in both parentheses, $7+x^3$ and $7-x^3$ can not be further factored and so we are done. We can easily check our work by multiplication:

$$\left(7+x^3\right)\left(7-x^3\right)=49-7x^3+7x^3-x^6=49-x^6$$

and so our answer, $\boxed{\left(7+x^3\right)\left(7-x^3\right)}$ is correct.

Please note that there is another method to solve this problem that might be more strategic. When dealing with algebraic expressions, we prefer to have the unknown first, in descending order of degrees, and then the number. Another option we have is to factor out -1 rigth away. This way we are dealing with a much more familiar situation

$$49-x^6=-x^6+49=-1\left(x^6-49\right)=-\left(\left(x^3\right)^2-7^2\right)=-\left(x^3+7\right)\left(x^3-7\right)$$

Example 3. Completely factor the expression $2p^4-162$.

Solution: We start with the greatest common factor (or GCF).

$$\begin{aligned}
2p^4-162 &= 2\left(p^4-81\right) && \text{re-write both quantities as squares}\\
&= 2\left(\left(p^2\right)^2-9^2\right) && \text{factor via the difference of squares theorem}\\
&= 2\left(p^2+9\right)\left(p^2-9\right) && \text{the second factor will factor again, the first will not!}\\
&= 2\left(p^2+9\right)\left(p^2-3^2\right) && \text{factor via the difference of squares theorem}\\
&= 2\left(p^2+9\right)(p+3)(p-3)
\end{aligned}$$

We check by multiplication:

$$\begin{aligned}
2\left(p^2+9\right)\underbrace{(p+3)(p-3)}_{\text{FOIL}} &= 2\left(p^2+9\right)\left(p^2-3p+3p-9\right)=2\underbrace{\left(p^2+9\right)\left(p^2-9\right)}_{\text{FOIL}}\\
&= 2\left(p^4-9p^2+9p^2-81\right)=2\left(p^4-81\right)=2p^4-162
\end{aligned}$$

Thus our solution, $\boxed{2\left(p^2+9\right)(p+3)(p-3)}$ is correct.

Example 4. Solve each of the following equations. Make sure to check your solution.

a) $x^2=9$ b) $x^4=9x^3$ c) $8x^3=50x^2$ d) $8p^3=50p$

Solution: a) Since the equation $x^2=9$ is of a higher degree than one, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned}
x^2 &= 9 && \text{subtract 9}\\
x^2-9 &= 0 && \text{factor via the difference of squares theorem}\\
x^2-3^2 &= 0\\
(x+3)(x-3) &= 0
\end{aligned}$$

A product can only be zero if one of its factors is zero. $(x+3)(x-3)=0$ means that either $x-3=0$ or $x+3=0$. We solve these linear equations separately and obtain $\boxed{3 \text{ and } -3}$. We check: $3^2=9$ and $(-3)^2=9$.

Note: one could ask why the four steps if we could just conclude from $x^2 = 9$ that then $x = \pm 3$. This shortcut, called the square root property, is perfectly fine, as long as we remember that there are *two* numbers whose square is 9: the numbers 3 and -3. It is a common but serious error to go from $x^2 = 9$ to $x = 3$. One advantage of the difference of squares theorem is that it will not allow us to forget about the negative solution.

b) The equation $x^4 = 9x^3$ is of a higher degree than one. This is indeed a degree four equation. Therefore, our only method to solve it is to reduce one side to zero, factor, and then apply the zero product rule. It is preferred not to create a negative leading coefficient, and so we will subtract $9x^3$ from both sides.

$$\begin{aligned} x^4 &= 9x^3 && \text{subtract } 9x^3 \\ x^4 - 9x^3 &= 0 && \text{factor out the GCF} \\ x^3(x-9) &= 0 \end{aligned}$$

We apply the zero property. $x^3(x-9) = 0$ or $x \cdot x \cdot x \cdot (x-9) = 0$ means that either $x = 0$ or $x - 9 = 0$. We solve these linear equations separately and obtain $\boxed{0 \text{ and } 9}$. We check: $0^2 = 9 \cdot 0$ and $9^2 = 9 \cdot 9$ and so our solution is correct.

c) The equation $8x^3 = 50x^2$ is of a higher degree than one, so our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} 8x^3 &= 50x^2 && \text{subtract } 50x^2 \\ 8x^3 - 50x^2 &= 0 && \text{the GCF is } 2x^2 \\ 2x^2(4x-25) &= 0 \end{aligned}$$

We now apply the zero product rule. If this product is zero, then either $2x^2 = 0$ or $4x - 25 = 0$. We solve these equations for x.

$$\begin{aligned} 2x^2 &= 0 && \text{or} && 4x - 25 = 0 \\ 2 \cdot x \cdot x &= 0 && \text{or} && 4x = 25 \\ x &= 0 && \text{or} && x = \frac{25}{4} \end{aligned}$$

We check both solutions. If $x = 0$, then $\text{LHS} = 8 \cdot 0^3 = 8 \cdot 0 = 0$ and $\text{RHS} = 50 \cdot 0^2 = 50 \cdot 0 = 0$. And if $x = \frac{25}{4}$, then

$$\text{LHS} = 8\left(\frac{25}{4}\right)^3 = \frac{8}{1} \cdot \frac{15\,625}{64} = \frac{15\,625}{8} \quad \text{and} \quad \text{RHS} = 50\left(\frac{25}{4}\right)^2 = \frac{50}{1} \cdot \frac{625}{16} = \frac{15\,625}{8}$$

Thus both solutions, $\boxed{0 \text{ and } \frac{25}{4}}$ are correct.

d) Since the equation $8p^3 = 50p$ is of a degree higher than one, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} 8p^3 &= 50p && \text{subtract } 50p \\ 8p^3 - 50p &= 0 && \text{the GCF is } 2p \\ 2p\left(4p^2 - 25\right) &= 0 && \\ 2p\left((2p)^2 - 5^2\right) &= 0 && \text{factor via difference of squares theorem} \\ 2p(2p+5)(2p-5) &= 0 && \end{aligned}$$

We now apply the zero product rule. If this product is zero, then either $2p = 0$ or $2p + 5 = 0$ or $2p - 5 = 0$. We solve each equation for p.

$$\begin{array}{lllll} 2p+5 = 0 & \text{or} & 2p-5=0 & \text{or} & 2p=0 \\ 2p = -5 & \text{or} & 2p=5 & \text{or} & p=0 \\ p = -\dfrac{5}{2} & \text{or} & p=\dfrac{5}{2} & & \end{array}$$

We check all three solutions. If $p = -\dfrac{5}{2}$, then

$$\text{LHS} = 8\left(-\frac{5}{2}\right)^3 = \frac{8}{1} \cdot \frac{-125}{8} = -125 \text{ and } \text{RHS} = 50\left(-\frac{5}{2}\right) = \frac{50}{1} \cdot \frac{-5}{2} = \frac{-250}{2} = -125$$

If $p = \dfrac{5}{2}$, then

$$\text{LHS} = 8\left(\frac{5}{2}\right)^3 = \frac{8}{1} \cdot \frac{125}{8} = 125 \quad \text{and} \quad \text{RHS} = 50\left(\frac{5}{2}\right) = \frac{50}{1} \cdot \frac{5}{2} = \frac{250}{2} = 125$$

and if $p = 0$, then

$$\text{LHS} = 8 \cdot 0^3 = 8 \cdot 0 = 0 \quad \text{and} \quad \text{RHS} = 50 \cdot 0 = 0$$

Thus all three solutions, $\boxed{-\dfrac{5}{2},\ 0,\ \text{and}\ \dfrac{5}{2}}$ are correct.

Example 5. Find all numbers that satisfy the following condition: if we square the number, we get back the same number.

Solution: Let us denote this number by x. Then the equation is simply $x^2 = x$. This is a quadratic equation, so we will reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} x^2 &= x \\ x^2 - x &= 0 \\ x(x-1) &= 0 \end{aligned}$$

We solve the linear equations $x = 0$ and $x - 1 = 0$ and obtain the solutions 0 and 1. Clearly both of these numbers square equals to the number. What is more important, we also proved that no other number has this property.

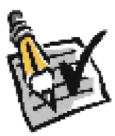
Example 6. Find all numbers that satisfy the following condition: if we raise the number to the third power, the result is four times the original number.

Solution: Let us denote the number by x. The equation is then $x^3 = 4x$. We solve this equation.

$$\begin{aligned} x^3 &= 4x && \text{reduce one side to zero} \\ x^3 - 4x &= 0 && \text{factor out the GCF} \quad (\text{it is } x) \\ x\left(x^2 - 4\right) &= 0 && \text{factor via the difference of squares theorem} \\ x(x+2)(x-2) &= 0 && \text{apply the zero property} \end{aligned}$$

$$\begin{array}{ccccc} x = 0 & \text{or} & x+2=0 & \text{or} & x-2=0 \\ x = 0 & \text{or} & x=-2 & \text{or} & x=2 \end{array}$$

Thus there are three numbers, 0, 2 and -2, satisfying the property. We check: $0^3 = 4 \cdot 0$, $2^3 = 4 \cdot 2$, and $-2^3 = 4(-2)$. Thus our answer is: $\boxed{0,\ 2,\ \text{and}\ -2.}$



Practice Problems

1. Factor out the greatest common factor from each of the following.

   a) $10a^2b^2 - 15ab^3 + 25a^2b^3c$ b) $6x^3 - 3x^2 - 15x^4$ c) $a^2 - a^3 + a^4$ d) $6a^2b + 12a^3b - 30a^3b^2$ e) $x^5 - 2x^4 + 4x^3$ f) $3xy(a-3) + 8t(a-3) - 200x^5(a-3)$

2. Factor out -1 from each of the following.

   a) $x^3 - x^5 + 2$ b) $-x^2 + 3x - 1$ c) $-x^2 + 3x - 5$

3. Factor each of the following via the difference of squares theorem.

   a) $x^2 - 49$ b) $9a^2 - 25$ c) $x^2 - 1$ d) $y^6 - 100$

4. Completely factor each of the following.

   a) $5a^2 - 45$ b) $2m^4 - 2n^4$ c) $2x^4 - 8x^2$ d) $3a - 12ab^2$ e) $x^3 - x$ f) $5x^3y^4 - 80x^3$ g) $a^2(x-1) - 9(x-1)$ h) $18a^2x^2 - 50x^2$ i) $a^2 - (x-1)^2$ j) $-16 + a^4$ k) $600ab^2 - 6ab^4$ l) $36x^2y^3 + 4x^4y^3$ m) $-2x^4 + 162$ n) $5a^3b^2 - 15ab$

5. Solve each of the following equations. Make sure to check your solutions.

   a) $(w+5)(w-1) = 0$ b) $x(x-2)(x+3) = 0$ c) $2(x-2)(x+3) = 0$ d) $x^2 = 4$ e) $x^2 + 6x = 0$ f) $3x^3 = 75x^2$ g) $3x^3 = 75x$ h) $45a^4 = 20a^2$

15.2 Linear Inequalities

We will study solving linear inequalities. Let us first recall a few definitions.

> **Definition**: An **inequality** is a statement in which two expressions (algebraic or numeric) are connected with one of $<$, $\leq$, $>$, or $\geq$. A **solution** of an inequality is a number that, when substituted into the variable in the inequality, makes the statement of inequality true. To **solve an inequality** is to find *all* solutions of it. The set of all solutions is also called the solution set.

Example 1. Consider the inequality $-3x+8 < -2(x-1)+5$. In case of each of the numbers given, determine whether it is a solution of the inequality or not.

a) -3 b) 4 c) 1 d) 8

Solution: a) We substitute the given number into the variable and check whether the inequality statement is true.

$\text{LHS} = -3(-3)+8 = 17$ and $\text{RHS} = -2(-3-1)+5 = -2(-4)+5 = 8+5 = 13$

The statement $17 < 13$ is false. Thus -3 is $\boxed{\text{not a solution}}$ of the inequality.

b) We substitute 4 into the variable and check whether the inequality statement is true.

$\text{LHS} = -3 \cdot 4+8 = -12+8 = -4$ and $\text{RHS} = -2(4-1)+5 = -2 \cdot 3+5 = -6+5 = -1$

The statement $-4 < -1$ is true. Thus 4 is $\boxed{\text{a solution}}$ of the inequality.

c) We substitute 1 into the variable and check whether the inequality statement is true.

$\text{LHS} = -3 \cdot 1+8 = -3+8 = 5$ and $\text{RHS} = -2(1-1)+5 = -2 \cdot 0+5 = 0+5 = 5$

The statement $5 < 5$ is false. Thus 1 is $\boxed{\text{not a solution}}$ of the inequality.

d) We substitute 8 into the variable and check whether the inequality statement is true.

$\text{LHS} = -3 \cdot 8+8 = -24+8 = -16$ and $\text{RHS} = -2(8-1)+5 = -2 \cdot 7+5 = -14+5 = -9$

The statement $-16 < -9$ is true. Thus 8 is $\boxed{\text{a solution}}$ of the inequality.

Inequalities have many, many solutions. To express these much larger solution sets, we developed interval notation. The steps of solving a liner inequality are almost identical to those of solving linear equations. However, there is a very important difference. Consider the true inequality $3 \leq 7$. If we add or subtract the same number from both sides, the inequality will remain true. The result is the same if we multiply both sides by a positive number. But if we multiply both sides of $3 \leq 7$ by -2, we get $-6 \leq -14$, which is false.

> **When multiplying or dividing both sides of an inequality, we must reverse the inequality sign.**

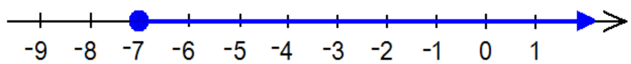
Example 2. Solve the inequality $\dfrac{-3x+1}{2} \leq 11$

Solution: The steps are identical to those of solving equations, except for when multiplying or dividing by a negative number.

$$\frac{-3x+1}{2} \leq 11 \qquad \text{multiply by 2}$$

$$\begin{aligned} -3x+1 &\leq 22 && \text{subtract 1} \\ -3x &\leq 21 && \text{divide by } -3 \implies \text{MUST reverse inequality sign} \\ x &\geq -7 \end{aligned}$$

Notice that we reversed the inequality sign when we divided by -3. Our solution set is the set of all numbers greater than or equal to -7. We can present this set as an interval: $[-7,\infty)$. We can also depict the solution set on the number line:



Although we will not be asked to do so, we can check inequalities too. If we randomly pick a number inside our solution set and substitute it into the inequality, the inequality statement must be true. If we randomly pick a number outside our solution set and substitute it into the inequality, the inequality statement must be false. Finally, if substitute the boundary point (the one that separates the solutions from the non-solutions) the two sides should be equal. For an easy to substitute number inside our solution set, we choose $x=0$. Substituting it into the inequality, we get $\frac{1}{2} \leq 11$, which is true. For a number outside of our solution set, we choose $x=-10$. This value results in $\frac{31}{2} \leq 11$, which is false. Finally, setting $x=-7$, we get the two sides equal.

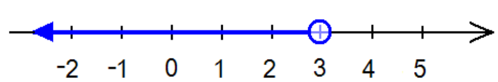
Example 3. Solve the given inequality. Present the solution set using interval notation and plot it on a number line.

$$\frac{5x+1}{4} - \frac{2x-1}{5} > 2x-3$$

Solution: We will bring both sides to the common denominator, and then clear all denominators by multiplying by that number. In fact, we can speed up the process by just multiplying both sides by the common denominator. In this case, that common denominator is 20.

$$\begin{aligned} \frac{5x+1}{4} - \frac{2x-1}{5} &> 2x-3 && \text{multiply by 20} \\ 5(5x+1)-4(2x-1) &> 20(2x-3) && \text{expand products} \\ 25x+5-8x+4 &> 40x-60 && \text{combine like terms} \\ 17x+9 &> 40x-60 && \text{subtract } 17x \\ 9 &> 23x-60 && \text{add 60} \\ 69 &> 23x && \text{divide by 23} \\ 3 &> x \end{aligned}$$

Our solution is $(-\infty,3)$. We depict the solution set on a number line:



In case it is not clear how we got from the first line to the second line, here is the detailed computation:

$$20\left(\frac{5x+1}{4} - \frac{2x-1}{5}\right) = 20 \cdot \frac{5x+1}{4} - 20 \cdot \frac{2x-1}{5} = \frac{20(5x+1)}{4} - \frac{20(2x-1)}{5} = 5(5x+1)-4(2x-1)$$

We recommend however to perform these steps on the margin or mentally.

Sample Problems

Solve each of the following inequalities. Graph the solution set.

1. $-7 > -5x + 3$
2. $3(x-2) \leq 2x + 1$
3. $5(4x-1) - (x-3) \geq -x - 2$
4. $\dfrac{m+4}{2} - \dfrac{4m+3}{5} > 2$

Practice Problems

Solve each of the following inequalities. Graph the solution set.

1. $x - 17 > -4x + 3$
2. $-3x + 5 \leq 12$
3. $5y + 3 < y - 7$
4. $-2x - (3x-1) \geq 2(5-3x)$
5. $\dfrac{2}{3}x - 1 \geq x$
6. $5 - (3a - 2) < -2$
7. $5x - 2 > 3(x-1) - 4x + 1$
8. $-3(x-2) \leq -2x + 5$
9. $3x - 2(x-1) < -2x - 1$
10. $-w + 13 \geq 2w + 1$
11. $2x + 5 > \dfrac{3x-1}{2} - \dfrac{2x+1}{3}$
12. $5(x-1) - 3(x+1) \geq 3x - 8$
13. $3(x-4) + 5(x+8) \leq 2(x-1)$
14. $2x + 6 > \dfrac{3x-1}{5} - \dfrac{7-x}{3}$
15. $-\dfrac{2}{5}(x+1) + \dfrac{1}{2}(x-4) \geq \dfrac{3}{10}x$
16. $\dfrac{3x-1}{4} + \dfrac{8-4x}{3} \leq -3 - x$
17. $\dfrac{x-2}{5} - \dfrac{x}{2} < x - 16$
18. $\dfrac{2x+1}{3} + 2 \geq x + \dfrac{3-x}{2}$

Problem Set 15

1. Simplify each of the following.

   a) $\dfrac{2-\frac{1}{3}}{2+\frac{1}{3}} \div \left(2\frac{1}{2}\right)$ b) $\dfrac{5^{101}-5^{100}}{5^{99}}$

2. Evaluate the expression $\dfrac{3a-4b+6ab-2}{3a-2}$ if

   a) $a=-\frac{1}{6}$ and $b=1\frac{1}{2}$ b) $a=\frac{2}{3}$ and $b=\frac{1}{2}$ c) $a=2$ and $b=-\frac{1}{2}$

3. Simplify each of the following expressions.

   a) $\left(-2ab^3\right)^2\left(-bab^4\right)$ b) $\left(\left(-2ab^3\right)\left(-bab^4\right)\right)^2$ c) $\left(-2ab^2\right)^3\left(-ba^4b\right)$ d) $(x+3)^2-(x+1)^2$ e) $\dfrac{2^{500}\cdot 9^{250}}{6^{100}}$ f) $\dfrac{4^8\cdot 9^6}{8^5\cdot 27^3}$

4. Completely factor each of the following.

   a) $2x^2-18$ b) $12a^2x^2-75x^2$ c) x^2+9 d) $5a^7-5a^3$ e) $-x^3+16x$ f) $x^{16}-25$

5. Completely factor each of the following. You should use the difference of squares theorem as your first step.

   a) $(2x+5)^2-(2x+1)^2$ b) $\left(5x^2-2x+1\right)^2-\left(3x^2-2x-1\right)^2$ c) $(a+2b-3c)^2-(a-2b-c)^2$

6. Solve each of the given equations.

   a) $x^2=16x$ b) $x^2=16$ c) $x^4=16$

7. Solve each of the given equations.

   a) $(3x-1)(x+1)-2(x-2)^2=14x-9$

   b) $5m^6=80m^2$

   c) $(2x-3(4x+5(-x+2)-3))=2(3(x-5)+1)$

   d) $3((9x-1)-5(2x+1))=-18$

   e) $5(2x+3)=(x+4)^2-(x-1)^2$

   f) $\dfrac{5}{6}\left(x+\dfrac{1}{3}\right)=\dfrac{1}{36}+\dfrac{2}{3}x$

   g) $(3-4(5-(6-x)+1)-1)+1=x^2+3$

   h) $3(x-2)^2-(2x-1)^2=15-(x+6)^2$

   i) $x(x+1)^2(x-5)=0$

   j) $(2x-5)^2=(5x-2)^2$

8. a) Solve the inequality $(3x-1)^2-(2x+5)^2<24-5x(4-x)$. Let A denote the solution set for this inequality. Present A using interval notation.

   b) Solve the inequality $x-\dfrac{1}{4}\geq\dfrac{1}{2}(3x-5)-\dfrac{3}{4}(2x+1)$. Let B denote the solution set for this inequality. Present B using interval notation.

   c) Find $A\cup B$

   d) Find $A\cap B$

9. A cab driver charges customers \$2 for the first mile and then \$0.15 for each additional mile. After we take a drive, we owe \$11.

   a) How far did we drive?

   b) How much should we pay if we would like to include a tip of 15%?

10. I'm thinking of a number. When I subtract 3 from four times the opposite of this number, I get -19. What number am I thinking of?

11. I'm thinking of a number. If I subtract 3 and square, the result is 18 greater then the product of this number and -6. What number am I thinking of?

12. Andy and Brett dine together. They share the \$45 bill as follows. Brett pays 17 dollar less than twice the amount paid by Andy. How much do they each pay?

13. Express each of the following as a single change.

    a) First a 20% increase and then a 20% decrease.
    b) First a 25% increase and then a 20% decrease.
    c) First a 20% increase and then a 25% decrease.
    d) First a 10% increase and then a 20% decrease.
    e) Three consecutive 50% decreases

14. Consider the lines $3x = -y+2$ and $y = \frac{1}{2}x - 5$.

    a) Graph the two lines in the same coordinate system. Find both coordinates of the point where the lines intersect each other.
    b) Use algebraic methods to check your solution.
    c) Graph the line $x = -2$ in the same coordinate system. The three lines determine a triangle. What is the area of this triangle?

15. Temperature can be measured in celsius and in Farnheit. The conversion formula is as follows: if the temperature is F Farenheits, then the same temperature is C celsius, where

$$F = \frac{9}{5}C + 32$$

    a) Convert 35 celsius to farenheit.
    b) Convert 122 farenheit to celsius.
    c) Solve the formula above for C.
    d) Is there a temperature for which the farenheit measure is the same number as the celsius measure?

16. During the first week of the summer camp, 60% of the students were boys and 40% were girls. On the weekend, 30 more boys arrived and 30 of the girls left. Now 25% of the students were girls. How many students were in the camp during the first week?

17. A square has sides s. Suppose that we obtain a rectangle by increasing one side by 3 units and decreasing another side by 3 units. How does the area change if

    a) $s = 4$ b) $s = 8$ c) $s = 13$
    d) Can you explain the relationship between your results from the previous parts?

18. Compute the value of each of the following expressions. Do not use a calculator. You might use the difference of squares theorem.

    a) $2020^2 - 2019^2$
    b) $95^2 - 85^2$
    c) $185^2 - 15^2$
    d) $55^2 - 45^2$
    e) $12345004321^2 - 12345004320 \cdot 12345004322$
    f) $2500000010 \cdot 2500000004 - 2500000007^2$
    g) $16^{50} - \left(2^{100}+1\right)\left(2^{100}-1\right)$

Chapter 16

16.1 Integer Exponents

Part 1 - The History Thus Far and the Problem

Recall what we know about exponentiation thus far. Exponential notation expresses repeated multiplication.

Definition: We define 2^7 to denote the factor 2 multiplied by itself repeatedly, such as

$$\underbrace{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2}_{\text{7 factors}} = 2^7$$

When mathematicians agreed to this definition, that was a free choice. They could have gone with other definitions. Once this definition exists, however, certain properties are automatically true, and we have no other option but to recognize them as true. They just fell into our laps.

Theorem 1. If a is any number and m, n are any positive integers, then $a^n \cdot a^m = a^{n+m}$

Theorem 2. If a is any non-zero number and m, n are any positive integers, then $\dfrac{a^n}{a^m} = a^{n-m}$

Theorem 3. If a is any number and m, n are any positive integers, then $(a^n)^m = a^{nm}$

Theorem 4. If a, b are any numbers and n is any positive integer, then $(ab)^n = a^n b^n$

Theorem 5. If a, b are any numbers, $b \neq 0$, and n is any positive integer, then $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Again, the definition, immediately followed by the theorems. And then there was a quiet. Another opening for a free choice.

Consider the expression 2^x. The problem is that the definition of exponentiation only allows for a positive integer value of x. The expression 2^x is meaningful for $x = 2$ or 9 or 100, but it is not meaningful for values of x such as -3 or $\frac{3}{5}$ or 3.2. In short, the world of exponents was just the set of all natural numbers. Mathematicians usually don't like that. The best case scenario, the ultimate hope is that the definition of exponents could be extended to any number for x. That way, 2^x would be meaningful, no matter what the value of x is.

So, one of the issues was the desire to grow our world of exponents beyond the set of all natural numbers. This will be achieved in several steps. Today, we are only focusing on enlarging the world of exponents from $\mathbb{N}$ to $\mathbb{Z}$ (i.e. from the set of all natural numbers to the set of all integers).

The other issue was that as we enlarge our world, we pay especial attention that the new definitions will not conflict with the mathematics we already have. This principle comes up often in our choices, and it is sometimes called the **expansion principle**.

Definition: In many situations, mathematicians attempt to increase, to enlarge our world. The **expansion principle** is that when we enlarge our mathematics by adding new definitions, we do so in such a way that the new definitions never create conflicts with the mathematics we already have.

Part 2 - Integer Exponents

Suppose we want to define 2^0. The repeated multiplication definition can not be applied to zero, so we have complete freedom to define 2^0. As it turns out, if we insist on a definition that does not conflict with Rule 2, $\frac{a^n}{a^m} = a^{n-m}$, then we do not have all that many choices for 2^0. Let us think of zero as the result of the subtraction $3-3$, and that we would like to define 2^0 so that Rule 2 is still true.

$$2^0 = 2^{3-3} \overset{\text{rule 2}}{=} \frac{2^3}{2^3} = \frac{8}{8} = 1$$

This is an expansion principle proof. It did not prove that the value of 2^0 is or must be zero. It showed much less; that if we wanted to define 2^0 without harming Rule 2 in the example given, then the only possible value for 2^0 is 1. The reader should imagine a team of mathematicians making first sure that no part of our good old math is hurt if we define $2^0 = 1$. And as it turned out, this is exactly the case.

This computation can be repeated with many different bases. For example,

$$5^0 = 5^{2-2} \overset{\text{rule 2}}{=} \frac{5^2}{5^2} = \frac{25}{25} = 1 \quad \text{or} \quad (-3)^0 = (-3)^{2-2} \overset{\text{rule 2}}{=} \frac{(-3)^2}{(-3)^2} = \frac{9}{9} = 1$$

The only base that is problematic is 0. Indeed, division by zero is not allowed and Rule 2, $\frac{a^n}{a^m} = a^{n-m}$ does not work with $a = 0$. If we try to perform the same computation with zero, we ultimately end up in $\frac{0}{0}$ which is undefined.

Theorem 6. If a is any non-zero number, then $a^0 = 1$.

0^0 is undefined.

Please note that as we extend our world of exponents, old issues might re-surface. For example, $(-3)^0 = 1$ but $-3^0 = -1$ is an important distinction, but not a new one.

Now that we have defined zero exponent, we will similarly try to define negative integer exponents such as 2^{-3}.

Again, the original definition can not be applied. We cannot write down the factor two negative three times. So we have a freedom here to define 2^{-3} in any way we wish. In this decision, we will again use the expansion principle: that we would like to keep our old rules after having 2^{-3} defined.

We will again use Rule 2, $\frac{a^n}{a^m} = a^{n-m}$ and write -3 as a subtraction between two positive integers.

$$2^{-3} = 2^{1-4} \overset{\text{Rule 2}}{=} \frac{2^1}{2^4} = \frac{2}{16} = \frac{1}{8} = \frac{1}{2^3} \quad \text{or, more elegantly, } 2^{-3} = 2^{1-4} \overset{\text{Rule 2}}{=} \frac{2^1}{2^4} = \frac{\cancel{2}}{\cancel{2}\cdot 2\cdot 2\cdot 2} = \frac{1}{2^3}$$

When we discovered this rule, we saw that it was true because of cancellation. In case of a negative exponent, we have the same cancellation, it's just that we run out of factors in the numerator first. The computation can be repated with any base except for zero.

> **Theorem 7.** If a is any non-zero number, and n is any positive integer, then $a^{-n} = \frac{1}{a^n}$.
>
> 0^{-n} is undefined.

Example 1. Simplify each of the following expressions. Use only positive exponents in your answer.

a) 5^{-2} b) a^{-5} c) $\frac{1}{3^{-2}}$ d) $\left(\frac{2}{3}\right)^{-3}$ e) $\frac{1}{x^{-3}}$ f) $2x^{-3}$

Solution: a) Recall our new rule, $a^{-n} = \frac{1}{a^n}$. We apply this rule: $5^{-2} = \frac{1}{5^2} = \boxed{\frac{1}{25}}$.

b) We can use the same rule again: $a^{-5} = \boxed{\frac{1}{a^5}}$.

c) In this case, the expression with the negative exponent is in the denominator.

The short story is that $\frac{1}{3^{-2}} = 3^2 = 9$. The long story is that we apply our new rule $a^{-n} = \frac{1}{a^n}$ and then we divide by mutiplying by the reciprocal.

$$\frac{1}{3^{-2}} = \frac{1}{\frac{1}{3^2}} = \frac{\frac{1}{1}}{\frac{1}{3^2}} = \frac{1}{1}\cdot\frac{3^2}{1} = \frac{9}{1} = \boxed{9}$$

So, $\frac{1}{a^{-n}}$ can be re-written as a^n.

d) In this case, the expression with the negative exponent is already a fraction.

The short story is that $\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$. The long story is that we apply our new rule $a^{-n} = \frac{1}{a^n}$ and then we divide by mutiplying by the reciprocal.

$$\left(\frac{2}{3}\right)^{-3} = \frac{1}{\left(\frac{2}{3}\right)^3} = \frac{\frac{1}{1}}{\frac{8}{27}} = \frac{1}{1}\cdot\frac{27}{8} = \boxed{\frac{27}{8}}$$

This computation shows that $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$.

e) The short story is that $\frac{1}{a^{-n}}$ can be re-written as a^n. The computation below justifies this step.

$$\frac{1}{x^{-3}} = \frac{1}{\frac{1}{x^3}} = \frac{\frac{1}{1}}{\frac{1}{x^3}} = \frac{1}{1} \cdot \frac{x^3}{1} = \frac{x^3}{1} = \boxed{x^3}$$

So, $\frac{1}{a^{-n}}$ can be re-written as a^n.

f) It is a common mistake to interpret $2x^{-3}$ as $(2x)^{-3}$. Without the parentheses, we perform the exponentiation before the multiplication. Therefore, the correct computation is

$$2x^{-3} = 2 \cdot x^{-3} = \frac{2}{1} \cdot \frac{1}{x^3} = \boxed{\frac{2}{x^3}}.$$

Theorem: The following statements are practical applications of the rule $a^{-n} = \frac{1}{a^n}$ and frequently occur in computations.

$$\frac{1}{a^{-n}} = a^n \qquad \text{and} \qquad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$$

Proof: As the computation shows, we apply the rule $a^{-n} = \frac{1}{a^n}$ and then perform the division by multiplying by the reciprocal.

$$\frac{1}{a^{-n}} = \frac{1}{\frac{1}{a^n}} = \frac{\frac{1}{1}}{\frac{1}{a^n}} = \frac{1}{1} \cdot \frac{a^n}{1} = \frac{a^n}{1} = a^n \quad \text{and} \quad \left(\frac{a}{b}\right)^{-n} = \frac{1}{\left(\frac{a}{b}\right)^n} = \frac{\frac{1}{1}}{\frac{a^n}{b^n}} = \frac{1}{1} \cdot \frac{b^n}{a^n} = \frac{b^n}{a^n} = \left(\frac{b}{a}\right)^n \quad \blacksquare \; (\textit{end of proof})$$

Example 2. Re-write the expression $\frac{a^3 b^{-5}}{c^{-2} d^4}$ using only positive exponents.

Solution: We re-write the expressions with negative exponents using the rule $a^{-n} = \frac{1}{a^n}$.

$$\frac{a^3 b^{-5}}{c^{-2} d^4} = \frac{a^3 \cdot \frac{1}{b^5}}{\frac{1}{c^2} \cdot d^4} = \frac{\frac{a^3}{1} \cdot \frac{1}{b^5}}{\frac{1}{c^2} \cdot \frac{d^4}{1}} = \frac{\frac{a^3}{b^5}}{\frac{d^4}{c^2}} = \frac{a^3}{b^5} \cdot \frac{c^2}{d^4} = \boxed{\frac{a^3 c^2}{b^5 d^4}}.$$

Notice the pattern here. If a factor with a negative exponent is in the numerator, we can re-write it with a positive exponent in the denominator. Also, if a factor with a negative exponent is in the denominator, we can re-write it with a positive exponent in the numerator.

Theorem: $\frac{a^{-n} b^m}{c^p d^{-q}} = \frac{b^m d^q}{a^n c^p}$ where a, c, d are any non-zero numbers and n, m, p, q are positive integers.

The definitions of a^0 and a^{-n} were developed with the intention that the previous rules (1 through 5) will remain true. Keep that in mind in case of computations with more complex exponential expressions.

Example 3. Simplify each of the given expressions. Present your answer using only positive exponents.

a) $\left(a^{-2}\right)^{-5}$ b) $\dfrac{\left(-x^{-2}\right)^{-3}}{x^{-6}\left(-x\right)^{-4}}$ c) $\dfrac{a^{-3}}{a^{-8}}$ d) $\dfrac{a^{-2}b^{-3}}{a^{-5}b^{3}}$ e) $\dfrac{\left(2a^{-4}b^{3}\right)^{-5}}{\left(3a^{3}b^{-2}\right)^{0}}$

Solution: a) It is much preferred to first simplify the exponent. Repeated exponentiation means multiplication in the exponent.

$$\left(a^{-2}\right)^{-5}=a^{-2(-5)}=\boxed{a^{10}}$$

b) Let us re-write the solo negative signs as multiplications by -1. Then we will use the rules of exponents to simplify the exponents. Only after that will we address negative exponents.

$$\frac{\left(-x^{-2}\right)^{-3}}{x^{-6}\left(-x\right)^{-4}}=\frac{\left(-1\cdot x^{-2}\right)^{-3}}{x^{-6}\left(-1\cdot x\right)^{-4}}=\frac{\left(-1\right)^{-3}\left(x^{-2}\right)^{-3}}{x^{-6}\left(-1\right)^{-4}x^{-4}}=\frac{\left(-1\right)^{-3}x^{6}}{x^{-6}\left(-1\right)^{-4}x^{-4}}$$

Now we get rid of all negative exponents by moving the factors. A factor with exponent -5 in the numerator can be re-written as a factor with exponent 5 in the denominator, and vica versa.

$$\frac{\left(-1\right)^{-3}x^{6}}{x^{-6}\left(-1\right)^{-4}x^{-4}}=\frac{\left(-1\right)^{4}x^{6}x^{6}x^{4}}{\left(-1\right)^{3}}=\frac{1\cdot x^{16}}{-1}=\boxed{-x^{16}}$$

c) Solution 1: apply the rule $\dfrac{a^n}{a^m}=a^{n-m}$. $\dfrac{a^{-3}}{a^{-8}}=a^{-3-(-8)}=a^{-3+8}=\boxed{a^5}$

Solution 2: First we get rid of negative exponents and then apply the rule $\dfrac{a^n}{a^m}=a^{n-m}$.

$$\frac{a^{-3}}{a^{-8}}=\frac{a^{8}}{a^{3}}=a^{8-3}=\boxed{a^5}$$

d) First we get rid of negative exponents.

$$\frac{a^{-2}b^{-3}}{a^{-5}b^{3}}=\frac{a^{5}}{a^{2}b^{3}b^{3}}=\frac{a^{5}}{a^{2}b^{6}}=\boxed{\frac{a^{3}}{b^{6}}}$$

e) We can save a lot of work by noticing that the denominator is just 1, because any non-zero quantity raised to the power zero is 1, and so $\left(3a^{3}b^{-2}\right)^{0}=1$.

$$\frac{\left(2a^{-4}b^{3}\right)^{-5}}{\left(3a^{3}b^{-2}\right)^{0}}=\frac{2^{-5}\left(a^{-4}\right)^{-5}\left(b^{3}\right)^{-5}}{1}=\frac{2^{-5}a^{20}b^{-15}}{1}=\frac{a^{20}}{2^{5}b^{15}}=\boxed{\frac{a^{20}}{32b^{15}}}$$

Part 3 - Scientific Notation Revisited

When we first saw scientific notation, we learned to use it to handle uncomfortbly large numbers.

Recall the definition of scientific notation:

> **Definition:** We can write numbers in scientific notation. This means to write a number as a product of two numbers. The first number is between 1 and 10 (can be 1 but must be less than 10), and the second number is a 10−power. For example, the scientific notation for $428\,600\,000\,000$ is 4.286×10^{11}.

With negative exponents, we can also use scientific notation to handle extremely small numbers. For example, the mass of an electron is 0.00000000000000000000000000091094 grams. Instead of hurds of trailing zeroes, now we are faced with many zeroes after the decimal point. This number can be re-written as $9.1094 \cdot 10^{-28}$.

Example 4. Re-write the number 0.0000000317 using scientific notation.

Solution: The first number in scientific notation needs to be between 1 and 10. In this case, this number is 3.17. We just need to figure out the 10−power in the second part. We count how many decimal places we move the decimal from 0.0000000317 to 3.17. We count 8 decimal places. So the correct answer is $\boxed{3.17 \cdot 10^{-8}}$.

Example 5. Suppose that $A = 3.8 \cdot 10^{15}$ and $B = 6.5 \cdot 10^{-8}$. Perform each of the following operations. Present your answer using scientific notation.

a) B^2 b) AB^2 c) $\dfrac{B}{A}$

Solution: a) We will apply rules of exponents.

$$B^2 = \left(6.5 \cdot 10^{-8}\right)^2 = 6.5^2 \cdot \left(10^{-8}\right)^2 = 42.25 \cdot 10^{-16}$$

This number is not in scientific notation because 42.25 is too large for the first part of scientific notation. Recall that the first factor must be between 1 and 10. So we re-write 42.25 as $4.225 \cdot 10$.

$$B^2 = 42.25 \cdot 10^{-16} = 4.225 \cdot 10 \cdot 10^{-16} = 4.225 \cdot 10^{1+(-16)} = \boxed{4.225 \cdot 10^{-15}}$$

b) We will apply rules of exponents.

$$AB^2 = \left(3.8 \cdot 10^{15}\right)\left(6.5 \cdot 10^{-8}\right)^2 = 3.8 \cdot 10^{15} \cdot 6.5^2 \cdot \left(10^{-8}\right)^2 = 3.8 \cdot 10^{15} \cdot 42.25 \cdot 10^{-16}$$
$$= (3.8 \cdot 42.25) \cdot \left(10^{15} \cdot 10^{-16}\right) = 160.55 \cdot 10^{15+(-16)} = 160.55 \cdot 10^{-1} = 16.055$$

This number is not in scientific notation because 16.055 is too large for the first part of scientific notation.

$$AB^2 = 16.055 = \boxed{1.6055 \cdot 10}.$$

c) $$\frac{B}{A} = \frac{6.5 \cdot 10^{-8}}{3.8 \cdot 10^{15}} = \left(\frac{6.5}{3.8}\right) \cdot 10^{-8-15} = \boxed{1.7105 \cdot 10^{-23}}.$$

Sample Problems

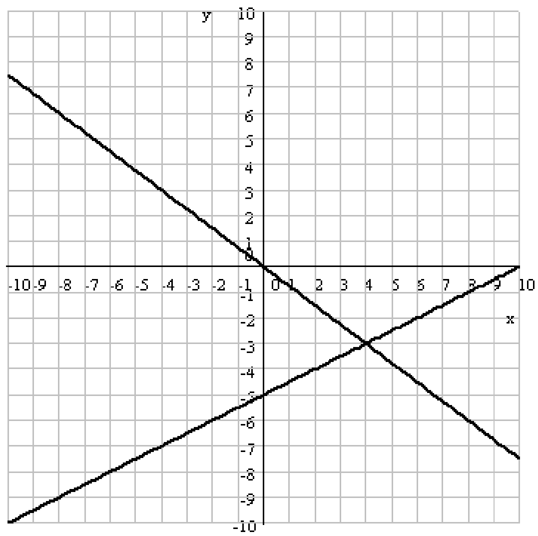
Simplify each of the following. Assume that all variables represent positive numbers. Present your answer without negative exponents.

1. 3^{-2}
2. $\dfrac{1}{2^{-3}}$
3. m^{-4}
4. $\dfrac{1}{x^{-5}}$
5. $a^8 \cdot a^{-1}$
6. $p^3\left(p^{-7}\right)p^8$
7. $\dfrac{x^{-4}}{x^{-9}}$
8. $\dfrac{50a^{12}}{10a^{-3}}$
9. $\dfrac{t^{-3}}{t^4}$
10. x^0
11. $-x^0$
12. $(-x)^0$
13. $\left(b^{-5}\right)\left(b^2\right)\left(b^{-1}\right)$
14. $\dfrac{1}{\left(b^{-5}\right)\left(b^2\right)\left(b^{-1}\right)}$
15. $\dfrac{m^{-2}}{m^{-5}}$
16. $\dfrac{x^3y^{-5}}{z^{-4}}$
17. $\dfrac{18q^3}{6q^{-3}}$
18. $\left(\dfrac{2}{3}\right)^{-3}$
19. $2y^{-3}$
20. $(2y)^{-3}$
21. $\left(-\dfrac{3}{5}\right)^{-2}$
22. $\dfrac{a^3b^{-5}}{a^{-2}b^3}$
23. $\left(3m^3\right)^{-2}$
24. $\left(-2ab^{-3}\right)^{-3}$
25. $\dfrac{\left(k^3\right)^{-3}}{\left(k^{-5}\right)^2}$
26. $\left(\dfrac{2a^{-3}b^5}{-3a^3b^{-2}}\right)^{-2}\left(a^3b^{-5}\right)^{-3}$
27. $\left(-2a^{-3}\right)\left(-2a^{-2}b\right)^{-4}$
28. $\dfrac{\left(-3p^3q^5\right)^2}{\left(2q^0p^3\right)^{-1}}$
29. $\left(\dfrac{2a^{-2}b^3}{-2^2\left(a^{-1}b\right)^{-3}}\right)^{-2}$
30. $\left(-\dfrac{x^3y^0x^{-5}}{y^{-3}}\right)^{-2}$
31. $\left(-\dfrac{x^3y^7x^{-5}}{y^{-3}}\right)^{0}$
32. $\dfrac{x^{-1}+y^{-1}}{x^{-2}-y^{-2}}$
33. $\dfrac{\left(-2a^{-2}\right)^{-2}b^3a^0\left(-aba^{-2}b^{-2}\right)^{-3}}{2a^2\left(-2a^{-2}b\right)^{-2}ab^0}$
34. $\left(\dfrac{-a^2\left(b^{-1}a\right)^{-5}}{b^7\left(-ab^2\right)^{-3}}\right)^{-2}$
35. $\dfrac{\left(x^{-2}\right)^{-2}y^3x^0\left(-2yx^0y^{-2}x^{-2}\right)^0}{yx^5\left(y^{-2}x\right)^{-3}\left(2x^{-1}yx^3\right)^{-1}}$
36. Suppose that $x = 8.5\cdot 10^{-12}$ and $y = 7.5\cdot 10^{7}$. Perform each of the following operations. Present your answer using scientific notation.

    a) xy b) x^3 c) xy^2 d) $\dfrac{x}{y}$ e) $\dfrac{y}{x^5}$

16.2 Systems of Linear Equations: Elimination

Recall that the graph of an equation is the set of all points whose coordinates form a solution to the equation. Consider the lines $y=-\frac{3}{4}x$ and $x-2y=10$. We graph these lines in the same coordinate system. What can we say about the point where the two lines intersect each other? In this particular case, we can read the coordinates of this point: $(4,-3)$.



This point is on the line $y=-\frac{3}{4}x$ if and only if its coordinates form a solution of the equation. We check:

$$\text{LHS}=-3 \quad \text{and} \quad \text{RHS}=-\frac{3}{4}(4)=-3 \checkmark$$

Therefore, this point is on the line $y=-\frac{3}{4}x$.

Similarly, this point is on the line $x-2y=10$ if and only if its coordinates form a solution of the equation.

$$\text{LHS}=4-2(-3)=4+6=10 \quad \text{and} \quad \text{RHS}=10 \checkmark$$

Therefore, this point is also on the line $x-2y=10$.

Two different lines cannot have more than one point in common. Two points uniquely determine a line. Therefore, if two lines have two points in common, they are really the same line. In this sense, the intersection point is special. Algebraically, the intersection point is the unique point whose coordinates are a solution for the equations of both lines.

> **Definition:** Two equations in x and y form a **system of equations**. The **solution**(s) of the system are the point(s) whose coordinates form a solution of both equations. To **solve** a system means to find all solutions of it.

In our example above, $(-4,3)$ is the only solution of the system $\begin{cases} y=-\frac{3}{4}x \\ x-2y=10 \end{cases}$.

In this case, the intersection was a point whose both coordiantes happen to be integers. (Such points are called lattice points.) In case we are less lucky, we will need more precise tools for solving than graphing the two equations in the same cordinate system.

There are several algebraic methods to solve a system of linear equations.

Solving Linear Systems Using Elimination.

This method is sometimes called the addition method. We will refer to it as elimination. The basic principle behind this method is the following. We can multiply both sides of an equation by the same non-zero number- and this step will not change the solution set. We can also add equations: If $x=y$ and $A=B$, then $x+A=y+B$.

Example 1. Solve the given system of linear equations using elimination. $\begin{cases} 3x-5y=11 \\ 2x+3y=20 \end{cases}$

Solution: We will multiply both sides of one or both equations with the goal to end up with one unknown with opposite coefficients. This is possible for both x and y. To elimininate y, we will multiply the first equation by 3 and the second equation by 5. That way the coeffcients of y will be 15 and -15. Once we add the two equations, y will be eliminated.

$$\begin{cases} 3x-5y=11 & \text{multiply by 3} \\ 2x+3y=20 & \text{multiply by 5} \end{cases} \implies \begin{cases} 9x-15y=33 \\ 10x+15y=100 \end{cases}$$

We will eliminate y by adding the two equations (i.e adding the left-hand side to left-hand side and right-hand side to right-hand side.) Then the equation will become an equation in only x, so we can solve for it.

$$\begin{array}{rrcl} & 9x-15y & = & 33 \\ + & 10x+15y & = & 100 \\ \hline & 19x & = & 133 \quad \text{divide by 19} \\ & x & = & 7 \end{array}$$

Now that we know the value of x, we can use either one of the two equations to find the value of y. The second equation, in its original form, will be transformed from $2x+3y=20$ to $2\cdot 7+3y=20$. Now we can easily solve for y.

$$\begin{aligned} 2\cdot 7+3y &= 20 \\ 14+3y &= 20 \quad \text{subtract} \\ 3y &= 6 \\ y &= 2 \end{aligned}$$

Thus, the solution of this system is $x=7$ and $y=2$, or, in short, $(7,2)$. We check: the solution of a system is a simultaneous solution of both equations.

Checking $3x-5y=11$

$$\begin{aligned} \text{LHS} &= 3\cdot 7-5\cdot 2=21-10=11 \\ \text{RHS} &= 11 \checkmark \end{aligned}$$

Checking $2x+3y=20$

$$\begin{aligned} \text{LHS} &= 2\cdot 7+3\cdot 2=14+6=20 \\ \text{RHS} &= 20 \checkmark \end{aligned}$$

Therefore, our solution, $\boxed{(7,2)}$ is correct.

Example 2. Solve the given system of linear equations using elimination. $\begin{cases} 2x-y=-19 \\ -x+3y=12 \end{cases}$

Solution: We will multiply both sides of one or both equations with the goal to end up with one unknown with opposite coefficients. This is possible for both x and y. To elimininate x, we will leave the first equation as it is, and multiply the second equation by 2. This way the coeffcients of x will be 2 and 2. Then, when we add the two equations, x will be eliminated, and we can solve the equation for y.

$$\begin{cases} 2x-y=-19 & \\ -x+3y=12 & \text{multiply by 2} \end{cases} \implies \begin{cases} 2x-y=-19 \\ -2x+6y=24 \end{cases}$$

We now add the two equations:

$$\begin{cases} \quad 2x-y = -19 \\ + \quad -2x+6y=24 \end{cases}$$
$$\begin{aligned} 5y &= 5 \quad \text{divide by 5} \\ y &= 1 \end{aligned}$$

Now that we know the value of y, we can use either one of the two equations to find the value of x. The first equation, in its original form, will be transformed from $2x-y=-19$ to $2x-1=-19$. Now we can easily solve for x.

$$\begin{aligned} 2x-1 &= -19 \qquad \text{add 1} \\ 2x &= -18 \qquad \text{divide by 2} \\ x &= -9 \end{aligned}$$

Thus, the solution of this system is $x=-9$ and $y=1$, or, in short, $(-9,1)$. We check: the solution of a system is a simultaneous solution of both equations.

Checking $2x-y=-19$

$$\begin{aligned} \text{LHS} &= 2(-9)-1=-18-1=-19 \\ \text{RHS} &= -19 \checkmark \end{aligned}$$

Checking $-x+3y=12$

$$\begin{aligned} \text{LHS} &= -(-9)+3\cdot 1=9+3=12 \\ \text{RHS} &= 12 \checkmark \end{aligned}$$

Therefore, our solution, $\boxed{(-9,1)}$ is correct.

Most real-world problems boil down to systems of equations. In this sense, solving systems of equations is one of the most important tasks in problem solving.

Example 3. There is an animal farm where chickens and cows live. All together, there are 53 heads and 174 legs. How many chickens and how many cows are there on the farm?

Solution: We will denote the number of chickens by x and the number of cows by y. The first equation will express the number of heads. x many chickens come with x many heads, and y many cows come with y many heads. The second equation will express the number of legs. x many chickens come with $2x$ many legs, and y many cows come with $4y$ many heads.

$$\begin{cases} x+y=53 \\ 2x+4y=174 \end{cases}$$

Before we start solving the system, let us notice that we can simplify the second equation by dividing both sides by 2. We can often make our life easier with simplifications such as this one.

$$\begin{cases} x+y=53 \\ x+2y=87 \end{cases}$$

To eliminate x, we will multiply the first equation by -1 leave the second equation as is. Then we add the two equations.

$$\begin{cases} \quad -x-y=-53 \\ + \quad x+2y=87 \end{cases}$$
$$y=34$$

Now that we know the value of y, we use the first equation to find x.

$$\begin{aligned} x+34 &= 53 \quad \text{subtract 34} \\ x &= 19 \qquad \Longrightarrow x=19,\ y=34 \end{aligned}$$

Thus we have $\boxed{\text{19 chickens and 34 cows}}$. We check: the number of heads is $19+34=53$, and the number of legs is $2\cdot 19+4\cdot 34=38+136=174$. So our solution is correct.

Example 4. We invested \$10 000 into two bank accounts. One account earns 14% per year, the other account earns 8% per year. How much did we invest into each account if after the first year, the combined interest from the two accounts is \$1238?

Solution: Let us denote the amount invested at 14% by x and the amount invested at 8% by y. The two equations will express the total amount invested, and the total interest earned.

$$\begin{aligned} x+y &= 10\,000 \qquad \text{the amounts invested add up to \$10 000} \\ 0.14x+0.08y &= 1238 \qquad \text{the interests earned add up to \$1238} \end{aligned}$$

We solve the system of equation by elimination. But let us first make the second equation simpler:

$$\begin{aligned} 0.14x+0.08y &= 1238 \qquad \text{multiply by 100} \\ 14x+8y &= 123\,800 \qquad \text{divide by 2} \\ 7x+4y &= 61\,900 \end{aligned}$$

We now have

$$\begin{aligned} x+y &= 10\,000 \\ 7x+4y &= 61\,900 \end{aligned}$$

We will multiply the first equation by -4 to eliminate y. Then we add the two equations.

$$\begin{aligned} -4x-4y &= -40\,000 \\ 7x+4y &= \ \ 61\,900 \\ \hline 3x &= 21\,900 \qquad \text{divide by 3} \\ x &= 7300 \end{aligned}$$

Thus we invested \$7300 at 14%. The other amount can be found using the first equation:

$$\begin{aligned} 7300+y &= 10\,000 \\ y &= 2700 \end{aligned}$$

We invested $\boxed{\text{\$7300 at 14\% and \$2700 at 8\%}}$. We check: the amounts add up to $\$7300+\$2700=\$10\,000$. The interest from the accounts are:

$$14\% \text{ of } 7300 \text{ is } 0.14\,(7300)=1022 \text{ and } 8\% \text{ of } 2700 \text{ is } 0.08\,(2700)=216$$

Since $1022+216=1238$, our solution is correct.

Example 5. We have a jar of coins, all pennies and dimes. All together, we have 372 coins, and the total value of all coins in the jar is \$20.91. How many pennies are there in the jar?

Solution: Let us denote the number of pennies by x and the number of dimes by y. The first equation will express the number of the coins. This equation is therefore $x+y=372$. To express the value of all coins, x many pennies are worth $0.01x$ and y many dimes are worth $0.1y$. The total value of all coins is then

$$0.01x+0.1y=20.91$$

In order to clear the decimals, we may multiply both sides by 100. Then we have

$$x+10y=2091$$

Let us notice that this is the same equation that we would obtain if we expressed the value of all coins in cents and not in dollars. So our system is now

$$\begin{aligned} x+y &= 372 \\ x+10y &= 2091 \end{aligned}$$

We will eliminate x by multiplying the first equation by -1 and leaving the second equation as is. Then we add the two equations and solve for y.

$$\begin{aligned} -x-y &= -372 \\ x+10y &= 2091 \\ \hline 9y &= 1719 \quad \text{divide by } 9 \\ y &= 191 \end{aligned}$$

Thus we have 191 dimes. The number of pennies can be found using any of the two equations. We will use the simplest one, the original first equation.

$$\begin{aligned} x+191 &= 372 \quad \text{subtract } 191 \\ x &= 181 \end{aligned}$$

Thus we have $\boxed{\text{181 pennies and 191 dimes}}$. We check: the number of all coins is $181+191=372$, and the value of the coins is $0.01\cdot 181+0.1\cdot 191=1.81+19.1=20.91$. Thus our solution is correct.

Practice Problems

1. Solve each of the following system of linear equations.

a) $\begin{cases} 2x - y = -8 \\ x + 2y = -9 \end{cases}$

b) $\begin{cases} 2(p-1) - 3(q-1) = 24 \\ p + q = -6 \end{cases}$

c) $\begin{cases} 3x - y = 10 \\ \frac{1}{3}x - y = 2 \end{cases}$

d) $\begin{cases} \frac{1}{2}x + \frac{1}{4}y = -1 \\ \frac{1}{2}y - \frac{1}{3}x = 6 \end{cases}$

e) $\begin{cases} 3x - y = 4 \\ 2(y-3) = -2(x+1) \end{cases}$

f) $\begin{cases} 2a + b = 17 \\ a + b = 5 \end{cases}$

g) $\begin{cases} 3x - 2y = 2 \\ 2x + 3y = 5 \end{cases}$

h) $\begin{cases} x - 2y = 5 \\ 2x + 3y = 10 \end{cases}$

i) $\begin{cases} 0.5x - 1.2y = -1.21 \\ x + 3.2y = 2.06 \end{cases}$

2. Given the equations of two straight lines, find both coordinates of all intersection points.

a) $2x - 5y = -41$ and $x + y = 4$

d) $3x - y = 9$ and $-\frac{2}{3}x + \frac{1}{2}y = -2$

3. There is an animal farm where chickens and cows live. All together, there are 60 heads and 164 legs. How many chickens and how many cows are there on the farm?

4. We invested \$6000 into two bank accounts. One account earns 7% per year, the other account earns 11% per year. How much did we invest into each account if after the first year, the combined interest from the two accounts is \$520?

5. We have 51 coins, all dimes and quarters, in the total value of \$7.05. How many quarters and how many dimes are there?

6. We invested \$7600 in two bank accounts. One account earns 9% per year, the other account earns 13% per year. How much did we invest into each account if after the first year we have a total of \$8508 in the accounts?

Problem Set 16

1. Simplify each of the following.

   a) $\dfrac{2^{-1}+5^{-1}\cdot 3}{2^{-1}-5^{-1}\cdot 3}$ b) $-\dfrac{2}{5}+\dfrac{1}{4}\left(-\dfrac{3}{2}\right)^2$ c) $1-\dfrac{2}{3-\dfrac{4}{5-\dfrac{1}{6}}}$ d) $\dfrac{1}{3}-\left(\left(-\dfrac{2}{3}\right)^2-\dfrac{5}{6}\right)$

2. Let $U=\{1,2,3,4,5,6,7,8,9,10\}$. Find each of the given sets.

   a) $\{x\in U : x\geq 8 \text{ and } x \text{ is even}\}$ c) $\{x\in U : x<3 \text{ and } x>7\}$ e) $\{x\in U : x\geq 4 \text{ and } x>6\}$

   b) $\{x\in U : x\geq 8 \text{ or } x \text{ is even}\}$ d) $\{x\in U : x<3 \text{ or } x>7\}$ f) $\{x\in U : x\geq 4 \text{ or } x>6\}$

3. Compute the prime factorization of each of the following.

   a) 2500 b) 72^{50}

4. Use the prime factorization to find the greatest common factor and least common multiple of 72 and 960.

5. Label each of the following statements as true or false.

   a) Every positive integer has an even number of factors.

   b) The product of two consecutive integers is always even.

   c) If an integer is the product of two different prime numbers, then it has exactly four divisors.

   d) If we square an odd number and then subtract one, the result is always divisible by 4.

   e) There is no prime number that is divisible by 5.

   f) If n is a perfect square, then all exponents in the prime-factorization of n are even.

   g) If A and B are any sets such that $A\cup B=A$, then $A=B$.

   h) If A and B are any sets such that $A\cup B=A$, then $B\subseteq A$.

6. Which one is greater?

   a) 2^{14} or 8^5 b) 10^{100} or 3^{200} c) 6^{20} or 20^6

7. Simplify each of the following. Express your answer using only positive exponents.

   a) $x^6\cdot x^{-7}$ c) $\left(-x^6\right)\cdot\left(-x^{-7}\right)$ e) $\left(2xy^{-3}\right)^{-3}$ g) $\left(x^{-3}\right)^{-2}$ i) $\left(-5a^{-2}b\right)^2$

   b) $(-x)^6\cdot(-x)^{-7}$ d) $\left(x^6\right)^{-7}$ f) $\dfrac{\left(x^3\right)^{-8}}{x^3\cdot x^{-8}}$ h) $\left(\dfrac{\left(x^{-2}\right)^5}{\left(x^3\right)^{-2}}\right)^{-1}$ j) $\left(-5a^{-2}b\right)^3$

   k) $\dfrac{\left(2x^7y^{-3}\right)^{-2}\left(-2xy^{-2}x^4\right)^3}{\left(-x^3y^{-2}\right)^{-4}}$ l) $\dfrac{2a^{-3}\left(-2a^{-1}b\right)^3\left(-a^2b^5\right)^{-2}}{4b^{-5}\left(-2a^{-6}b\right)^2}$

8. Suppose that $x=2\,500\,000\,000$ and $y=0.000\,004$. Write each of the following in scientific notation.

   a) x b) y c) xy^2 d) x^2y^3 e) $\dfrac{1}{y}$

9. Simplify each of the following.

   a) $\left(x^2-2x+3\right)+\left(x^2+5x-1\right)$ c) $-2\left(x^2-2x+3\right)-8\left(x^2+5x-1\right)$ e) $(2x-1)^3$

   b) $\left(x^2-2x+3\right)-\left(x^2+5x-1\right)$ d) $\left(x^2-2x+3\right)\left(x^2+5x-1\right)$

10. Expand each of the following.

    a) $(a-b)(a+b)$ b) $(a-b)\left(a^2+ab+b^2\right)$ c) $(a-b)\left(a^3+a^2b+ab^2+b^3\right)$

11. Completely factor each of the following.

    a) $18a^3b-50ab^3$
    b) $x^2(x-2)-(x-2)$
    c) $x^2(x-1)+9x-9$
    d) $100x-10x^2$
    e) x^2+x
    f) $12x^3\left(y^2-1\right)-3x\left(y^2-1\right)$
    g) $3p^2qy^3-48p^2qy$
    h) $(2x+5)^2-(8x-3)^2$
    i) $(a+2b-3c)^2-(a-2b-3c)^2$
    j) x^4-1

12. Solve each of the following equations. Make sure to check your solutions.

    a) $2x-5(x-3)=(x+1)^2-(x-2)^2$
    b) $6x+x^2=0$
    c) $x(x+1)(3x-7)(x+5)^2=0$
    d) $-x(x-2)+3(x-1)^2=3$
    e) $\frac{1}{2}(x-3)+\frac{1}{2}(x+1)=3x-1$
    f) $\frac{2}{3}(x-1)-\frac{1}{2}(x+5)=\frac{1}{6}(x-2)$
    g) $\frac{3}{4}x-\frac{2}{5}-\left(\frac{1}{2}-\frac{x}{4}\right)=x-\frac{9}{10}$
    h) $x^5=36x^4$
    i) $x^5=36x^3$

13. Solve each of the following inequalities.

    a) $\frac{2x-1}{3}-\frac{x-1}{2}\geq -x+6$ b) $(x-3)^2-(2x+1)^2\geq 8-3x^2$ c) $-\frac{3}{5}x+\frac{1}{2}<\frac{2}{5}$

14. Solve each of the given systems of linear equations.

    a) $\begin{cases} 2x+3y=-7 \\ 2y-x=14 \end{cases}$ b) $\begin{cases} 3x-5y=-23 \\ x+4y=32 \end{cases}$ c) $\begin{cases} \frac{1}{3}x-\frac{1}{2}y=-7 \\ -\frac{2}{3}x+\frac{1}{4}y=5 \end{cases}$ d) $\begin{cases} 2x-y=-5 \\ 3x+y=5 \end{cases}$

    a) $(-8,3)$ b) $(4,7)$ c) $(-3,12)$ d) $(0,5)$

15. Write an equation whose only solutions are -2 and 6.

16. In a class of 30, students discuss their summer activities. 20 students report that they traveled to other cities or countries in the summer. 18 students report that they visited local beaches or waterparks. If 7 students claimed to do none of the above, how many students did both traveling and local beaches of waterparks?

17. $A(-3,-1)$, $B(-5,4)$, and $C(5,8)$ are three vertices of a rectangle.

    a) Find the fourth vertex of the rectangle.
    b) Find the coordinates of the point at which diagonals AC and BD intersect each other.

18. A farmer has 120 animals, all chickens and cows. How many of each live on the farm if the number of legs is 314?

19. One side of a rectangle is ten feet shorter than five times another side. Find the sides of the rectangle if we also know that its perimeter is 52 ft.

20. We have 72 coins in a jar, all dimes and nickels. How many of each coin do we have if the totalo value of all coins is \$4.75?

21. If we increase the side of a square by 2 units, its area will increase by 12 unit^2. How long are the sides of the original square?

22. We have invested \$10 000 in two accounts. One accout earns a 4% interest each year. The other account earns a 3% interest each year. How much was invested in each account if after one year, the combined interest was \$338.

23. If we square a number, we get four times the opposite of twice the number. Find this number.

24. The tickets for the field trip were purchased yesterday for both students and instructors. Children tickets cost \$11, adult tickets cost \$19. The number of children ticket purchased was five more than four times the number of adults tickets purchased. How many of each were purchased if all of the tickets cost a total of \$685 dollars?

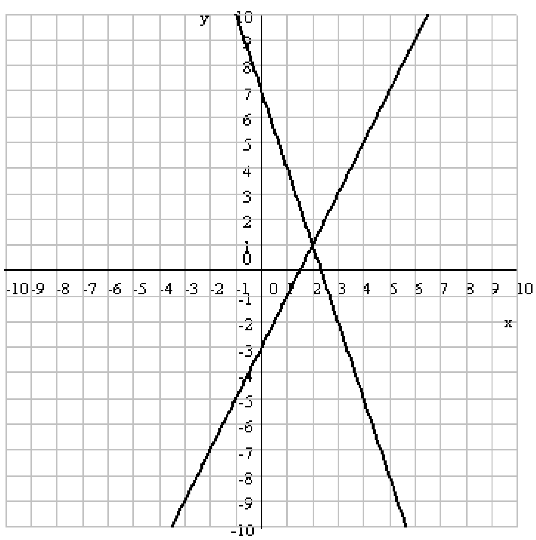
25. The tickets for the field trip were purchased yesterday for both students and instructors. Children tickets cost \$11, adult tickets cost \$19. How many of each were purchased if we purchased 40 tickets for a total of \$472 dollars?

Chapter 17

17.1 Systems of Linear Equations: Substitution

As we are progressing in algebra, we have learned how to solve linear equations and inequalities. The solution set of a linear equation was usually very simple: a single number. The set of all solutions of an inequality is much more complicated. We can no longer just list all elements in the solution set, and so we needed to develop new notation: interval notation.

Straight lines are even more complicated solution sets. They are solution set of a linear equation in two variables. Consider, for example, the graph of the equation $y = 2x - 3$. Every point on the graph of this line have coordinates that form a solution to the equation $y = 2x - 3$. For example, points such as $(7, 11)$ and $(10, 17)$, and $(0, -3)$ are all in this solution set, because $11 = 2 \cdot 7 - 3$, $17 = 2 \cdot 10 - 3$, and $-3 = 2 \cdot 0 - 3$. As a matter of fact, x can be any real number and then there is a unque real number that will work for y. This set is the set of points $P(x, 2x - 3)$. The most meaningful representation of this set might just be its graph.



Consider now two equations in x and y. Our example will be $y = 2x - 3$ and $y = -3x + 7$. If we graphed the two equations in the same coordinate system, we would see two straight lines. If the lines are not parallel, they intersect each other in a point. In this case, this point appears to be $(2, 1)$.
Can we use algebraic methods to see if the point $(2, 1)$ is the intersection point? The intersection point is the only point that is contained in both lines. Indeed, $1 = 2 \cdot 2 - 3$, and so $x = 2, y = 1$ is a solution of $y = 2x - 3$. Also, $1 = -3 \cdot 2 + 7$ and so $x = 2$, $y = 1$ is also a solution of $y = -3x + 7$. Therefore, the point $(2, 1)$ is on both lines and must be the intersection point.

We also say that the ordered pair $(2, 1)$ is a symultaneous solution of both equations.

Definition: Two equations in x and y form a **system of equations**. The **solution**(s) of the system are the point(s) whose coordinates form a solution of both equations. To **solve** a system means to find all solutions of it.

In our example above, $(2,1)$ is the only solution of the system $\begin{cases} y=2x-3 \\ y=-3x+7 \end{cases}$.

In this case, the intersection was a point whose both coordiantes happen to be integers. Such points are called lattice points. In case we are less lucky, we will need more precise tools for solving than graphing the two equations in the same cordinate system. There are several algebraic methods to solve a system of linear equations. Here we will explore a technique called substitution.

Solving Linear Systems using Substitution

The basic idea of this method is to absorb the information of one equation and to substitute that into the other equation, thereby reducing the number of unknowns to one.

Example 1. Solve the given system of linear equations using substitution. $\begin{cases} 2x-y=-19 \\ -x+3y=12 \end{cases}$

Solution: We first inspect the two equations and look for coefficients such as 1 or -1. In this case, the coefficient of y is -1 in the first equation. We solve for y in this equation. We can't solve for y and obtain a number, we can only solve for it in terms of x.

$$\begin{aligned} 2x-y &= -19 && \text{add } y \\ 2x &= y-19 && \text{add } 19 \\ 2x+19 &= y \end{aligned}$$

The information from the first equation can be expressed as $\boxed{y=2x+19}$. This is going to be what we substitute into the other equation by substituting $2x+19$ into y. This way, the equation $x+3y=12$ will become $-x+3(2x+19)=12$. This is now an equation in only one variable for which we can solve.

$$\begin{aligned} -x+3(2x+19) &= 12 \\ -x+6x+57 &= 12 \\ 5x+57 &= 12 && \text{subtract } 57 \\ 5x &= -45 && \text{divide by } 5 \\ x &= -9 \end{aligned}$$

Now that we know the value of x, we return to what we used for substitution and get the value of the other unknown.

$$y=2x+19=2(-9)+19=-18+19=1$$

Therefore, the solution of this system is $x=-9$ and $y=1$, or, in short, $(-9,1)$. We check: the solution of a system is a simultaneous solution of both equations.

Checking $2x-y=-19$

$$\begin{aligned} \text{LHS} &= 2(-9)-1=-18-1=-19 \\ \text{RHS} &= -19 \checkmark \end{aligned}$$

Checking $-x+3y=12$

$$\begin{aligned} \text{LHS} &= -(-9)+3\cdot 1=9+3=12 \\ \text{RHS} &= 12 \checkmark \end{aligned}$$

Therefore, our solution, $\boxed{(-9,1)}$ is correct.

Of course, not all linear systems contain easy coefficients such as 1 or -1.

Example 2. Solve the given system of linear equations. $\begin{cases} 3x-5y=11 \\ 2x+3y=20 \end{cases}$

Solution: We will solve for x in the second equation and substitute the information into the first equation. First, we solve for x in $2x+3y=20$.

$$\begin{aligned} 2x+3y &= 20 && \text{subtract } 3y \\ 2x &= -3y+20 && \text{divide by 2} \\ x &= \frac{-3y+20}{2} \end{aligned}$$

This is the information we will substitute into the first equation. $3x-5y=11$ will become $3\left(\frac{-3y+20}{2}\right)-5y=11$. We solve this equation for y.

$$\begin{aligned} 3\left(\frac{-3y+20}{2}\right)-5y &= 11 && \text{multiply by 2} \\ 3(-3y+20)-10y &= 22 && \text{distribute 3} \\ -9y+60-10y &= 22 && \text{combine like terms} \\ -19y+60 &= 22 && \text{subtract 60} \\ -19y &= -38 && \text{divide by } -19 \\ y &= 2 \end{aligned}$$

Now that we know that y is 2, we find x using the expression we used for the substitution.

$$x=\frac{-3y+20}{2}=\frac{-3\cdot 2+20}{2}=\frac{-6+20}{2}=\frac{14}{2}=7$$

Thus, the solution of this system is $x=7$ and $y=2$, or, in short, $(7,2)$. We check: the solution of a system is a simultaneous solution of both equations.

Checking $3x-5y=11$

$$\begin{aligned} \text{LHS} &= 3\cdot 7-5\cdot 2=21-10=11 \\ \text{RHS} &= 11\ \checkmark \end{aligned}$$

Checking $2x+3y=20$

$$\begin{aligned} \text{LHS} &= 2\cdot 7+3\cdot 2=14+6=20 \\ \text{RHS} &= 20\ \checkmark \end{aligned}$$

Therefore, our solution, $\boxed{(7,2)}$ is correct.

Most real-world problems boil down to systems of equations. In this sense, solving systems of equations is one of the most important tasks in problem solving.

Example 3. There is an animal farm where chickens and cows live. All together, there are 53 heads and 174 legs. How many chickens and how many cows are there on the farm?

Solution: We will denote the number of chickens by x and the number of cows by y. The first equation will express the number of heads. x many chickens come with x many heads, and y many cows come with y many heads. The second equation will express the number of legs. x many chickens come with $2x$ many legs, and y many cows come with $4y$ many heads.

$$\begin{cases} x+y=53 \\ 2x+4y=174 \end{cases}$$

Before we start solving the system, let us notice that we can simplify the second equation by dividing both sides by 2. We can often make our life easier with simplifications such as this one.

$$\begin{cases} x+y=53 \\ x+2y=87 \end{cases}$$

We will solve for x in the first equation and substitute that expression into the second equation.

$$x=53-y \quad \Longrightarrow \quad (53-y)+2y=87$$

We solve this equation for y.

$$\begin{aligned} 53-y+2y &= 87 \\ 53+y &= 87 \qquad \text{subtract 53} \\ y &= 34 \end{aligned}$$

Now that we know the value of y, we can easily find x.

$$x=53-34=19 \quad \Longrightarrow x=19, \ \ y=34$$

Thus we have $\boxed{\text{19 chickens and 34 cows}}$. We check: the number of heads is $19+34=53$, and the number of legs is $2\cdot 19+4\cdot 34=38+136=174$. So our solution is correct.

Example 4. We invested \$10 000 into two bank accounts. One account earns 14% per year, the other account earns 8% per year. How much did we invest into each account if after the first year, the combined interest from the two accounts is \$1238?

Solution: Let us denote the amount invested at 14% by x and the amount invested at 8% by y. The two equations will express the total amount invested, and the total interest earned. Then the interest earned form the first account is 14% of x, and that of the second account is 8% of y. Recall that 14% of x can be written as $0.14x$ and 8% of y as $0.08y$.

$$\begin{cases} x+y=10\,000 & \text{the amounts invested add up to \$10 000} \\ 0.14x+0.08y=1238 & \text{the interests earned add up to \$1238} \end{cases}$$

We solve the system of equation by substitution, but let us first make the second equation simpler:

$$\begin{aligned} 0.14x+0.08y &= 1238 \qquad \text{multiply by 100} \\ 14x+8y &= 123\,800 \quad \text{divide by 2} \\ 7x+4y &= 61\,900 \end{aligned}$$

We now have

$$\begin{cases} x+y=10\,000 \\ 7x+4y=61\,900 \end{cases}$$

We will solve for y in the first equation and substitute the result into the second equation.

$$x+y=10\,000 \quad \Longrightarrow \quad y=10\,000-x$$

Now the equation $7x+4y=61\,900$ becomes $7x+4(10\,000-x)=61\,900$. We can solve this equation for x.

$$\begin{aligned} 7x+4(10\,000-x) &= 61\,900 && \text{distribute 4} \\ 7x+40\,000-4x &= 61\,900 && \text{combine like terms} \\ 3x+40\,000 &= 61\,900 && \text{subtract 40 000} \\ 3x &= 21\,900 && \text{divide by 3} \\ x &= 7300 \end{aligned}$$

We can now easily find y using $y=10\,000-x$.

$$y=10\,000-x=y=10\,000-7300=2700$$

Our solution, $x=7300$ and $y=2700$ means that we invested $\boxed{\$7300 \text{ at } 14\% \text{ and } \$2700 \text{ at } 8\%}$. We check: the amounts add up to $\$7300+\$2700=\$10\,000$. The interest from the accounts are:

$$14\% \text{ of } 7300 \text{ is } 0.14(7300)=1022 \text{ and } 8\% \text{ of } 2700 \text{ is } 0.08(2700)=216$$

Since $1022+216=1238$, our solution is correct.

Example 5. We have a jar of coins, all pennies and dimes. All together, we have 372 coins, and the total value of all coins in the jar is \$20.91. How many pennies are there in the jar?

Solution: Let us denote the number of pennies by x and the number of dimes by y. The first equation will express the number of the coins. This equation is therefore $x+y=372$. To express the value of all coins, x many pennies are worth $0.01x$ and y many dimes are worth $0.1y$. The total value of all coins is then

$$0.01x+0.1y=20.91$$

In order to clear the decimals, we may multiply both sides by 100. Then we have

$$x+10y=2091$$

Let us notice that this is the same equation that we would obtain if we expressed the value of all coins in cents and not in dollars. So our system is now

$$\begin{cases} x+y=372 \\ x+10y=2091 \end{cases}$$

We solve for x in the first equation and substitute that into the second equation.

$$x+y=372 \quad \Longrightarrow \quad x=372-y$$

Now the other equation, $x+10y=2091$ becomes

$$\begin{aligned} 372-y+10y &= 2091 && \text{combine like terms} \\ 9y+372 &= 2091 && \text{subtract 372} \\ 9y &= 1719 && \text{divide by 9} \\ y &= 191 && \Longrightarrow \quad x=372-y=372-191=181 \end{aligned}$$

The solution $x=181, y=191$ means that we have $\boxed{\text{181 pennies and 191 dimes}}$. We check: the number of all coins is $181+191=372$, and the value of the coins is $0.01\cdot 181+0.1\cdot 191=1.81+19.1=20.91$. Thus our solution is correct.

Practice Problems

1. Solve each of the following system of linear equations.

   a) $\begin{cases} 3x+y=-4 \\ x-3y=-8 \end{cases}$

   b) $\begin{cases} 5(p-1)-2(q-1)=22 \\ p-q=8 \end{cases}$

   c) $\begin{cases} a+3b=10 \\ 3b-5a=22 \end{cases}$

   d) $\begin{cases} \frac{1}{2}x+\frac{1}{4}y=5 \\ \frac{1}{2}y-\frac{1}{3}x=-6 \end{cases}$

   e) $\begin{cases} 2x-y=1 \\ 2(y-3)=6(x-1) \end{cases}$

   f) $\begin{cases} 2a+3b=-16 \\ (a+3)^2=a^2+2b+27 \end{cases}$

   g) $\begin{cases} 2x+3y=3 \\ 5x-2y=4 \end{cases}$

   h) $\begin{cases} 3x-2y=-8 \\ -2x+3y=12 \end{cases}$

   i) $\begin{cases} 2r-0.5s=-1.7 \\ 1.5r+s=0.65 \end{cases}$

2. Given the equations of two straight lines, find both coordinates of all intersection points.

   a) $2x-5y=-41$ and $x+y=4$

   b) $x+y=-5$ and $2x-y=-7$

   c) $y=\frac{3}{4}x-2$ and $2y=x$

   d) $5x-y=-35$ and $y=-\frac{3}{4}x+\frac{1}{2}$

   e) $y=-\frac{2}{3}x+7$ and $x+2y=6$

3. There is an animal farm where chickens and cows live. All together, there are 52 heads and 134 legs. How many chickens and how many cows are there on the farm?

4. We invested \$9700 into two bank accounts. One account earns 7% per year, the other account earns 12% per year. How much did we invest into each account if after the first year, the combined interest from the two accounts is \$1004?

5. We have 54 coins, all dimes and quarters, in the total value of \$10.05. How many quarters and how many dimes are there?

6. We invested \$7800 into two bank accounts. One account earns 9% per year, the other account earns 10% per year. How much did we invest into each account if after the first year we have a total of \$8549 in the accounts?

17.2 Factoring by Trial and Error

For all polynomials, factoring is unique. For example, the expression $x^2 - 16$ can *only* be factored as $(x+4)(x-4)$. As long as we insist to completely factor a polynomial, there is just one correct form. This is different from equations that can have more than one solution.

Because of its uniqueness, we are not forced to develop mathods to systematically find all factored form; if we found one, we found *it*. Because of this, we are allowed to use trial and error to 'stumble into' the factored form. **Trial and error** (or by inspection) refers to a factoring method where we make educated guesses that allow us to quickly factor a quadratic expression.

In what follows, we will only focus on expressions in which the coefficient of the quadraic term is 1. Suppose we expand the expression $(x+a)(x+b)$.

$$(x+a)(x+b) = x^2 + bx + ax + ab = x^2 + (b+a)x + ab$$

The result is $x^2 + (a+b)x + ab$. Notice that the linear coefficient is the sum of a and b, and the number term is the product of a and b. We can use these facts to quickly factor a quadratc expression such as $x^2 + 7x + 12$. We will start with the easiest case: when all signs are +.

Case 1: All signs are + in the expression to be factored.

Example 1. Completely factor the expression $x^2 + 7x + 12$.

Solution: If $x^2 + 7x + 12$ is factored into $(x+a)(x+b)$, then factoring is just a matter of finding a and b. Consider the equation $a+b=7$. This equation has infinitely many solutions. For any value of a there is a value of b that works. If $a=1$, then $b=6$, if $a=10$, then $b=-3$, and so on. This is not the case with the equation $ab=12$. As long as we are looking for integer values, there is a finite list of how the product of two numbers is 12. Because of this, we will **always start with the number term** that is the product of a and b.

We can quickly list all the pairs of positive numbers with a product of 12.

12	
1	12
2	6
3	4

Now we consider the three pairs as candidates for a and b. We are looking for the pair with sum 7. Clearly that is 3 and 4. Once we found a and b with product 12 and sum 7, we have the factored form: $x^2 + 7x + 12 = \boxed{(x+3)(x+4)}$

Naturally, things are not always as simple. Consider now the second case, when the last term is positive but the second term is negative.

Case 2: The third sign is + in the expression to be factored, and the second sign is negative.

Example 2. Completely factor the expression $x^2 - 17x + 30$.

Solution: We are looking for two numbers a and b with a product of 30 and a sum of -17. A positive product indicates that a and b are either both positive or both negative. Because of the negative second sign, both positive is impossible. Therefore, we are looking for two negative numbers.

As always, we start with the equation $ab = 30$. We list all the pairs of negative numbers with a product of 30.

30	
-1	-30
-2	-15
-3	-10
-5	-6

Now we consider these pairs as candidates for a and b. We are looking for the pair with sum -17. Clearly that is -2 and -15. Once we found a and b with product 30 and sum -17, we have the factored form: $x^2-17x+30=\boxed{(x-2)(x-15)}$

Case 3: The third sign is - in the expression to be factored.

Example 3. Completely factor the expression $x^2-2x-48$.

Solution: We are looking for two numbers a and b with a product of -48 and a sum of -2. A negative product indicates that one of a and b is positive and the other is negative. Now we inspect the second sign. If the sum of a positive and a negative number is negative, then between the two of them, the negative one has the greater absolute value. These observations will make our task much easier.

As always, we start with the equation $ab=-48$. We list all the pairs of positive numbers with a product of 48, and put a $-$ sign in front of the greater one.

-48	
1	-48
2	-24
3	-16
4	-12
6	-8

Now we consider these pairs as candidates for a and b. We are looking for the pair with sum -2. Clearly that is -8 and 6. Once we found a and b with product -48 and sum -2, we have the factored form: $x^2-2x-48=\boxed{(x-8)(x+6)}$

In the next example, the second term is positive.

Example 4. Completely factor the expression $x^2+11x-60$.

Solution: We are looking for two numbers a and b with a product of -60 and a sum 11. A negative product indicates that one of a and b is positive and the other is negative. Now we inspect the second sign. If the sum of a positive and a negative number is positive, then between the two of them, the negative one has the smaller absolute value. These observations will make our task much easier.

We start with the equation $ab=-60$. We list all the pairs of positive numbers with a product of 60, and put a $-$ sign in front of the smaller one.

-60	
-1	60
-2	30
-3	20
-4	15
-5	12
-6	10

Now we consider these pairs as candidates for a and b. We are looking for the pair with sum 11. Clearly that is -4 and 15. Therefore, $x^2+11x-60=\boxed{(x+15)(x-4)}$

We can check by mutiplication: $(x+15)(x-4)=x^2-4x+15x-60=x^2+11x-60$, so our solution is correct.

This method is quick and easy. However, it only works for simple expressions that start with x^2. Consider for example the product $(2x+3)(x+5)=2x^2+10x+3x+15=2x^2+13x+15$. The middle term is clearly not the sum of 3 and 5. Because of uniqueness of factoring, trial and error is still a good approach, but the middle term is no longer just the sum.

Example 5. Solve the equation $(x-2)(x-4)=24$

Solution: We might be tempted to use the factored form on the left-hand side, but it cannot be used because the other side is not zero. So we need to expand the product, reduce one side to zero and then factor.

$$\begin{aligned}(x-2)(x-4) &= 24 && \text{expand product}\\ x^2-6x+8 &= 24 && \text{subtract 24}\\ x^2-6x-16 &= 0\end{aligned}$$

To factor $x^2-6x-16$, we need to find two integers a and b with product -16 and sum -6. The negative sign in -16 indicates that one is positive, the other one is negative. The negative sign in -6 indicates that the negative number has the greater absolute value.

	-16
1	-16
2	-8
4	-4

The only pair with sum -6 is -8 and 2. Therefore, $x^2-6x-16=(x-8)(x+2)$. Applying the zero product rule, we obtain $x=8$ and $x=-2$.

We check: if $x=8$, then LHS $=(8-2)(8-4)=6\cdot 4=24=$ RHS ✓

and if $x=-2$, then LHS $=(-2-2)(-2-4)=-4(-6)=24=$ RHS ✓

So both $\boxed{8 \text{ and } -2}$ work.

If there is a leading coefficient, this method only works if it is also the greatest common factor and can be factored out.

Example 6. One side of a rectangle is 6 feet shorter than twice another side. Find the sides of the rectangle if we also know that its area is $140\,\text{ft}^2$.

Solution: If we label one side by x, the other side is $2x-6$. The equation will express the are of the rectangle.

$$\begin{aligned}x(2x-6) &= 140 && \text{distribute } x\\ 2x^2-6x &= 140 && \text{subtract 140}\\ 2x^2-6x-140 &= 0 && \text{factor out 2}\\ 2\left(x^2-3x-70\right) &= 0\end{aligned}$$

We will factor $x^2-3x-70$. We are lookin for two integers with product -70 and sum -3. These are easily found: -10 and 7. Therefore, $x^2-3x-70=(x-10)(x+7)$. Back to the equation:

$$2(x-10)(x+7)=0 \implies x_1=10 \text{ and } x_2=-7$$

The two solutions of this equation are 10 and -7. Since we are looking for a distance and distances cannot be negative, -7 is easily ruled out. If the shorter side is x, then the other side is $2x-6=2\cdot 10-6=14$. So the two sides are $\boxed{10\,\text{ft and } 14\,\text{ft}}$ long.

Example 7. Find all numbers that are exactly six less than their own square.

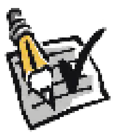
Solution: If we label such a number by x, then the equation will be $x^2 = x - 6$.

$$\begin{aligned} x^2 &= x - 6 \qquad \text{subtract } x \text{ and add } 6 \\ x^2 - x - 6 &= 0 \end{aligned}$$

We quickly find -3 and 2 as two numbers with sum -1 and product -6.

$$(x-3)(x+2) = 0 \quad \Longrightarrow \quad x_1 = 3 \text{ and } x_2 = -2$$

We check: 3 is indeed 6 less than 9, and -2 is indeed less than 4. So our asnwer is $\boxed{-2 \text{ and } 3}$. Perhaps even more importantly, we also proved that there is no other number with this property.



Practice Problems

1. Completely factor each of the following using the trial and error method.

   a) $x^2 + 2x - 15$
   b) $x^2 - 12x + 32$
   c) $x^2 - 2x - 3$
   d) $x^2 - 10x + 25$
   e) $x^2 + 9x + 20$
   f) $x^2 - x - 20$
   g) $x^2 - 5x + 6$
   h) $x^2 - 5x - 6$
   i) $2x^2 - 8x - 42$
   j) $-3x^2 - 3x + 6$

2. Solve each of the following equations. Make sure to check your solutions.

   a) $(w+5)(w-1) = 0$
   b) $(w+5)(w-1) = 55$
   c) $(x-2)(x+3) = 50$
   d) $5x^3 = 10x^2 + 75x$
   e) $(2x-1)^2 - x = 3x(x-1)$
   f) $(x+5)^2 + (x-1)^2 = (x+6)^2 + 2$

3. Find all numbers satisfying the given conditions.

   a) The square of the number is twenty greater than the number.

   b) The sum of the square of the number and three times the number is 70.

4. a) One side of a rectangle is twelve feet shorter than three times another side. Find the sides of this rectangle if we also know that the area of this rectangle is $420\,\text{ft}^2$.

   b) One side of a rectangle is twelve feet longer than three times another side. Find the sides of this rectangle if we also know that the area of this rectangle is $288\,\text{ft}^2$.

Chapter 18

18.1 Linear Equations 3

Part 1 - More Fractions

We will further study solving linear equations. Let us first recall a few definitions.

> **Definition**: An **equation** is a statement in which two expressions (algebraic or numeric) are connected with an equal sign. A **solution** of an equation is a number (or an ordered set of numbers) that, when substituted into the variable(s) in the equation, makes the statement of equality of the equation true. To **solve an equation** is to find *all* solutions of it. The set of all solutions is also called the solution set.

Example 1. Solve each of the given equations.

a) $\dfrac{2x-5}{3}-\dfrac{x-2}{5}=x-5$ b) $\dfrac{2}{3}(x-1)-\dfrac{1}{2}\left(x+\dfrac{3}{5}\right)=-x+\dfrac{37}{10}$

Solution: a) The main idea here is that we can clear denominators of fractions in equations by muliplying by a suitable number. As always, we will multiply both sides by a suitable number.

$$\begin{aligned}
\frac{2x-5}{3}-\frac{x-2}{5} &= x-5 && \text{we write everything as a fraction}\\
\frac{2x-5}{3}-\frac{x-2}{5} &= \frac{x-5}{1} && \text{bring all three fractions to the common denominator}\\
\frac{5(2x-5)}{15}-\frac{3(x-2)}{15} &= \frac{15(x-5)}{15} && \text{clear denominators by multiplying by 15}\\
5(2x-5)-3(x-2) &= 15(x-5) && \text{remove parentheses}\\
10x-25-3x+6 &= 15x-75 && \text{combine like terms}\\
7x-19 &= 15x-75 && \text{subtract } 7x\\
-19 &= 8x-75 && \text{add 75}\\
56 &= 8x && \text{divide by 8}\\
7 &= x
\end{aligned}$$

We check: if $x=7$, then

$$\text{LHS}=\frac{2\cdot 7-5}{3}-\frac{7-2}{5}=\frac{14-5}{3}-\frac{5}{5}=\frac{9}{3}-1=3-1=2 \quad \text{and} \quad \text{RHS}=7-5=2$$

and so our solution, $\boxed{x=7}$ is correct.

Note that it is faster if we multiply both sides by the common multiple without bringing the terms to the common denominator. Here is the recommended computation:

$$\begin{aligned} \frac{2x-5}{3}-\frac{x-2}{5} &= \frac{x-5}{1} && \text{multiply both sides by 15} \\ 5(2x-5)-3(x-2) &= 15(x-5) \end{aligned}$$

Why is this step correct? Here is the breakdown:

$$\frac{2x-5}{3}-\frac{x-2}{5}=\frac{x-5}{1}$$

$$15\left(\frac{2x-5}{3}-\frac{x-2}{5}\right)=15\left(\frac{x-5}{1}\right)$$

$$\frac{15}{1}\cdot\frac{2x-5}{3}-\frac{15}{1}\cdot\frac{x-2}{5}=\frac{15}{1}\cdot\frac{x-5}{1}$$

$$\frac{15(2x-5)}{3}-\frac{15(x-2)}{5}=\frac{15(x-5)}{1}$$

$$5(2x-5)-3(x-2)=15(x-5)$$

b) There are several methods available to solve this equation. The method presented here is focusing on how similar this equation is to the previous example.

$$\begin{aligned} \frac{2}{3}(x-1)-\frac{1}{2}\left(x+\frac{3}{5}\right) &= -x+\frac{37}{10} && x=1x=\frac{5}{5}x \text{ and } -x=-1x=-\frac{10}{10}x \\ \frac{2}{3}(x-1)-\frac{1}{2}\left(\frac{5x}{5}+\frac{3}{5}\right) &= -\frac{10x}{10}+\frac{37}{10} \\ \frac{2}{3}\cdot\frac{x-1}{1}-\frac{1}{2}\cdot\frac{5x+3}{5} &= \frac{-10x+37}{10} \\ \frac{2(x-1)}{3}-\frac{5x+3}{10} &= \frac{-10x+37}{10} && \text{bring all fractions to the common denominator} \\ \frac{20(x-1)}{30}-\frac{3(5x+3)}{30} &= \frac{3(-10x+37)}{30} && \text{clear denominator by multiplying by 30} \\ 20(x-1)-3(5x+3) &= 3(-10x+37) && \text{remove parentheses} \\ 20x-20-15x-9 &= -30x+111 && \text{combine like terms} \\ 5x-29 &= -30x+111 && \text{add } 30x \\ 35x-29 &= 111 && \text{add 29} \\ 35x &= 140 && \text{divide by 35} \\ x &= 4 \end{aligned}$$

We check: if $x=4$, then

$$\begin{aligned} \text{LHS} &= \frac{2}{3}(4-1)-\frac{1}{2}\left(4+\frac{3}{5}\right)=\frac{2}{3}\cdot 3-\frac{1}{2}\left(\frac{20}{5}+\frac{3}{5}\right)=2-\frac{1}{2}\cdot\frac{23}{5}=2-\frac{23}{10}=\frac{20}{10}-\frac{23}{10}=\frac{-3}{10} \\ \text{RHS} &= -4+\frac{37}{10}=-\frac{40}{10}+\frac{37}{10}=-\frac{3}{10} \end{aligned}$$

and so our solution, $\boxed{x=4}$ is correct.

Just as before, we can get from $\frac{2(x-1)}{3}-\frac{5x+3}{10}=\frac{-10x+37}{10}$ immediately to

$$2\cdot 10(x-1)-3(5x+3)=3(-10x+37)$$

by multiplying both sides by 30 without bringing the fractions to lowest terms.

Example 2. Solve each of the given equations. Make sure to check your solutions.

a) $\frac{1}{2}m-1=\frac{5}{4}m-\frac{1}{4}$ b) $\frac{2}{3}x-4-\frac{1}{6}(x+6)=\frac{1}{2}(x-10)$ c) $\frac{2}{3}(x-1)-\frac{1}{2}\left(x+\frac{3}{5}\right)=-x+\frac{37}{10}$

Solution:

a)

$$\frac{1}{2}m-1=\frac{5}{4}m-\frac{1}{4} \qquad \text{subtract } \frac{1}{2}m \qquad \text{margin work: } \frac{5}{4}-\frac{1}{2}=\frac{5}{4}-\frac{2}{4}=\frac{3}{4}$$

$$-1=\frac{3}{4}m-\frac{1}{4} \qquad \text{add } \frac{1}{4} \qquad -1+\frac{1}{4}=\frac{-4}{4}+\frac{1}{4}=-\frac{3}{4}$$

$$-\frac{3}{4}=\frac{3}{4}m \qquad \text{divide by } \frac{3}{4} \qquad -\frac{3}{4}\div\frac{3}{4}=-\frac{3}{4}\cdot\frac{4}{3}=-1$$

$$-1=m$$

So the only solution of this equation is -1. We check; if $m=-1$,

$$\text{LHS}=\frac{1}{2}(-1)-1=-\frac{1}{2}-1=\frac{-1}{2}-\frac{2}{2}=-\frac{3}{2} \quad \text{and}$$

$$\text{RHS}=\frac{5}{4}(-1)-\frac{1}{4}=-\frac{5}{4}-\frac{1}{4}=-\frac{6}{4}=-\frac{3}{2} \qquad \Longrightarrow \qquad \text{LHS}=\text{RHS}$$

So our solution, $\boxed{m=-1}$ is correct.

b) We first eliminate the parentheses by applying the distributive law.

$$\frac{2}{3}x-4-\frac{1}{6}(x+6) = \frac{1}{2}(x-10) \qquad \text{eliminate parentheses}$$

$$\frac{2}{3}x-4-\frac{1}{6}x-1 = \frac{1}{2}x-5 \qquad \text{combine like terms} \qquad \text{margin work: } \frac{2}{3}-\frac{1}{6}=\frac{4-1}{6}=\frac{3}{6}=\frac{1}{2}$$

$$\frac{1}{2}x-5 = \frac{1}{2}x-5 \qquad \text{subtract } \frac{1}{2}x$$

$$-5 = -5$$

Our last line indicates that this equation is an identity; all numbers are solutions of it.

c) There are several methods available. The method presented here is focusing on how this equation requires the same steps as before, only, each step will take more work. The computations for each step are shown separate, on the margin. We start by distributing $\frac{2}{3}$ and $-\frac{1}{2}$.

$$\frac{2}{3}(x-1)-\frac{1}{2}\left(x+\frac{3}{5}\right)=-x+\frac{37}{10} \qquad \text{distribute} \quad -\frac{1}{2}\cdot\frac{3}{5}=-\frac{3}{10}$$

$$\frac{2}{3}x-\frac{2}{3}-\frac{1}{2}x-\frac{3}{10}=-x+\frac{37}{10} \qquad \text{combine like terms:} \quad \frac{2}{3}x-\frac{1}{2}x=\frac{4-3}{6}x=\frac{1}{6}x$$

$$\text{and } -\frac{2}{3}-\frac{3}{10}=\frac{-20-9}{30}=-\frac{29}{30}$$

$$\frac{1}{6}x-\frac{29}{30}=-x+\frac{37}{10} \qquad \text{add } x \quad \frac{1}{6}+1=\frac{1}{6}+\frac{6}{6}=\frac{7}{6}$$

$$\frac{7}{6}x-\frac{29}{30}=\frac{37}{10} \qquad \text{add } \frac{29}{30}: \quad \frac{37}{10}+\frac{29}{30}=\frac{111+29}{30}=\frac{140}{30}=\frac{14}{3}$$

$$\frac{7}{6}x=\frac{14}{3} \qquad \text{divide by } \frac{7}{6}: \quad \frac{14}{3}\div\frac{7}{6}=\frac{14}{3}\cdot\frac{6}{7}=\frac{2\cdot 7\cdot 3\cdot 2}{3\cdot 7}=\frac{4}{1}$$

$$x=4$$

We check: if $x=4$, then

$$\text{LHS} = \frac{2}{3}(4-1)-\frac{1}{2}\left(4+\frac{3}{5}\right)=\frac{2}{3}\cdot 3-\frac{1}{2}\left(\frac{20}{5}+\frac{3}{5}\right)=2-\frac{1}{2}\cdot\frac{23}{5}=2-\frac{23}{10}=\frac{20}{10}-\frac{23}{10}=\frac{-3}{10}$$

$$\text{RHS} = -4+\frac{37}{10}=-\frac{40}{10}+\frac{37}{10}=-\frac{3}{10}$$

and so our solution, $\boxed{x=4}$ is correct.

Part 2 - Linear Equations after Multiplying Expressions

Now that we can multiply algebraic expressions, we can solve equations that include such products.

Example 3. Solve the given equation. Make sure to check your solution.

$$(2x-3)^2-(x+1)(3x-5)=11-(x-1)(3-x)$$

Solution: We carefully expand the indicated products and combine like terms. Notice that even after we expanded $(x+1)(3x-5)$ and $(x-1)(3-x)$, we still need to keep them in parentheses because we are subtracting them. We will first work out the products.

$$(2x-3)^2 = (2x-3)(2x-3)=4x^2-6x-6x+9=4x^2-12x+9$$
$$(x+1)(3x-5) = 3x^2-5x+3x-5=3x^2-2x-5$$
$$(x-1)(3-x) = 3x-x^2-3+x=-x^2+4x-3$$

We are now ready to begin solving the equation.

$$(2x-3)^2-(x+1)(3x-5) = 11-(x-1)(3-x)$$
$$4x^2-12x+9-\left(3x^2-2x-5\right) = 11-\left(-x^2+4x-3\right) \qquad \text{to subtract is to add the opposite}$$

$$\begin{aligned}
4x^2-12x+9-3x^2+2x+5 &= 11+x^2-4x+3 && \text{combine like terms}\\
x^2-10x+14 &= x^2-4x+14 && \text{subtract } x^2\\
-10x+14 &= -4x+14 && \text{add } 10x\\
14 &= 6x+14 && \text{subtract } 14\\
0 &= 6x && \text{divide by } 6\\
0 &= x
\end{aligned}$$

We check: if $x=0$, then

$$\begin{aligned}
\text{LHS} &= (2\cdot 0-3)^2-(0+1)(3\cdot 0-5)=(-3)^2-1(-5)=9+5=14\\
\text{RHS} &= 11-(0-1)(3-0)=11-(-1)3=11+3=14
\end{aligned}$$

and so our solution, $\boxed{x=0}$ is correct.

Example 4. The area of a square would increase by 93 unit^2 if we increased each of its sides by 3 units. How long are the sides of the square?

Solution: Let us denote the side of the square by x. Then its area is x^2. If we increase its sides by 3 units, the new side would by $x+3$, and the new area $(x+3)^2$. The equation would express the difference between the two areas.

$$\begin{aligned}
(x+3)^2 &= x^2+93\\
x^2+6x+9 &= x^2+93 && \text{subtract } x^2\\
6x+9 &= 93 && \text{subtract } 9\\
6x &= 84 && \text{divide by } 6\\
x &= 14
\end{aligned}$$

Thus the square has sides $\boxed{\text{14 units long}}$. We check: The area of the smaller square is $14^2=196$ unit^2. If we increase the sides by 3 units, they would be 17 units. Then the area if $17^2=289$ unit^2. The difference between the two areas is indeed 93unit^2, as $289-196=93$. Thus our solution is correct.

Sample Problems

Solve each of the following equations. Make sure to check your solutions.

1. $2x+3=4x+9$
2. $3w-5=5(w+1)$
3. $3y-9=-2y+4$
4. $4-x=3(x-7)$
5. $7(j-5)+9=2(-2j+5)+5j$
6. $3(x-5)-5(x-1)=-2x+1$
7. $\dfrac{3-x}{4}-\dfrac{10-3x}{5}=x+2$
8. $\dfrac{3x+17}{2}=x-1+\dfrac{x+19}{2}$
9. $\dfrac{2}{3}(x-1)=\dfrac{3}{5}(x-4)+1$
10. $\dfrac{2}{3}(x-7)=\dfrac{4}{5}(x+1)$
11. $\dfrac{x+2}{4}-\dfrac{x-3}{5}=20-x$
12. $(x-3)^2-(2x-5)(x+1)=5-(x-1)^2$
13. $(x+1)^2-(2x-1)^2+(3x)^2=6x(x-2)$
14. $12-(2p-1)(p+1)=-2(-p+5)^2$
15. If we increase all sides of a square by 3 units, the area of the square increase by 75 units. How long are the sides of the square?

Practice Problems

Solve each of the following equations. Make sure to check your solutions.

1. $5x-3=x+9$
2. $-x+13=2x+1$
3. $-2x+4=5x-10$
4. $5x-7=6x+8$
5. $8x-1=3x+19$
6. $-7x-1=3x-21$
7. $3(x-4)=2(x+5)$
8. $4(5x+1)=6x+4$
9. $a-3=5(a-1)-2$
10. $3y-2=-2y+18$
11. $8(x-3)-3(5-2x)=x$
12. $5(x-1)-3(x+1)=3x-8$
13. $-2x-(3x-1)=2(5-3x)$
14. $3(x-4)+5(x+8)=2(x-1)$
15. $5(x-1)-3(-x+1)=8x$
16. $\dfrac{3x-1}{5}-\dfrac{7-x}{3}=2x+6$
17. $\dfrac{3x-1}{4}+\dfrac{8-4x}{3}=-3-x$
18. $\dfrac{3x-2}{5}+\dfrac{x+4}{3}=\dfrac{14(x+1)}{15}$
19. $\dfrac{3}{8}x+1\dfrac{4}{5}=\dfrac{1}{4}x+1\dfrac{3}{10}$
20. $\dfrac{2x+1}{3}-\dfrac{3-x}{2}=x-2$
21. $\dfrac{2}{3}x-1=-\dfrac{2}{3}\left(x+\dfrac{1}{2}\right)$
22. $2(b+1)-5(b-3)=2(b-7)+1$
23. $3(2x-1)-5(2-x)=4(x-1)+5$
24. $3(2x-7)-2(5x+2)=-5x-30$
25. $3(x-4)-4(x-3)=3(x-2)+2(3-x)$
26. $2x(3x-1)-x(5x-2)=(x-1)^2$
27. $y^2-(y-1)^2+(y-2)^2=(y-3)(y-5)$
28. $(3x)^2-(x+3)(5x-3)=(5-2x)^2-16$
29. $(w+4)(1-2w)=3w-2(w-3)^2$
30. $(2x-3)^2-3(x-2)^2=10-(x-2)(7-x)$
31. $(2-w)^2-(2w-3)^2+7=(w-2)(5-3w)$
32. $3(a+11)-a(8-3a)=3(a-2)^2$
33. $-5(2x-1)-(4-x)^2=3-(x+1)^2$
34. $5(-3-x)-3x(x-2)=x-3(x+2)(x-5)$
35. $2(-m-2)^2-(m-2)^2=8m+(m+2)^2$
36. $(3a-5)(2-a)-(2a-1)(a+3)=-5a^2-7$
37. $\dfrac{1}{2}(x-3)^2-\dfrac{1}{2}(x+1)^2=4(x-7)$
38. If we increase all sides of a square by 6 units, the square's area increasee by 144 unit^2. How long are the sides of the square?

18.2 Square Roots of Integers

Caution! Currently, square roots are defined differently in different textbooks. In conversations about mathematics, approach the concept with caution and first make sure that everyone in the conversation uses the same definition of square roots.

The square of an integer, such as $0, 1, 4, 9, 16, 25, \ldots$ is called a **perfect square**. In this section we will only discuss the square root of perfect squares (and their opposites). The square root of other numbers such as 2, 3, or 10 also exist and is studied later in intermediate algebra. For now, we will be dealing with perfect squares only.

The main idea of square roots is simple. Consider the number 36. What number's square equals to 36? However, there is a complication here. Both 6 and -6, when squared, result in 36. This causes two issues. First, since both 6 and -6 square to 36, no real number, when squared, results in -36. Thus $\sqrt{-36}$ is undefined. This is our second exmple for undefined result, the first being division by zero. The second issue is that mathematicians wanted square root to be an operation, with a unique result. So they defined the square root of 36 to be the *positive* number that when squared, we get 36. So, we define $\sqrt{36}$ to be 6. If we wanted to denote -6, that is not the square root of 36, but rather its opposite, so we will write $-\sqrt{36}$.

> **Definition**: Let N be a non-negative number. Then the **square root of** N (notation: $\sqrt{N}$) is the non-negative number that, if we square, the result is N. If N is negative, then $\sqrt{N}$ is undefined.

For example, $\sqrt{25} = 5$. On the other hand, $\sqrt{-25}$ is undefined. The definition uses the expression non-negative becuse $\sqrt{0}$ exists and is 0.

Example 1. Evaluate each of the given numerical expressions.

a) $\sqrt{49}$ b) $-\sqrt{49}$ c) $\sqrt{-49}$ d) $-\sqrt{-49}$

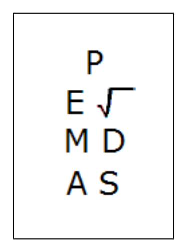
Solution: a) $\sqrt{49} = \boxed{7}$

b) $-\sqrt{49} = \boxed{-7}$

c) $\sqrt{-49} = \boxed{\text{undefined}}$

d) $-\sqrt{-49} = \boxed{\text{undefined}}$

How do square roots fit into the order of operations agreement? With this new operation, we need to revisit this old topic and see how square roots fit into it. The answer is, **taking the square root is equally strong with exponentiation**, so we perform it before all multiplications, divisions, additions and subtractions. When there are both exponents and square roots in an expression, or several square roots, we perform them left to right.



Example 2. Evaluate each of the given numerical expressions.

a) $\dfrac{2\sqrt{49}-(-1)^3}{\sqrt{9}}$ b) $\sqrt{100}-2\left(\sqrt{36}-3\left(\sqrt{25}-\sqrt{16}\right)\right)$

Solution: a) If we re-write the expression using the division sign instead of the horizontal bar, we get $\left(2\sqrt{49}-(-1)^3\right)\div\sqrt{9}$. Thus we will work out the dividend first. There, we have a multiplication, an exponentiation, and a subtraction. Square root and exponentiation are equally strong, so will perform them left to right.

$$\frac{2\sqrt{49}-(-1)^3}{\sqrt{9}} = \qquad \text{square root upstairs}$$

$$= \frac{2\cdot 7-(-1)^3}{\sqrt{9}} \qquad \text{exponentiation}$$

$$= \frac{2\cdot 7-(-1)}{\sqrt{9}} \qquad \text{multiplication}$$

$$= \frac{14-(-1)}{\sqrt{9}} \qquad \text{subtraction}$$

$$= \frac{15}{\sqrt{9}} \qquad \text{square root}$$

$$= \frac{15}{3} \qquad \text{division}$$

$$= \boxed{5}$$

b) There are several pairs of parentheses. We will start with the innermost one.

$$\sqrt{100}-2\left(\sqrt{36}-3\left(\sqrt{25}-\sqrt{16}\right)\right) \qquad \text{square roots, left to right}$$

$$= \sqrt{100}-2\left(\sqrt{36}-3\left(5-\sqrt{16}\right)\right)$$

$$= \sqrt{100}-2\left(\sqrt{36}-3(5-4)\right) \qquad \text{subtraction}$$

$$= \sqrt{100}-2\left(\sqrt{36}-3\cdot 1\right) \qquad \text{square root}$$

$$= \sqrt{100}-2(6-3\cdot 1) \qquad \text{multiplication}$$

$$= \sqrt{100}-2(6-3) \qquad \text{subtraction}$$

$$= \sqrt{100}-2\cdot 3 \qquad \text{square root}$$

$$= 10-2\cdot 3 \qquad \text{multipliction}$$

$$= 10-6 \qquad \text{subtraction}$$

$$= \boxed{4}$$

Square roots, when stretched over entire expressions, also serve as grouping symbols. This is what we called an *invisible parentheses*.

Example 3. Evaluate each of the following expressions.

a) $\sqrt{25}-\sqrt{16}$ c) $\sqrt{16}+9$ e) $\sqrt{-\sqrt{100}\div(-2)+\sqrt{(-6)^2-5\sqrt{16}}}$

b) $\sqrt{25-16}$ d) $\sqrt{16+9}$

Solution: a) $\sqrt{25}-\sqrt{16}=5-4=\boxed{1}$

b) $\sqrt{25-16}=\sqrt{9}=\boxed{3}$

These two examples illustrate that the order here matters, so we must be careful not to confuse these two types of expressions.

c) $\sqrt{16}+9=4+9=\boxed{13}$

d) $\sqrt{16+9}=\sqrt{25}=\boxed{5}$

These two examples illustrate that we have to be precise with our notation and extend the square root sign over the entire expression as in d) to avoid confusing the two types of expressions.

e) The entire expression is enclosed in a square root, so that will be the last operation to perform. Inside, there is another square root stretched over an entire expression. That serves as parentheses, so we start there. Inside, we start with exponentiations and square roots, left to right.

$$
\begin{aligned}
&\sqrt{-\sqrt{100}\div(-2)+\sqrt{(-6)^2-5\sqrt{16}}}= && \text{exponentiation}\\
&=\sqrt{-\sqrt{100}\div(-2)+\sqrt{36-5\sqrt{16}}} && \text{square root}\\
&=\sqrt{-\sqrt{100}\div(-2)+\sqrt{36-5\cdot 4}} && \text{multiplication}\\
&=\sqrt{-\sqrt{100}\div(-2)+\sqrt{36-20}} && \text{subtraction}\\
&=\sqrt{-\sqrt{100}\div(-2)+\sqrt{16}} && \text{square roots, left to right}\\
&=\sqrt{-10\div(-2)+\sqrt{16}} && \\
&=\sqrt{-10\div(-2)+4} && \text{division}\\
&=\sqrt{5+4} && \text{addition}\\
&=\sqrt{9} && \text{square root}\\
&=\boxed{3} &&
\end{aligned}
$$

Practice Problems

1. Evaluate each of the following expressions.

   a) $\sqrt{100}$ b) $\sqrt{-100}$ c) $-\sqrt{100}$ d) $-\sqrt{49}$ e) $-\sqrt{-49}$ d) $\sqrt{1}$ e) $\sqrt{0}$

2. Evaluate each of the following expressions.

   a) $\sqrt{\sqrt{81}+\sqrt{49}}$ b) $\sqrt{\sqrt{64}+1}$ c) $\sqrt{\sqrt{11-\sqrt{4}}+\sqrt{1}}$ d) $\sqrt{\sqrt{\sqrt{100}-1}+1}$

3. Evaluate each of the following expressions.

   a) $\sqrt{25}-2\sqrt{9-(-3)^3}=-7$ b) $\sqrt{\sqrt{36}+5\sqrt{9}-\frac{\sqrt{100}}{2}}$ c) $\sqrt{\sqrt{4\sqrt{64}-\sqrt{49}}-\sqrt{1}}$

4. Evaluate each of the following expressions.

   a) $\sqrt{16-2\sqrt{-4^2-\sqrt{1}+2\cdot 3^2\div\sqrt{4}\cdot 6-1}}$ b) $\sqrt{3\sqrt{25}-\left(2\sqrt{100}-5\sqrt{16}-\sqrt{36}\div 6\right)}$

Chapter 19

19.1 Slope of a Line

Let us first recall a definition.

Definition: Equations that are in x, or in y, or in x and y can be graphed. **The graph of such an equation is the set of all points $P(x,y)$ for which the coordinates x and y form a solution of the equation.**

The slope of a line is a very important concept.

Definition: The slope of a line (usually denoted by m) is a signed ratio expressing the steepness of the line. Given points $A(x_1,y_1)$ and $B(x_2,y_2)$, the slope of the line connecting these points is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

This formula is called the **slope formula**.

Example 1. In each case, find the slope of the line determined by the two points given.

a) $(5,-1)$ and $(1,3)$ b) $(2,1)$ and $(6,3)$ c) $(8,-1)$ and $(-1,-1)$ d) $(7,3)$ and $(7,-4)$

Solution: a) We find the slope determined by the points. $(5,-1) = (x_1,y_1)$ and $(1,3) = (x_2,y_2)$ using the slope formula.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-(-1)}{1-5} = \frac{4}{-4} = -1$$

So the slope is $\boxed{-1}$.

b) We find the slope determined by the points. $(2,1) = (x_1,y_1)$ and $(6,3) = (x_2,y_2)$ using the slope formula.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{6-2} = \frac{2}{4} = \frac{1}{2}$$

So the slope is $\boxed{\frac{1}{2}}$.

c) We find the slope determined by the points. $(8,-1) = (x_1,y_1)$ and $(-1,-1) = (x_2,y_2)$ using the slope formula.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-(-1)}{-1-8} = \frac{0}{-9} = 0$$

So the slope is $\boxed{0}$. Note that **the slope of all horizontal lines is zero**.

d) We find the slope determined by the points. $(7,3) = (x_1, y_1)$ and $(7,-4) = (x_2, y_2)$ using the slope formula.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4-3}{7-7} = \frac{-7}{0} = \text{undefined}$$

So this line has $\boxed{\text{no slope}}$. Note that **vertical lines have no slope**.

Discussion: Suppose we are finding the slope of a line connecting $A(-5,2)$ and $B(1,10)$. Ann is using point A as (x_1, y_1) and point B as (x_2, y_2). Beth is using point B as (x_1, y_1) and point A as (x_2, y_2). Compute the slope both ways. Do we get the same answer? Why or why not?

Theorem: Suppose that the equation of a line is $y = mx + b$. Then the slope of this line is m, the coefficient of x in the equation.

For example, the line $y = 2x - 3$ has slope $m = 2$, the line $y = -\frac{2}{3}x + 1$ has slope $m = -\frac{2}{3}$.

Caution! If the line's equation is not in this form, the coefficient of x is NOT the slope. For example, the slope of the line $2x - 3y = -12$ is NOT 2.

The equation $y = mx + b$ is called the **slope-intercept form** of the line.

Example 2. In each case, find the slope of the line given.

a) $y = -3x + 8$ b) $y = \frac{3}{4}x - 2$ c) $y = -2$ d) $x = 7$

Solution: a) Since the equation is in the slope-intercept form, the slope of the line is the coefficient of x, -3. So $\boxed{m = -3}$.

b) Since the equation is in the slope-intercept form, the slope of the line is the coefficient of x, $\frac{3}{4}$. So $\boxed{m = \frac{3}{4}}$.

c) This equation is horizontal, so its slope is zero, $\boxed{m = 0}$. There are two ways to remind ourselves of this fact: when in doubt, find any two points on the line, say $(1,-2)$ and $(5,-2)$ and then apply the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-(-2)}{5-1} = \frac{0}{4} = 0$$

Antoher way is to think of the equation $y = -2$ as $y = 0x - 2$ and then the coefficient of x is zero.

d) This equation is vertical, so it does not have a slope. There are two ways to remind ourselves of this fact: when in doubt, find any two points on the line, say $(7,1)$ and $(7,2)$ and then apply the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-1}{7-7} = \frac{0}{0} = \text{undefined}$$

So this line has $\boxed{\text{no slope}}$. (Antoher way is to think of the equation $x = 7$ as an equation in which we can not solve for y, because it doesn't even appear in the equation. Or, $x = 0y + 7$ and then solving for y would result in having to divide by zero.)

The slope-intercept form of a line's equation enables us to plot a line quickly and with very little computation. Every non-vertical line has a slope-intercept form. In case of an integer m, we can think of m as a fraction: $\frac{m}{1}$. If m is already a fraction, great. If $m = \frac{p}{q}$ where p and q are integers, then from one nice lattice point to another, we will make q steps to the right, and p steps up (if p is positive) or down (if p is negative).

$$m = 4 = \frac{4}{1} \implies \text{1 step to the right, 4 steps up}$$

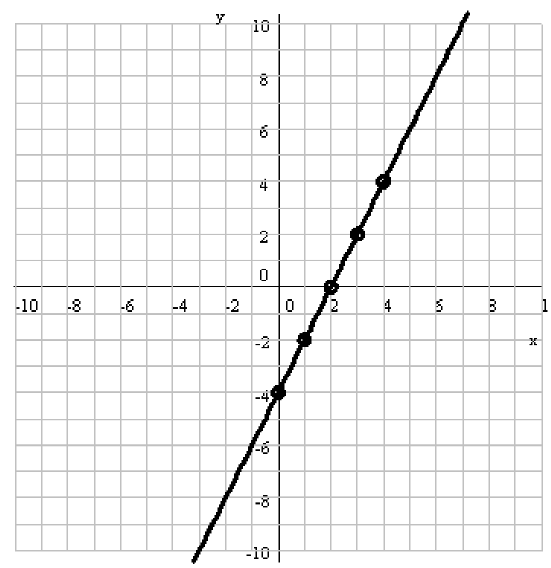
$$m = -1 = \frac{-1}{1} \implies \text{1 step to the right, 1 step down}$$

$$m = \frac{2}{3} \implies \text{3 steps to the right, 2 steps up}$$

$$m = -\frac{5}{6} = \frac{-5}{6} \implies \text{6 steps to the right, 5 steps down}$$

Example 3. Graph the line $y = 2x - 4$

Solution: This line is given in its slope-intercept form. The slope of this line is the coefficient of x. It is $m = 2$.



We graph the $y-$intercept of the line. We obtain the $y-$intercept by substituting $x = 0$ into the equation. Clearly, the $y-$intercept is $(0, -4)$. In general, the $y-$intercept of the line $y = mx + b$ is $(0, b)$. We graph that point first.

We graph additional points using the slope of the line, $m = 2$. If the slope is a not fraction, we can always divide it by 1 to create one.

$$2 = \frac{2}{1} \implies \frac{\text{2 up}}{\text{1 to the right}}$$

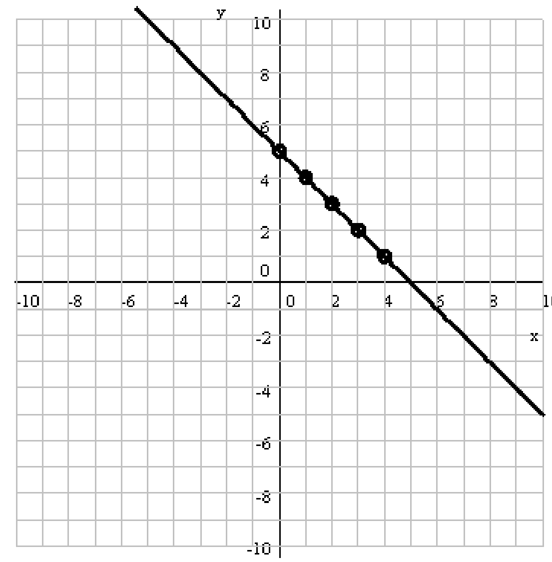
We start at the $y-$intercept, $(0, -4)$ and to plot additional points, we step 1 to the right, 2 up. We repeat this process several times to plot enough points.

Example 4. Graph the line $y = -x + 5$.

Solution: The slope of this line is $m = -1$, and the $y-$intercept of it is $(0,5)$. We graph the $y-$intercept first and then graph additional points using the slope of the line.
The slope of this line is $m = -1$. If the slope is a not fraction, we can always divide it by 1 to create one.

$$-1 = \frac{-1}{1} \implies \frac{1 \text{ down}}{1 \text{ to the right}}$$

We start at the $y-$intercept, $(0,5)$ and to plot additional points, we step 1 to the right, 1 down. We repeat this process several times to plot enough points.



Example 5. Graph the line $x + 2y = 4$

Solution: Step 1. We bring the equation to its slope-intercept form, $y = mx + b$ by solving for y in $x + 2y = 4$.

$$\begin{aligned} x + 2y &= 4 && \text{subtract } x \\ 2y &= -x + 4 && \text{divide by 2} \\ y &= \frac{-x+4}{2} = \frac{-x}{2} + \frac{4}{2} = -\frac{1}{2}x + 2 \end{aligned}$$

The slope-intercept form of the equation is $y = -\frac{1}{2}x + 2$.

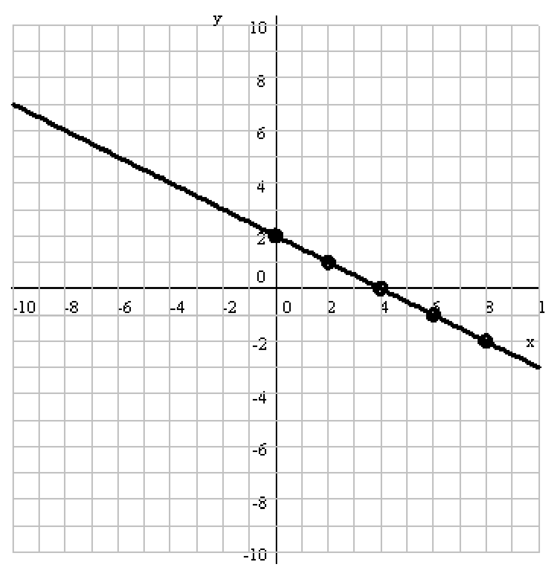
Step 2. We graph first the $y-$intercept of the line, which we obtain by substituting $x = 0$ into the slope-intercept form of the line, $y = -\frac{1}{2}x + 2$. Clearly, the $y-$intercept is $(0,2)$.

Step 3. Graph additional points using the slope of the line using the slope.

The slope of this line is $m = -\frac{1}{2}$. The denominator tells us how many units we step to the right, and the numerator tells us how many units we step down (since the slope is negative).

$$-\frac{1}{2} = \frac{-1}{2} \implies \frac{1 \text{ down}}{2 \text{ to the right}}$$

We start at the $y-$intercept, $(0,2)$ and to plot additional points, we step 2 to the right, 1 down. We repeat this process several times to plot enough points.



Example 6. Graph the line $3x-4y=12$.

Solution: Step 1. We bring the equation to its slope-intercept form, $y=mx+b$ by solving for y in $3x-4y=12$.

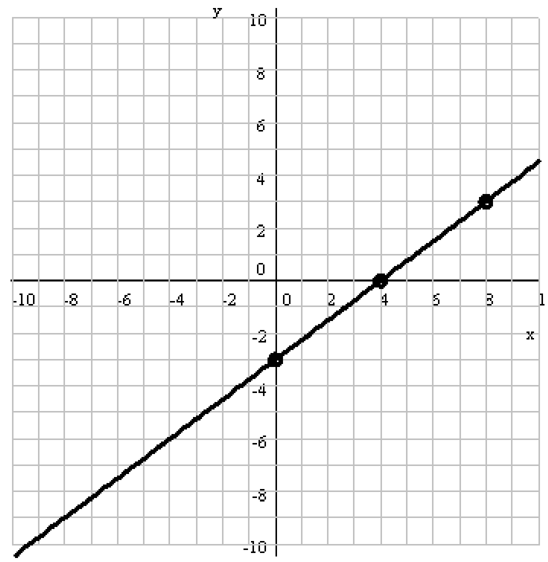
$$\begin{aligned} 3x-4y &= 12 && \text{add } 4y \\ 3x &= 4y+12 && \text{subtract } 12 \\ 3x-12 &= 4y && \text{divide by } 4 \\ \frac{3x-12}{4} &= y \\ y &= \frac{3x-12}{4} = \frac{3x}{4} - \frac{12}{4} = \frac{3}{4}x-3 \end{aligned}$$

The slope-intercept form of the equation is $y=\frac{3}{4}x-3$.

Step 2. We graph the $y-$intercept of the line. We obtain the $y-$intercept by substituting $x=0$ into the equation. Clearly, the $y-$intercept is $(0,-3)$.
Step 3. Graph additional points using the slope of the line.

$$m=\frac{3}{4} \implies \frac{3 \text{ up}}{4 \text{ to the right}}$$

We start at the $y-$intercept, $(0,-3)$ and to plot additional points, we step 4 to the right, and 3 up. We repeat this process several times to plot enough points.



> **Theorem**: Two lines are parallel if and only if they are both vertical or they have the same slope.

In other words, two lines, if they have slopes and are parallel, then the slopes must be the same.

> **Theorem**: Two lines are perpendicular if and only if one is vertical and the other is horizontal, or they both have slopes and they are negative reciprocals, $m_1m_2=-1$.

Practice Problems

1. In each case, determine the slope of the line connecting the points given.

   a) $(-2,7)$ and $(3,-3)$ b) $(7,-5)$ and $(3,-8)$ c) $(3,-2)$ and $(3,6)$ d) $(2,7)$ and $(10,7)$

2. Graph each of the following lines.

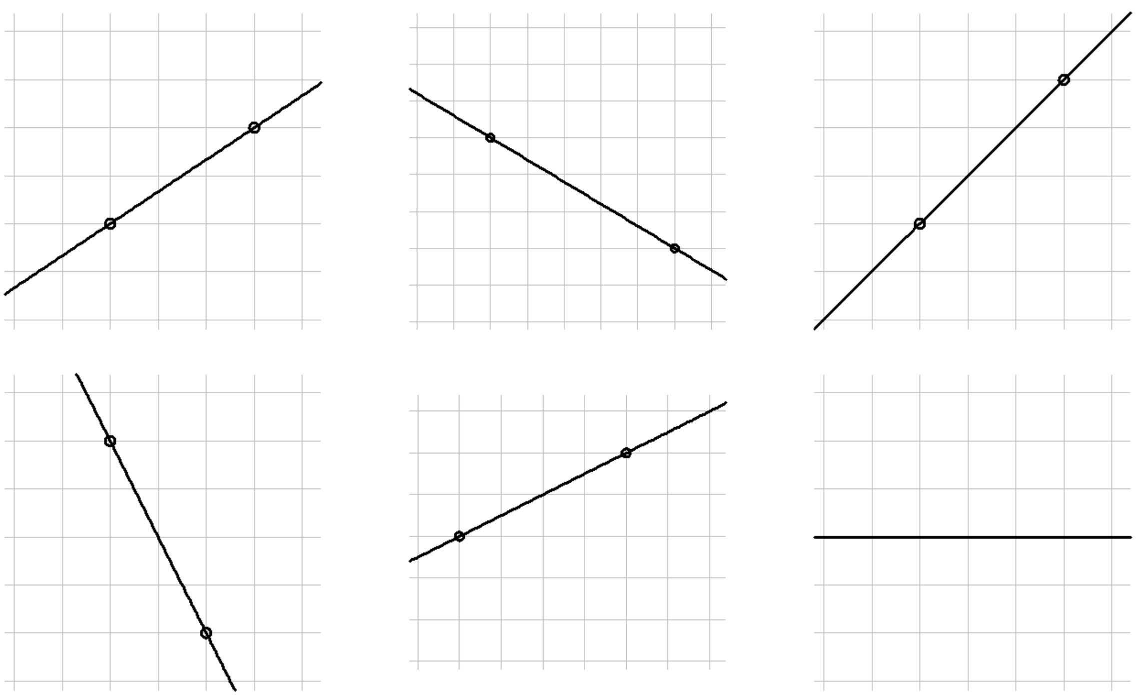
   a) $3x+2y=6$ d) $2x-3y=10$ g) $3x+5y=-30$

   b) $x=-4$ e) $y=1$ h) $2x-y=7$

   c) $y=\frac{2}{5}x-3$ f) $y=3x+6$ i) $y=\frac{1}{3}x$

3. Determine the slope of each of the following lines, based on their graphs.

a) b) c)



d) e) f)

Chapter 20

20.1 Factoring by Grouping

Grouping is a factoring technique that can be used in several situations. Grouping consists of strategically pairing (or grouping) terms, and then factoring out the greatest common factor or GCF three times. We can also say that factoring by grouping is a reversal of FOIL (First, Outer, Inner, Last). We should consider factoring by groupimg when we have four terms and perhaps also several variables or higher degrees.

Example 1. Completely factor the expression $15ax+6ay-10bx-4by$.

Solution: The first step is grouping the four terms into two pairs. The goal is to pair terms that have similar terms or coefficients with common divisors. The goal is to have as much of a GCF in a pair as possible.

In this case, we have two options that would both work. We could pair $15ax$ with $6ay$ because they share the common factor of $3a$. We could also pair $15ax$ with $-10bx$ because then $5x$ is a shared factor. The only pairing that would not work is to pair $15ax$ with $4by$ as these two terms have no common factor besides 1.

So, we first pair the first two terms and the second two tems. As we stated before, grouping is to factor out the common factor three times. First, we factor out the greatest common factor from $15ax+6ay$.

$$3a\,(5x+2y)-10bx-4by$$

This method can only work if, when factoring out the GCF from the second pair, what is left in the parentheses is identical to $5x+2y$. The greatest common factor of $-10bx-4by$ is $2b$. This means that we have two choices: either factor out $2b$ or $-2b$. We have to select the sign that guarantees that we have $5x+2y$ left in the parentheses. Therefore, we factor out $-2b$ from $-10bx-4by$.

$$3a\,(5x+2y)-2b\,(5x+2y)$$

At this point, $5x+2y$ is the common factor of the two terms and so we can factor that out.

$$3a\underbrace{(5x+2y)}_{\text{GCF}}-2b\underbrace{(5x+2y)}_{\text{GCF}}=(5x+2y)\,(3a-2b)$$

So the factored form is $\boxed{(5x+2y)\,(3a-2b)}$. When we check, we can see why this method is a reversal of FOIL:

$$(5x+2y)\,(3a-2b)=15ax-10bx+6ay-4by$$

The next example illustrates a commonly occuring situation, when one or more of the GCF is 1. Still, grouping will work just fine.

Example 2. Completely factor the expression $6mx-3x-2m+1$.

Solution: The first step is grouping the four terms into two pairs. The goal is to pair terms that have similar terms or coefficients with common divisors. The goal is to have as much of a GCF in a pair as possible. The terms here seem to share not much in common, but the first and third term share $2m$. So, we first rearrange the terms:

$$\begin{aligned}6mx-3x-2m+1 &= 6mx-2m-3x+1 \quad \text{the GCF in the first two terms is } 2m\\ &= 2m(3x-1)-3x+1\end{aligned}$$

The second pair shares no factor besides 1, but we already see the similarity between the expression left after factoring out the GCF from the first pair and the second pair. The only quation is, should we factor out 1 or -1? The goal is to have the same expression left in the parentheses. Therefore, we should factor out -1.

$$2m(3x-1)-3x+1 = 2m(3x-1)-1(3x-1)$$

Now $3x-1$ is the common factor.

$$2m(3x-1)-1(3x-1) = (2m-1)(3x-1)$$

So the factored form is $\boxed{(2m-1)(3x-1)}$. When we check, we can see why this method is a reversal of FOIL:

$$(2m-1)(3x-1) = 6mx-2m-3x+1$$

Sometimes we see only one variable, but with higher degrees. In this case, having four terms is still an important indication that grouping would work.

Example 3. Completely factor the expression $2\left(x^2-4\right)(5x+3) = 10x^3+6x^2-40x-24$.

Solution: Like with all factoring, we start with the GCF. In this case, all terms are divisible by 2, so we will factor it out. Then we group the terms by degrees.

$$\begin{aligned}10x^3+6x^2-40x-24 &= 2\left(5x^3+3x^2-20x-12\right) && \text{factor out } x^2 \text{ from first pair}\\ &= 2\left[x^2(5x+3)-20x-12\right] && \text{factor out } -4 \text{ from second pair}\\ &= 2\left[x^2(5x+3)-4(5x+3)\right] && \text{factor out } (5x+3)\\ &= 2\left(x^2-4\right)(5x+3)\end{aligned}$$

Although we are done with factoring by grouping, the expression $2\left(x^2-4\right)(5x+3)$ is not *completely* factored, because x^2-4 can be factored by the difference of squares theorem into $(x+2)(x-2)$. So the factored form is $\boxed{2(x+2)(x-2)(5x+3)}$. We can check by multiplying back:

$$2(x+2)(x-2)(5x+3) = 2\left(x^2-4\right)(5x+3) = 2\left(5x^3+3x^2-20x-12\right) = 10x^3+6x^2-40x-24$$

and so our solution is correct.

Another important application of grouping is factoring a general quadratic trinomial, ax^2+bx+c.

Example 4. Completely factor $14x^2-8x+21x-12$.

Solution: We group $14x^2$ with $21x$ and $-8x$ with -12.

$$\begin{aligned} 14x^2-8x+21x-1 &= 14x^2+21x-8x-12 && \text{factor out GCF from first pair} \\ &= 7x(2x+3)-8x-12 && \text{factor out GCF from second pair} \\ &= 7x(2x+3)-4(2x+3) && \text{factor ou the GCF } 2x+3 \\ &= (2x+3)(7x-4) \end{aligned}$$

So the factored form is $\boxed{(2x+3)(7x-4)}$. We can check by multiplying back:

$$(2x+3)(7x-4) = 14x^2-8x+21x-12$$

Note that general trinomials usually do not occur with two like terms such as $-8x+21x$. So, how could we factor $14x^2+13x-12$ by grouping? The trick is to strategically 'take apart' the linear term $13x$ to two like terms $-8x+21x$. But how do we know how to take apart the linear term? There is a systematic way to do that, and it will be discussed later. This factoring technique is called the AC-method.

Practice Problems

Completely factor each of the following.

1. $5x-6a+2ax-15$
2. $2a-4x^2+2ax^2-4$
3. $5ax^3-5bx^3+5ax^2y-5bx^2y$
4. $3a-3ax-3ay+3axy$
5. $18m-90n-2mx^2+10nx^2$
6. $p^2x^2-p^2y^2-q^2x^2+q^2y^2$
7. $2a^2by-6a^2bt-3a^2btx+a^2bxy$
8. $2a^2b-50b-25bm^3+a^2bm^3$
9. $6x^5-15x^4+2x-5$
10. $-24ax^9+8ax^7+6ax^3-2ax$
11. x^3+2x^2-4x-8
12. $6x^2+16x-3x-8$
13. $2x^2+5x-6x-15$
14. $x^2-2x-4x+8$
15. $6x^2+4x-3x-2$
16. $10x^2-25x-4x+10$

20.2 Rational Expressions

> **Definition:** A **rational expression** is a quotient of two polynomials.

Rational expressions are abstract fractions, such as $\frac{1}{x}$, $\frac{x}{x+1}$, or $\frac{2x-5}{x^2-4x+3}$. Because they are fractions, we can do the same thing to rational expressions as we did with fractions formed from numbers. To present fractions as final results, we always use the reduced form in which numerator and denominator do not share any common factors.

Recall the fundamental property of fractions: the only thing we can do to a fraction without changing its value is multiplying or dividing both numerator and denominator by the same number. In order to bring rational expressions to lowest terms, we factor both numerator and denominator and cancel out the common factors. It is important to remember that we can only cancel out common factors; we can not cancel out anything unless both numerator and denominator are factored.

Example 1. Simplify each of the given rational expressions.

a) $\frac{18ab^2cx}{12a^3b^2c^3x}$ b) $\frac{x^2-25}{x^2-6x+5}$ c) $\frac{2x^4+4x^3-6x^2}{4x^5-4x^4}$ d) $\frac{a-3}{3-a}$

Solution: a) Between 18 and 12, the common factor is 6. Between a and a^3, the common factor is a. Between c and c^3, the common factor is c. The factors b^2 and x can be completely cancelled out.

$$\frac{18ab^2cx}{12a^3b^2c^3x} = \frac{6ab^2cx\ (3)}{6ab^2cx\ (2a^2c^2)} = \frac{\cancel{6ab^2cx}\ (3)}{\cancel{6ab^2cx}\ (2a^2c^2)} = \boxed{\frac{3}{2a^2c^2}}$$

b) We first factor both numerator and denominator. $x^2-25 = (x+5)(x-5)$ by the difference of squares theorem. We can factor the denominator by trial and error and get that $x^2-6x+5 = (x-5)(x-1)$. Then we cancel out the common factors.

$$\frac{x^2-25}{x^2-6x+5} = \frac{(x+5)(x-5)}{(x-1)(x-5)} = \frac{(x+5)\cancel{(x-5)}}{(x-1)\cancel{(x-5)}} = \boxed{\frac{x+5}{x-1}}$$

c) We first factor both numerator and denominator and then cancel out factors common to both numerator and denominator. In both cases, we start with the greatest common factor or GCF.

$$2x^4+4x^3-6x^2 = 2x^2\left(x^4+2x-3\right) = 2x^2(x+3)(x-1)$$
$$4x^5-4x^4 = 4x^4(x-1)$$

The factors common to numerator and denominator are $2x^2$ and $x-1$. We can cancel those out.

$$\frac{2x^4+4x^3-6x^2}{4x^5-4x^4} = \frac{2x^2(x+3)(x-1)}{4x^4(x-1)} = \frac{2x^2(x+3)\cancel{(x-1)}}{4x^4\cancel{(x-1)}} = \boxed{\frac{x+3}{2x^2}}$$

d) This problem often presents a troubling situation because neither numerator nor denominator can be factored. But if we substitute a few values into a, we find that in each case, the result is -1. The expressions $a-3$ and $3-a$ are opposites. We can factor out from one of the two and then we can cancel out the comon linear factors.

$$\frac{a-3}{3-a} = \frac{a-3}{-1(-3+a)} = \frac{a-3}{-1(a-3)} = \frac{\cancel{(a-3)}}{-1\cancel{(a-3)}} = \frac{1}{-1} = \boxed{-1}$$

We can also perform operations on rational expressions. We can add and subtract them, just like we do with fractions but that will be a topic later. For now, we will perform much easier operations: multiplications and divisions. When we multiply two fractions, terms common to the numerator of one and the denominator of the other can also be cancelled out, just like in $\frac{5}{8}\cdot\frac{3}{10}=\frac{5}{8}\cdot\frac{3}{5\cdot 2}=\frac{3}{8\cdot 2}=\frac{3}{16}$.

Example 2. Perform the given operations and simplify.

a) $\frac{2a^2-18}{3a^2-6a-9}\cdot\frac{a^2+2a+1}{6a^2+6a}$ b) $\frac{5y^2+5y}{8y-16}\div\frac{10y^3+10y^2}{4y^2-16}$

Solution: a) First we factor numerators and denominators in both rational expressions.

$$2a^2-18=2\left(a^2-9\right)=2(a+3)(a-3)$$
$$3a^2-6a-9=3\left(a^2-2a-3\right)=3(a-3)(a+1)$$
$$a^2+2a+1=(a+1)(a+1)$$
$$6a^2+6a=6a(a+1)$$

Now we are ready to perform the multiplication.

$$\frac{2a^2-18}{3a^2-6a-9}\cdot\frac{a^2+2a+1}{6a^2+6a}=\frac{2(a+3)(a-3)}{3(a-3)(a+1)}\cdot\frac{(a+1)(a+1)}{6a(a+1)}=\frac{2(a+3)\cancel{(a-3)}}{3\cancel{(a-3)}(a+1)}\cdot\frac{(a+1)\cancel{(a+1)}}{6a\cancel{(a+1)}}$$
$$=\frac{2(a+3)(a+1)}{3(6a)(a+1)}=\frac{2(a+3)\cancel{(a+1)}}{3(6a)\cancel{(a+1)}}=\frac{2(a+3)}{3(6a)}$$
$$=\frac{2(a+3)}{3(2\cdot 3a)}=\frac{\cancel{2}(a+3)}{3(\cancel{2}\cdot 3a)}=\frac{a+3}{3(3a)}=\boxed{\frac{a+3}{9a}}$$

f) This problem indicates a division. Recall that to divide is to multiply by the reciprocal. After we re-write the problem as multiplication of rational expressions, this problem becomes very similar to the previous one.

$$\frac{5y^2+5y}{8y-16}\div\frac{10y^3+10y^2}{4y^2-16}=\frac{5y^2+5y}{8y-16}\cdot\frac{4y^2-16}{10y^3+10y^2}$$

As before, we start with four factoring exercises.

$5y^2+5y=5y(y+1)$ and $8y-16=8(y-2)$

$4y^2-16=4\left(y^2-4\right)=4(y+2)(y-2)$ and $10y^3+10y^2=10y^2(y+1)$

Therefore, $\frac{5y^2+5y}{8y-16}\cdot\frac{4y^2-16}{10y^3+10y^2}=\frac{5y(y+1)}{8(y-2)}\cdot\frac{4(y+2)(y-2)}{10y^2(y+1)}=\frac{5y\cancel{(y+1)}}{8\cancel{(y-2)}}\cdot\frac{4(y+2)\cancel{(y-2)}}{10y^2\cancel{(y+1)}}$

$$=\frac{5y}{8(y-2)}\cdot\frac{4(y+2)}{10y^2}=\frac{\cancel{5}\cancel{y}}{2\cdot\cancel{4}}\cdot\frac{\cancel{4}(y+2)}{\cancel{5}\cdot 2\cancel{y}\cdot y}=\boxed{\frac{y+2}{4y}}$$

Discussion: Consider the expression $\frac{a-3}{3-a}$ from example 1d). If we substitute a few values into a (say 2, -5, 5, and 0), the value of the expression is -1. This is not true for every number. Find a number that, when substituted into the expression, does not result in -1.

Sample Problems

1. Simplify each of the following.

a) $\frac{2a-5}{5-2a}$ b) $\frac{x^3-x}{x+1}$ c) $\frac{2x+1}{4x^2-1}$ d) $\frac{x^2-4x+3}{x^2+2x-15}$ e) $\frac{(x+5)-2}{5(x+2)-(x-2)}$

2. Perform the indicated operations and simplify.

a) $\frac{c}{5a}\cdot\frac{15a^2b}{3b^2c}$

b) $\frac{5x-30}{x^2-36}\cdot\frac{3x+18}{5}$

c) $\frac{x^2-3x}{x^2-8x+15}\cdot\frac{x^2-16x+15}{x^2-x}$

d) $\frac{x^2-9}{x^2-4x-21}\div\frac{4x-12}{3x-21}$

e) $\frac{x^2-10x+25}{x^2-10x+24}\left(\frac{x^2-2x-8}{x^2-6x+5}\div\frac{x-5}{x-1}\right)$

f) $\frac{6x^2y^2-36x^2y+48x^2}{py^2-2py-8p}\div\frac{-3y^2+24y-36}{p^2y+2p^2}$

Practice Problems

1. Simplify each of the following.

a) $\frac{2b-5}{10-4b}$ c) $\frac{4t^2-9}{4t-6}$ e) $\frac{3m+m^2-10}{11m+m^2+30}$ g) $\frac{-7x-11-3(x-2)}{5x-11+3(x+5)}$

b) $\frac{x^2-1}{x+1}$ d) $\frac{p^2-p}{p^2-1}$ f) $\frac{x^2-x-2}{x^2-5x+6}$

2. Perform the indicated operations and simplify.

a) $\frac{x}{5yz}\cdot\frac{10x^2y^3z}{4xy^2}$

b) $\frac{a^2-8a+16}{a}\cdot\frac{a^3}{4-a}$

c) $\frac{x^2-5x}{x^2-2x-15}\cdot\frac{x^2-9}{x^2-3x}$

d) $\frac{x^2-4x-21}{x^2-49}\div\frac{8x+x^2+15}{2x+x^2-35}$

e) $\frac{5y-35}{y^2-2y-35}\cdot\frac{3y+15}{5y-5}$

f) $\frac{2x^2-98}{x^2-6x-7}\div\frac{21+3x}{6x^2-6}$

Chapter 21

21.1 More on Linear Systems

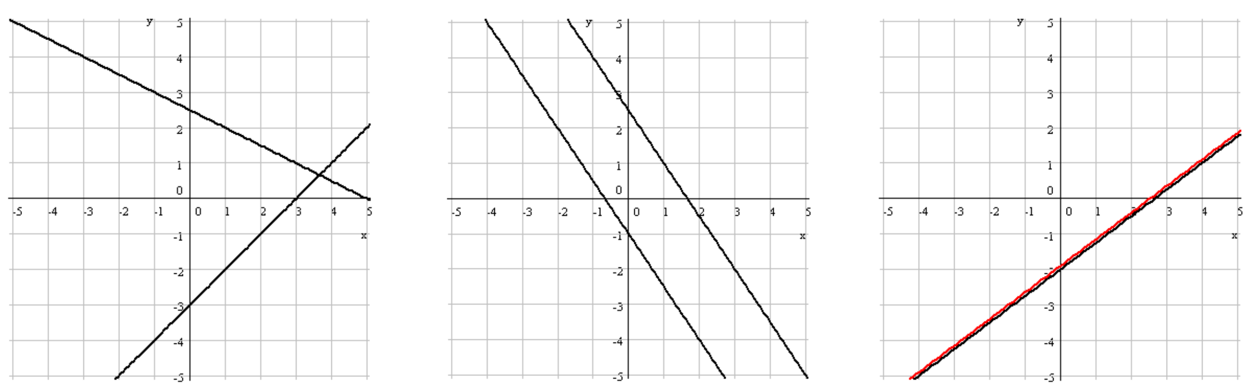
Recall that linear equations in x and y has many, many solutions, and one way to meaningfully represent these solutions is to graph them. Every linear equation can be graphed, and every point on the graph has coordinates that form a solution of the equation.

Therefore, a system of two linear equations can also be graphed, and the intersection point is a point whose coordinates form a simultaneous solution of both equations.

Most systems of linear equations have a unique solution (x, y). However, this is not always the case.

If we think of a system of linear equations as two lines, it is clear that geometrically, there are three distinct ways two lines in a plane can behave.

One intersection point | No intersection point | All points on the line are intersection points



Until now, we have only seen linear systems with a unique solution, corresponding to two lines intersecting each other in a uinque point. However, there are linear systems that end up with different results.

Example 1. Solve the given system of linear equations. $\begin{cases} x-2y=6 \\ y=\frac{1}{2}x+1 \end{cases}$

Solution: Since the second equation is already solved for y, we will use substitution, but elimination would also work fine. We substitute $y=\frac{1}{2}x+1$ into the first equation and solve the linear equation for x.

$$\begin{aligned} x-2\left(\frac{1}{2}x+1\right) &= 6 && \text{distribute 2} \\ x-x-2 &= 6 && \text{combine like terms} \\ -2 &= 6 \end{aligned}$$

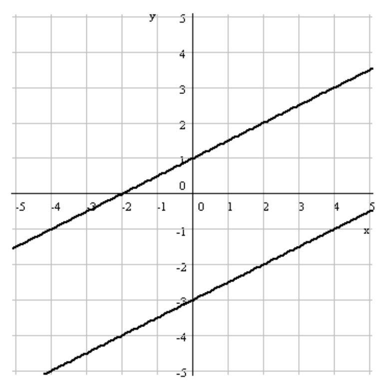
As we solve the equation for x, it disappears from the equation and we are left with an unconditionally false equation. We have seen this before when we solved linear equations. An equation such as $-2=6$ is called a contradiction, and it has no solution. What does this result mean for a system?

Let us solve the first equation for y.

$$\begin{aligned} x-2y &= 6 && \text{add } 2y \\ x &= 2y+6 && \text{subtract 6} \\ x-6 &= 2y && \text{divide by 2} \\ \frac{x-6}{2} &= y \implies y=\frac{x}{2}-\frac{6}{2}=\frac{1}{2}x-3 \end{aligned}$$

Let us look again at the system, but this time from a geometric point of view. Both equations represent a straight line.

$\begin{cases} y=\frac{1}{2}x-3 \\ y=\frac{1}{2}x+1 \end{cases}$ These lines are parallel because they have the same slope, $m=\frac{1}{2}$, and parallel lines have no intersection points.



In case of such a system, the last line is an unconditionally false equation, a contradiction. Such a system is called an **inconsistent system**, and there is no solution of it.

Example 2. Solve the given system of linear equations. $\begin{cases} 2x+6y=-12 \\ y=-\frac{1}{3}x-2 \end{cases}$

Solution: Since the second equation is already solved for y, we will use substitution, but elimination would also work fine. We substitute $y=-\frac{1}{3}x-2$ into the first equation and solve the linear equation for x.

$$\begin{aligned} 2x+6\left(-\frac{1}{3}x-2\right) &= -12 && \text{distribute 6} \\ 2x-2x-12 &= -12 && \text{combine like terms} \\ -12 &= -12 \end{aligned}$$

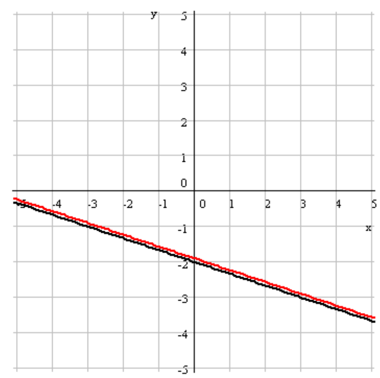
As we solve the equation for x, it disappears from the equation again, but this time we are left with an unconditionally true equation. We have seen this before when we solved linear equations. Such an eqaution is called an identity, and all real numbers are solutions of it.

Just as before, we solve the first equation for y.

$$\begin{aligned} 2x+6y &= -12 && \text{subtract } 2x \\ 6y &= -2x-12 && \text{divide by 6} \\ y &= \frac{-2x-12}{6} && \text{divide by 2} \\ y &= \frac{-2x-12}{6} = -\frac{2x}{6}-\frac{12}{6} = -\frac{1}{3}x-2 \end{aligned}$$

So, the linear system now look like this: $\left\{\begin{array}{l} y=-\frac{1}{3}x-2 \\ y=-\frac{1}{3}x-2 \end{array}\right.$

$\left\{\begin{array}{l} y=-\frac{1}{3}x-2 \\ y=-\frac{1}{3}x-2 \end{array}\right.$ These lines are identical and so every point on the line is a solution. We can express this as the set of points $\left(x, -\frac{1}{3}x-2\right)$.



In case of such a system, the last line is an unconditionally true equation, an identity. Such a system is called a **dependent system**, and all points on the line are solutions.

Practice Problems

1. Solve each of the following systems of linear equations.

a) $\left\{\begin{array}{l} x+3y=11 \\ 12y=-4x+7 \end{array}\right.$

b) $\left\{\begin{array}{l} 4y=6x+10 \\ 3x-2y=-5 \end{array}\right.$

c) $\left\{\begin{array}{l} (x-2)^2+y=(x+1)^2-y \\ 2y=6x-3 \end{array}\right.$

d) $\left\{\begin{array}{l} 5x-6y=-10 \\ \frac{1}{2}x-\frac{3}{5}y=1 \end{array}\right.$

2. Consider the linear system $\left\{\begin{array}{l} 2x+3y=A \\ x=-\frac{3}{2}y-12 \end{array}\right.$. Find the value of A so that the system

a) has no solution   b) has infinitely many solutions

21.2 The Pythagorean Theorem

The Pythagorean theorem is a fundamental theorem about the connection between the three sides of a right triangle. Let us recall first a few things we will need for this topic.

Standard labeling simplifies notation. According to standard labeling, side a is opposite point A and angle α. Similarly, side b is opposite of point B and angle β, and side c is opposite point C and angle γ.

The three angles in any triangle add up to $180°$.

In any triangle, the order between sides is the same as the order between the angles opposite them. For example, the longest side is always opposite the greatest angle, and vica versa: the greatest angle is opposite the longest side. The shortest side is opposite the smallest angle, and vica versa: the smallest angle is opposite of the shortest side. If two sides are equally long, the angles opposite them are also equal.

In a right triangle, the right angle is the single greatest angle in the triangle, because the other two angles must add up to $90°$ so they both are smaller than $90°$.

Therefore, right triangles have a single longest side and it is located opposite of the right angle.

Definition: In a right triangle, the side opposite the right angle is the longest side. We call this side the **hypotenuse** of the triangle.

We are now ready to state the Pythagorean theorem. It actually has two parts.

Theorem: (The Pythagorean theorem)

Part 1. If ABC is a right triangle with shorter sides a and b and hypotenuse c, then

$$a^2 + b^2 = c^2$$

Part 2. In any triangle with sides a, b, and c, if $a^2 + b^2 = c^2$, then the angle opposite side c measures $90°$.

The first part tells us how right triangles behave. The second part tells us that *only* right triangles have this behavior.

Example 1. In each case, determine whether the three sides given are sides in a right triangle or not.

a) 5 cm, 7 cm, and 9 cm b) 5 ft, 13 ft and 12 ft c) 53 units, 28 units, and 45 units.

Solution: We will use the second part of the Pythagorean theorem. Let us denote the longest side by c and the other two sides by a and b. If the statement $a^2 + b^2 = c^2$, is true, the triangle is a right triangle. If not, the triangle has no right angle in it.

a) The only side here that could be the hypotenuse, is the longest one, measuring 9 cm. Therefore, the two quantities we need to compare are $(5\,\text{cm})^2 + (7\,\text{cm})^2$ and $(9\,\text{cm})^2$.

$(5\,\text{cm})^2 + (7\,\text{cm})^2 = 25\,\text{cm}^2 + 49\,\text{cm}^2 = 74\,\text{cm}^2$ and $(9\,\text{cm})^2 = 81\,\text{cm}^2$

Since $74\,\text{cm}^2 \neq 81\,\text{cm}^2$, this triangle is not a right triangle.

b) The only side here that could be the hypotenuse, is the longest one, measuring 13 ft. Therefore, the two quantities we need to compare are $(5\,\text{ft})^2 + (12\,\text{ft})^2$ and $(13\,\text{ft})^2$.

$(5\,\text{ft})^2 + (12\,\text{ft})^2 = 25\,\text{ft}^2 + 144\,\text{ft}^2 = 169\,\text{ft}^2$ and $(13\,\text{ft})^2 = 169\,\text{ft}^2$

Since $169\,\text{ft}^2 = 169\,\text{ft}^2$, this triangle is a right triangle with hypotenuse 13 ft long.

c) The only side here that could be the hypotenuse, is the longest one, measuring 53 units. Therefore, the two quantities we need to compare are 28^2+45^2 and 53^2.

$28^2+45^2=784+2025=2809$ and $53^2=2809$

Since $2809=2809$, this triangle is $\boxed{\text{a right triangle.}}$

If we know two sides of a right triangle, the Pythagorean theorem enables us to find the third side.

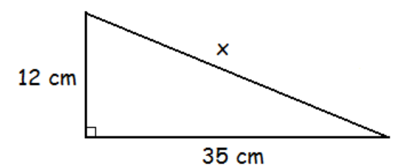
Example 2. In each case, find the missing side of the right triangle.

a) the shortest two sides are 12 cm and 35 cm

b) the longest two sides are 15 ft and 17 ft

Solution: a) We will use the first part of the Pythagorean theorem. Let us denote the hypotenuse by x. We state the Pythagorean theorem for this right triangle.

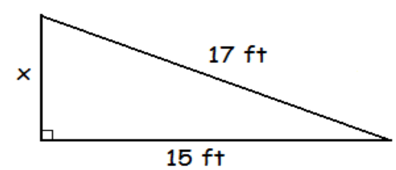
$$\begin{aligned} 12^2+35^2&=x^2\\ 144+1225&=x^2\\ 1369&=x^2\\ \pm 37&=x \end{aligned}$$



Since x represents a distance, and distances can never be negative, $x=-37$ is ruled out as a possible solution. Thus the hypotenuse of this triangle is $\boxed{37\,\text{cm}}$.

b) We will use the first part of the Pythagorean theorem. Let us denote the missing shortest side by x. We state the Pythagorean theorem for this right triangle.

$$\begin{aligned} x^2+15^2&=17^2\\ x^2+225&=289\\ x^2&=64\\ x&=\pm 8 \end{aligned}$$

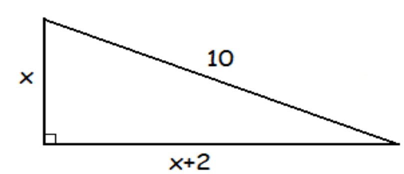


Since distances cannot be negative, $x=-8$ is ruled out as a possible solution. Thus the missing side is $\boxed{8\,\text{ft}}$ long.

In order to find missing sides, we do not always need to know the lengths of two sides if other information is given.

Example 3. The hypotenuse of a right triangle is 10 units long. The difference between the other two sides is 2 units. Find the missing sides of the right triangle.

Solution: If we label the shortest side by x, then the other missing side can be denoted by $x+2$. We state the Pythagorean theorem for this right triangle and solve the equation for x.



$x^2+(x+2)^2=10^2$ expand complete square

$x^2+x^2+4x+4=10$ combine like terms

$2x^2+4x+4=100$ subtract 100

$2x^2+4x-96=0$ factor out 2

$2\left(x^2+2x-48\right)=0$ factor (we used trial and error)

$2(x+8)(x-6)=0$ apply the zero product rule

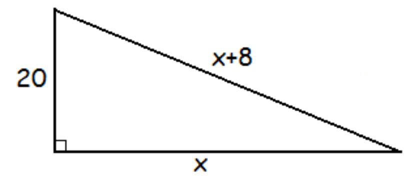
$x_1=-8,\quad x_2=6$

Since distances can never be negative, -8 is ruled out as a possible solution. The other solution, $x=6$ means that the other side, denoted by $x+2$ must be $6+2=8$ units long. Thus, this right triangle has sides $\boxed{\text{6, 8, and 10 units}}$ long.

We check: $6^2+8^2=36+64=100=10^2$, thus the triangle is indeed right. Also, $8-6=2$, so our answer is correct.

The following example will be a similar application problem, but the computation will be entirely different.

Example 4. The shortest side of a right triangle is 20 units long. The difference between the other two sides is 8 units. Find the missing sides of the right triangle.



Solution: If we label the shorter missing side by x, then the other missing side, the hypotenuse can be denoted by $x+8$. We state the Pythagorean theorem for this right triangle and solve the equation for x.

$$\begin{aligned}
20^2+x^2 &= (x+8)^2 && \text{expand complete square}\\
400+x^2 &= x^2+16x+64 && \text{subtract } x^2\\
400 &= 16x+64 && \text{subtract 64}\\
336 &= 16x && \text{divide by 16}\\
21 &= x
\end{aligned}$$

Therefore, the shorter missing side, denoted by x is 21 units long, and the hypotenuse, denoted by $x+8$ must be $21+8=29$ units long. So the three sides of this right triangle are $\boxed{\text{20, 21, and 29 units}}$ long. We check: the difference between 29 and 21 is indeed 8. For the Pythagorean theorem, $20^2+21^2=400+441=841$ and $29^2=841$, and so $20^2+21^2=29^2$. This means that this is indeed a right triangle, and so our answer is correct.

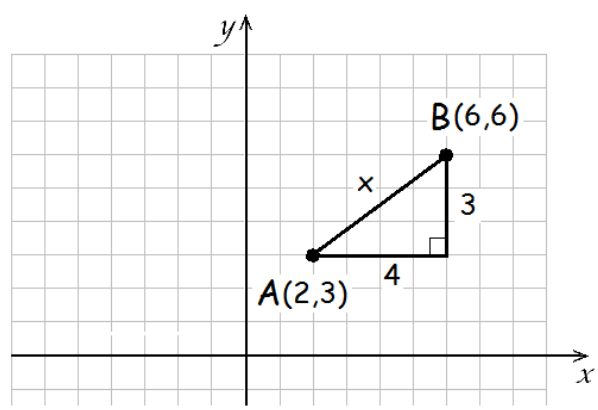
This problem turned out to be much easier because the equation was linear after x^2 was subtracted from both sides.

The following example is an extremely important application of the Pythagorean theorem.

Example 5. Find the distance between the points $A(2,3)$ and $B(6,6)$.

Solution: Let us plot the given points in a coordinate system as shown on the picture. The right triangle on the picture has shorter sides 4 and 3 units long. We denote the line segment connecting A and B (the hypotenuse) by x and state the Pythagorean theorem.

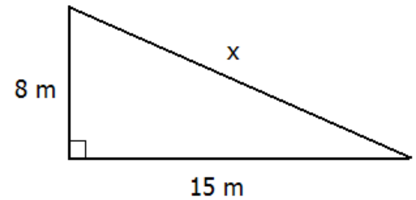
$$\begin{aligned}
4^2+3^2 &= x^2\\
16+9 &= x^2\\
25 &= x^2\\
\pm 5 &= x
\end{aligned}$$



Since distances can never be negative, -5 is ruled out and so our answer is $\boxed{\text{5 units}}$.

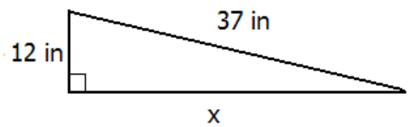
Sample Problems

1. Could the three line segments given below be the three sides of a right triangle? Explain your answer.
   a) 6 cm, 10 cm, and 8 cm
   b) 7 ft, 15 ft, and 50 ft
   c) 4 m, 5 m, and 6 m



2. Find the hypotenuse of the triangle shown on the figure.

3. Find the missing leg of the right triangle shown on the picture below.

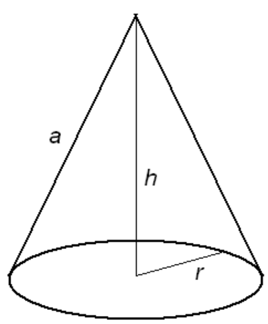


4. Find the distance between the points $(3,8)$ and $(8,-4)$.

5. The sides of an isosceles triangle are 42 units, 29 units, and 29 units long. Find the length of the height drawn to the 42 units long side.

6. The hypotenuse of a right triangle is 20 cm. The difference between the other two sides is 4 cm. Find the sides of the triangle.

7. Find the height h of the cone shown on the picture below, if the base has a radius of 10 m and $a = 26$ m.

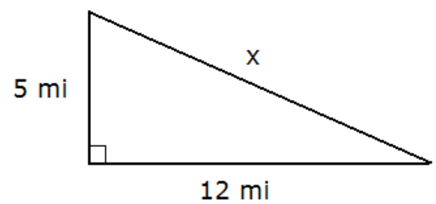


Practice Problems

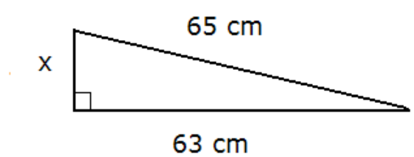
1. Could the three line segments given below be the three sides of a right triangle? Explain your answer.

   a) 3 cm, 7 cm, and 8 cm b) 37 ft, 12 ft, and 35 ft c) 6 m, 7 m, and 8 m

2. Find the hypotenuse of the triangle shown on the figure below.



3. Find the missing leg of the right triangle shown on the picture below.



4. The sides of an isosceles triangle are 25 m, 25 m, and 14 m long. Find the length of the height drawn to the 14 m long side.

5. Find the distance between the given points.

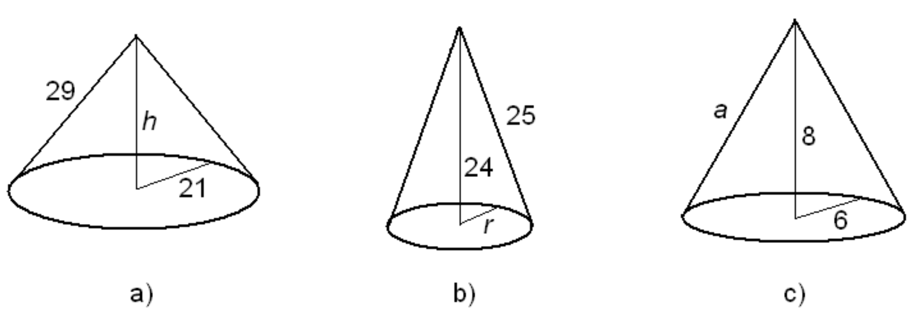
   a) $(-2,-3)$ and $(6,3)$ b) $(-9,-3)$ and $(15,4)$.

6. One leg of a right triangle is 9 cm. The difference between the other two sides is 1 cm. Find the length of all sides.

7. The hypotenuse of a right triangle is 50 in. The difference between the other two sides is 34 in. Find the length of all sides.

8. The shortest side of a right triangle is 16 units long. The difference between the other two sides is 4 units. Find the sides of this right riangle.

9. Find the missing lengths indicated on the picture below.



Chapter 22

22.1 Factoring by the AC-Method

Factoring by the AC-method is extremely useful because it adresses the most difficult situation, factoring a general quadratic expression with a leading coefficient, such as $6x^2 - 5x - 4$. Trial and error still works, but it is more difficult because there are more possibilities to be considered. The AC-method is a neat and powerful method to quickly factor a trinomial. The main steps are: we cleverly take apart the linear term into two parts and then factor by grouping.

In case of a general quadratic equation, we often denote the coefficients by a, b, and c, where the trinomial is $ax^2 + bx + c$. Notice the addition. This means that coefficients also carry the negative signs if there are any. In the case of $6x^2 - 5x - 4$, $a = 6$, $b = -5$, and $c = -4$.

Example 1. Completely factor the expression $6x^2 - 5x - 4$.

Solution: We will use the AC-method. The first step is to re-write the middle term, $-5x$ into a sum of two terms in such a way that grouping would work after that. To do that, we need to find two numbers p and q with a sum of -5 and a product that is the same as the product ac (hence the name, AC-method.) In this particular case, $ac = 6(-4) = -24$. So we need to find two numbers, p and q, such that

$$p + q = -5 \quad \text{and} \quad pq = -24$$

This sounds very similar to the trial and error method. Indeed, we will apply the same method to find p and q. A negative product pq indicates that one of p and q is positive and the other is negative. The negative sum -5 indicates that between the two numbers p and q, the negative one has a greater absolute value. These observations will make our task much easier. There are infinitely many integer pair solutions for $p + q$. There are just a few ways to solve the second equation for integers, and so we start there.

We list all the pairs of positive numbers with a product of 24, and take the opposite of the greater one in each pair. We are looking for the pair with sum -5.

	-24
1	-24
2	-12
3	-8
4	-6

Now we consider these pairs as candidates for p and q. We are looking for the pair with sum -5. Clearly that is -8 and 3. Once we found p and q with product -24 and sum -5, we can rewrite the middle term, $-5x$ as $-8x+3x$ or $3x-8x$ and factor by grouping.

$$\begin{aligned} 6x^2-5x-4 &= 6x^2-8x+3x-4 \\ &= 2x(3x-4)+1(3x-4) \\ &= (2x+1)(3x-4) \end{aligned}$$

and so the factored form is $\boxed{(2x+1)(3x-4)}$. We can check our solution by multiplying back: $(2x+1)(3x-4)=6x^2-8x+3x-4=6x^2-5x-4$ and so our solution is correct.

Example 2. Completely factor the expression $10x^2-19x+6$.

Solution: In this case, $a=10$, $b=-19$, and $c=6$. The product ac is 60. We are looking for two numbers p and q with a product of 60 and a sum of -19. A positive product indicates that both p and q are positive, or both of them negative. The negative sum indicates that both p and q are negative.

As always, we start with $pq=60$. We list all the pairs of negative numbers with a product of 60.

	60
-1	-60
-2	-30
-3	-20
-4	-15
-5	-12
-6	-10

The only pair with a sum -19 is -15 and -4. Now we know how to re-write the linear term, $-19x$ and then factor by grouping.

$$\begin{aligned} 10x^2-19x+6 &= 10x^2-15x-4x+6 && \text{factor by grouping} \\ &= 5x(2x-3)-2(2x-3) \\ &= (5x-2)(2x-3) \end{aligned}$$

Therefore, $10x^2-19x+6=\boxed{(5x-2)(2x-3)}$. We check by multiplying back:

$(5x-2)(2x-3)=10x^2-15x-4x+6=10x^2-19x+6$ and so our solution is correct.

What happens if we cannot find a suitable pair of numbers p and q? If that is the case, the trinomial cannot be factored using integer coefficients. At this point, we will just say that the expression is **prime** or **irreducible**. We are still responsible however, to factor out the GCF if there is any.

Example 3. Completely factor the expression $3x^2-4x+2$.

Solution: In this case, $ac=6$. Therefore, we are looking for two numbers p and q with a product of 6 and a sum -4. A positive product indicates that both p and q are negative or both are positive. The negative sum -4 indicates that both p and q are negative.

We start with the equation $pq=6$. We list all the pairs of negative numbers with a product of 6.

6	
-1	-6
-2	-3

The sum of the first pair is -7 and the sum of the second pair is -5. There are no pairs with sum -4.

This indicates that the trinomial cannot be factored. Therefore, our answer is $\boxed{3x^2-4x+2}$ for the factored form.

Sometimes the product ac is too rich in divisors to list all pairs. If ac is negative and the second term has a small coefficient, there is a neat shortcut to find the righ p and q. The following example illustrates this trick.

Example 4. Completely factor the expression $24x^2+x-10$.

Solution: In this case, $ac=-240$. We are looking for p and q with $pq=-240$ and $p+q=1$. The number -240 has quite a number of divisors, so we will find the right pair without listing all the pairs. This trick is not possible for all cases, but it will work here. The negative product pq indicates that one of p and q is positive and the other is negative. The positive sum 1 indicates that between the two numbers p and q, the positive one has a greater absolute value.

When we add a positive and a negative number, we subtract the absolute values. Considering the absolute values $|p|$ and $|q|$, their product is 240 and their difference is 1. That means that the right pair of factors are very close to each other. Consequently, both numbers must be fairly close to the square root of 240. We enter $\sqrt{240}$ into our calculator. This is not an integer, the calculator shows $\sqrt{240}=15.491933\ldots$. If two numbers are close to each other and their product is 240, then they both are fairly close to 15. So we round up to 16 and start looking for divisors counting backward: 16, 15, 14, etc. Except, we won't even reach 14 because we immediately bump into 15 and 16. Our p and q is 16 and -15. We can now easily factor by grouping.

$$\begin{aligned} 24x^2+x-10 &= 24x^2+16x-15x-10 \\ &= 8x(3x+2)-5(2x+2)=(8x-5)(3x+2) \end{aligned}$$

Example 5. One side of a rectangle is 2 feet shorter than three times another side. Find the sides of the rectangle if we also know that its area is $176\,\mathrm{ft}^2$.

Solution: If we label one side as x, then the other side can be written as $3x-2$. The equation will express the area of the rectangle.

$$\begin{aligned} x(3x-2) &= 176 \\ 3x^2-2x &= 176 \\ 3x^2-2x-176 &= 0 \end{aligned}$$

We will factor this trinomial by the AC-method. In this case, $ac=3(-176)=-528$. To factor $3x^2-2x-176$, we need to find two integers p and q with product -528 and sum -2. The negative product indicates that one of p and q is positive, the other one is negative. Therefore, the difference between their absolute values is 2. That is fairly small for a large product such as -528. If the two numbers multiplying each other to 528 are close to each other, they also must be close to $\sqrt{528}=22$.

978 251.... We roll up to 23 and start looking for pairs of integers multiplying to 528. We will try 23, 22, 21, and so on. We almost immediately find 22 and 24. Therefore p and q are -24 and 22. We can now easily factor the trinomial.

$$\begin{aligned} 3x^2-2x-176 &= 3x^2-24x+22x-176 \\ &= 3x(x-8)+11(2x-8) \\ &= (3x+11)(x-8) \end{aligned}$$

Applying the zero product rule, we solve $3x+11=0$ and $x-8=0$ ad obtain $x=-\dfrac{11}{3}$ or $x=8$. Since x represents a distance and distances cannot be negative, we rule out $-\dfrac{11}{3}$ and are left with $x=8$. If x is 8, then $3x-2=3\cdot 8-2=22$. Thus the sides of the rectangle are $\boxed{8\,\text{ft and } 22\,\text{ft}}$ long.
We check: 22 is indeed two less than three times 8, and the area of the rectangle is $8\,\text{ft}\,(22\,\text{ft})=176\,\text{ft}^2$, and so our solution is correct.

Quadratic equations often have two solutions. In the previous example, one solution of the equation was easily ruled out, but that is not always the case. Often times both solutions of the equation result in a meaningful solution. The next example illustrates this.

Example 6. Twice the square of a number is 35 greater than three times the number. Find all such numbers.

Solution: If we label this number by x, then the square of this number is x^2 and twice the square is $2x^2$. The equation will compare twice the square of the number and three times the number.

$$\begin{aligned} 2x^2 &= 3x+35 \\ 2x^2-3x-35 &= 0 \end{aligned}$$

We will factor $2x^2-3x-35$. The product ac is -70. Therefore, we are lookin for two integers p and q with product -70 and sum -3. These are easily found: -10 and 7. We now know how to take apart the middle term before grouping.

$$\begin{aligned} 2x^2-3x-35 &= 0 \\ 2x^2-10x+7x-35 &= 0 \\ 2x(x-5)+7(x-5) &= 0 \\ (2x+7)(x-5) &= 0 \end{aligned}$$

We solve the linear equations $2x+7=0$ and $x-5=0$ and obtain $x=-\dfrac{7}{2}$ and $x=5$. We check both:
If the number is 5, then twice its square is $2\cdot 5^2=50$, and three times the number is 15. Indeed, 50 is 35 greater than 15. So 5 works.
If the number is $-\dfrac{7}{2}$, then twice its square is $2\left(-\dfrac{7}{2}\right)^2=2\cdot\dfrac{49}{4}=\dfrac{49}{2}$. Three times the number is $3\left(-\dfrac{7}{2}\right)=-\dfrac{21}{2}$. The difference between $\dfrac{49}{2}$ and $-\dfrac{21}{2}$ is $\dfrac{49}{2}-\left(-\dfrac{21}{2}\right)=\dfrac{49+21}{2}=\dfrac{70}{2}=35$.

We found that both $\boxed{-\dfrac{7}{2} \text{ and } 5}$ works.

Practice Problems

1. Completely factor each of the following.

   a) $30x - 15y + 6ax - 3ay$

   b) $xy - y - x + 1$

   c) $6a^2b^2 - 4a^2bc - 10a^2c^2$

   d) $a^2m + 2a^2n - b^2m - 2b^2n$

   e) $x^2 - 4y^2 + m^2x^2 - 4m^2y^2$

   f) $b^2 - a + ab^2 - 1$

   g) $2m^2 - 18n^4 + 2m^2p^2 - 18n^4p^2$

   h) $a^2x^2 - a^2y^2 + b^2x^2 - b^2y^2$

   i) $3x^2 - 2x - 1$

   j) $6x^2 - 5x + 1$

   k) $14x - 12x^2 + 10$

   l) $5m^2 - 7mn + 2n^2$

   m) $29px - 21p^2 + 10x^2$

   n) $2x^4 - 3y^4 + x^2y^2$

2. Solve each of the following equations.

   a) $6x^4 - x^3 = 2x^2$

   b) $5a^2 + 5 = 26a$

   c) $11p + 35p^2 = 6$

3. One side of a rectangle is 4 in shorter than 3 times the other side. Find the sides of the rectangle if its area is 319 in^2.

Chapter 23

23.1 Factoring the Difference and Sum of Cubes

We have seen the difference of squares theorem that plays a fundamental role in factoring. Let us first recall the theorem.

> **Theorem:** (The Difference of Squares Theorem) For any quantities A and B,
>
> $$A^2 - B^2 = (A+B)(A-B)$$

Notice that $A+B$ and $A-B$ on the right-hand side are conjugates. The identical terms and alternating signs in front of B cause a cencellation between O and I in FOIL.

$$\begin{aligned}(A+B)(A-B) &= A^2 - AB + AB - B^2 \qquad -AB+AB=0 \\ &= \boxed{A^2 - B^2}\end{aligned}$$

This quite mechanical idea can be generalized to other products. Consider the following products.

Example 1. Expand each of the following.

a) $(A-B)\left(A^2+AB+B^2\right)$ b) $(A-B)\left(A^3+A^2B+AB^2+B^3\right)$

Solution: a) We will apply the distributive law.

$$\begin{aligned}(A-B)\left(A^2+AB+B^2\right) &= A\left(A^2+AB+B^2\right) - B\left(A^2+AB+B^2\right) \\ &= A^3+A^2B+AB^2-\left(A^2B+AB^2+B^3\right) \\ &= A^3+A^2B+AB^2-A^2B-AB^2-B^3 \\ &= \boxed{A^3-B^3}\end{aligned}$$

b) Similarly,

$$\begin{aligned}(A-B)\left(A^3+A^2B+AB^2+B^3\right) &= A\left(A^3+A^2B+AB^2+B^3\right) - B\left(A^3+A^2B+AB^2+B^3\right) \\ &= A^4+A^3B+A^2B^2+AB^3-A^3B-A^2B^2-AB^3-B^4 \\ &= \boxed{A^4-B^4}\end{aligned}$$

This pattern goes on with higher and higher exponents. Indeed, $A^n - B^n$ can be factored for every natural number $n \geq 2$.

Theorem: For any quantities A and B,

$$\begin{aligned}
A^3 - B^3 &= (A-B)\left(A^2+AB+B^2\right) \\
A^4 - B^4 &= (A-B)\left(A^3+A^2B+AB^2+B^3\right) \\
A^5 - B^5 &= (A-B)\left(A^4+A^3B+A^2B^2+AB^3+B^4\right) \\
&\vdots \\
A^n - B^n &= (A-B)\left(A^{n-1}+A^{n-2}B+A^{n-3}B^2+\ldots+A^2B^{n-3}+AB^{n-2}+B^{n-1}\right)
\end{aligned}$$

We will be focusing on the difference of cubes theorem:

$$A^3 - B^3 = (A-B)\left(A^2+AB+B^2\right)$$

We can not verify this yet, but the second, longer factor, A^2+AB+B^2 is irreducible, it can not be factored.

Example 2. Factor each of the following.

a) $x^3 - 125y^9$ b) $8p^3 - 27q^{12}$

Solution: a) We will factor the expression applying the difference of cubes theorem. A will be 'played' by x and B by $5y^3$, because $x^3 - 125y^9 = x^3 - \left(5y^3\right)^3$.

$$\begin{aligned}
A^3 - B^3 &= (A-B)\left(A^2+AB+B^2\right) \quad \text{where } A = x \text{ and } B = 5y^3 \\
x^3 - \left(5y^3\right)^3 &= \left(x-5y^3\right)\left(x^2+x\left(5y^3\right)+\left(5y^3\right)^2\right) \\
&= \boxed{\left(x-5y^3\right)\left(x^2+5xy^3+25y^6\right)}
\end{aligned}$$

b) Similarly, $A = 2p$ and $B = 3q^4$, because $8p^3 - 27q^{12} = (2p)^3 - \left(3q^4\right)^3$

$$\begin{aligned}
A^3 - B^3 &= (A-B)\left(A^2+AB+B^2\right) \quad \text{where } A = 2p \text{ and } B = 3q^4 \\
(2p)^3 - \left(3q^4\right)^3 &= \left(2p-3q^4\right)\left((2p)^2+(2p)\left(3q^4\right)+\left(3q^4\right)^2\right) \\
&= \boxed{\left(2p-3q^4\right)\left(4p^2+6pq^4+9q^8\right)}
\end{aligned}$$

Recall that the sum of two squares can never be factored. However, this is not the case with cubes, or fifth powers or seven powers etc. Consider the following products.

Example 3. Expand each of the following.

a) $(A+B)\left(A^2-AB+B^2\right)$ b) $(A+B)\left(A^4-A^3B+A^2B^2-AB^3+B^4\right)$

Solution: a) We will apply the distributive law.

$$\begin{aligned}
(A+B)\left(A^2-AB+B^2\right) &= A\left(A^2-AB+B^2\right)+B\left(A^2-AB+B^2\right) \\
&= A^3-A^2B+AB^2+A^2B-AB^2+B^3 \\
&= \boxed{A^3+B^3}
\end{aligned}$$

b) $(A+B)\left(A^4-A^3B+A^2B^2-AB^3+B^4\right)=$

$$\begin{aligned}&=A\left(A^4-A^3B+A^2B^2-AB^3+B^4\right)+B\left(A^4-A^3B+A^2B^2-AB^3+B^4\right)\\&=A^5-A^4B+A^3B^2-A^2B^3+AB^4+A^4B-A^3B^2+A^2B^3-AB^4+B^5\\&=\boxed{A^5+B^5}\end{aligned}$$

This pattern goes on with higher and higher exponents. Indeed, A^n+B^n can be factored for every odd natural number $n\geq 3$.

Theorem: For any quantities A and B, and odd natural number n,

$$\begin{aligned}A^3+B^3&=(A+B)\left(A^2-AB+B^2\right)\\A^5+B^5&=(A+B)\left(A^4-A^3B+A^2B^2-AB^3+B^4\right)\\A^7+B^7&=(A+B)\left(A^6-A^5B+A^4B^2-A^3B^3+A^2B^4-AB^5+B^6\right)\\&\vdots\\A^n+B^n&=(A+B)\left(A^{n-1}-A^{n-2}B+A^{n-3}B^2-\ldots+A^2B^{n-3}-AB^{n-2}+B^{n-1}\right)\end{aligned}$$

We will be focusing on the sum of cubes theorem:

$$A^3+B^3=(A+B)\left(A^2-AB+B^2\right)$$

Similar to the difference of cubes, the second, longer factor, A^2-AB+B^2 is irreducible, it can not be factored.

Example 4. Factor each of the following.

a) x^3+125y^9 b) $(x+2)^3+27$

Solution: a) We will factor the expression applying the sum of cubes theorem. A will be 'played' by x and B by $5y^3$, because $x^3+125y^9=x^3+\left(5y^3\right)^3$.

$$\begin{aligned}A^3+B^3&=(A+B)\left(A^2-AB+B^2\right)\quad\text{where } A=x \text{ and } B=5y^3\\x^3+\left(5y^3\right)^3&=\left(x+5y^3\right)\left(x^2-x\left(5y^3\right)+\left(5y^3\right)^2\right)\\&=\boxed{\left(x+5y^3\right)\left(x^2-5xy^3+25y^6\right)}\end{aligned}$$

b) Similarly, $A=x+2$ and $B=3$, because $27=3^3$. A will be 'played' by $x+2$ and B by 3.

$$\begin{aligned}A^3+B^3&=(A+B)\left(A^2-AB+B^2\right)\quad\text{where } A=x+2 \text{ and } B=3\\(x+2)^3+3^3&=(x+2+3)\left((x+2)^2-(x+2)(3)+3^2\right)\\&=(x+5)\left(x^2+4x+4-3x-6+9\right)\\&=\boxed{(x+5)\left(x^2+x+7\right)}\end{aligned}$$

Sample Problems

Completely factor each of the following.

1. $x^3 - 8y^3$
2. $125 - 27a^{12}$
3. $1000 + x^6$
4. $(x+1)^3 - 27$
5. $-2a^7 - 2a^4b^9$
6. $(a+2)^3 + (a-2)^3$
7. $a^6 - b^6$

Practice Problems

Completely factor each of the given expressions.

1. $8x^3 + 1000$
2. $(q+10)^3 + q^3$
3. $m^3 - (y+1)^3$
4. $a^5b - a^2bx^6$
5. $4a^3m + 2a^3n - 4b^3m - 2b^3n$
6. $3n^2x^3 - 12m^2y^3 - 12m^2x^3 + 3n^2y^3$
7. $2a^5 - 2a^2 - 2a^2b^2 + 2a^5b^2$
8. $(3a-1)^3 + (a-3)^3$
9. $(3a-1)^3 - (a-3)^3$
10. $2 - 2x^6$

Chapter 24

24.1 Fractions and Decimals

Part 1: Converting a Fraction to a Decimal

This is easy to do if we understand a formal, algebraic definition of a fraction. If a and b are integers, b not zero, then the **fraction** $\frac{a}{b}$ is the result of the division $a \div b$. In a sense, fractions are driving instructions. They do not tell us the value of the number, only, how to obtain it. To get a decimal, we simply perform the division.

Example 1. Convert $\frac{3}{8}$ to a decimal.

Solution: We perform the long division $3 \div 8$. The result is 0.375.

$$\begin{array}{cccccc}
 & & .3 & 7 & 5 \\ \hline
8 &)3 & .0 & 0 & 0 \\
- & 2 & 4 & & \\ \hline
 & & 6 & 0 & \\
 & - & 5 & 6 & \\ \hline
 & & & 4 & 0 \\
 & & - & 4 & 0 \\ \hline
 & & & & 0
\end{array}$$

Example 2. Convert $\dfrac{1927}{11}$ to a decimal.

Solution: We perform the division $1927 \div 11$. The result is $175.\overline{18}$. The bar over the last two digits indicates an infinitely many times repeating block.

$$
\begin{array}{cccccccccc}
 & & & 1 & 7 & 5 & .1 & 8 & 1 & 8... \\
\hline
11 &) & 1 & 9 & 2 & 7 & .0 & 0 & 0 & 0 \\
- & & 1 & 1 & & & & & & \\
\hline
 & & & 8 & 2 & & & & & \\
 & & - & 7 & 7 & & & & & \\
\hline
 & & & & 5 & 7 & & & & \\
 & & & - & 5 & 5 & & & & \\
\hline
 & & & & & 2 & 0 & & & \\
 & & & & - & 1 & 1 & & & \\
\hline
 & & & & & & 9 & 0 & & \\
 & & & & & - & 8 & 8 & & \\
\hline
 & & & & & & & 2 & 0 & \\
 & & & & & & - & 1 & 1 & \\
\hline
 & & & & & & & & 9 & 0 \\
 & & & & & & & - & 8 & 8 \\
\hline
 & & & & & & & & & 2
\end{array}
$$

Part 2: Converting a Terminating Decimal to Fraction

A decimal is **terminating** if it has a last digit. It is quite easy to turn a terminating decimal to a fraction of integers.

Example 3. Convert 0.45 to a reduced fraction.

Solution: Step 1. Write the number as a fraction of any kind first.

$$0.45 = \frac{0.45}{1}$$

We can mentally check it as division: any number divided by one results in the same number.

Step 2. We ask ourselves: "How many digits do we need to move the decimal point to the right in 0.45 to obtain an integer"? The answer is: two digits. Moving the decimal point to the right by two digits is the same as multiplication by 100. Thus, to fix the numerator, we need to multiply it by 100. Because we also want to preserve the value, we multiply both upstairs and downstairs by 100.

$$\frac{0.45}{1} = \frac{0.45 \cdot 100}{1 \cdot 100} = \frac{45}{100}$$

Step 3. We simplify the fraction by dividing numerator and denominator by the greatest common factor.

$$\frac{45}{100} = \frac{\cancel{5} \cdot 9}{\cancel{5} \cdot 20} = \frac{9}{20}$$

Example 4. Convert 0.0005 to a reduced fraction.

Solution: We will write the number as a fraction and then multiply numerator and denominator by 10000.

$$0.0005 = \frac{0.0005}{1} = \frac{0.0005 \cdot 10000}{1 \cdot 10000} = \frac{5}{10000} = \frac{\cancel{5} \cdot 1}{\cancel{5} \cdot 2000} = \frac{1}{2000}$$

Example 5. Convert 23.044 to a reduced fraction.

Solution:

$$23.044 = 23 + 0.044 = 23 + \frac{0.044}{1} = 23 + \frac{0.044 \cdot 1000}{1 \cdot 1000} = 23\frac{44}{1000} = 23\frac{4 \cdot 11}{4 \cdot 250} = 23\frac{11}{250}$$

Part 3: (The Fun Stuff)
Converting a Non-Terminating Decimal to Fraction

A decimal is non-terminating if it has infinitely many digits. If there is a repeating block, we denote it by a bar drawn over the repeating digit. For example, the number $2.\overline{35}$ denotes $2.35\,35\,35\,35......$

Example 6. Re-write each of the given repeating decimals without the bar notation.

a) $1.201\overline{7}$ b) $1.20\overline{17}$ c) $1.2\overline{017}$ d) $1.\overline{2017}$

Solution: Only the digit(s) under the bar are repeating. The rest is there as is.

a) $1.201\overline{7} = 1.201\,777777....$

b) $1.20\overline{17} = 1.20\,17\,17\,17\,17\,17\,17....$

c) $1.2\overline{017} = 1.2\,017\,017\,017\,017\,017....$

d) $1.\overline{2017} = 1.2017\,2017\,2017\,2017\,2017....$

Turning these decimals into fractions of integers is an interesting and fun application of linear equations.

Example 7. Convert the repeating decimal $7.\overline{4}$ to a fraction.

Solution: Step 1. We label our number x and write it without the bar notation. The dots are important: they indicate that we have infinitely many $4'$s there and not just three.

$$7.444... = x$$

Step 2. We multiply both sides of this equation by 10.

$$74.444... = 10x$$

Step 3. We write these equations together, starting with the second one.

$$\begin{aligned} 74.444... &= 10x \\ 7.444... &= x \end{aligned}$$

Step 3. (Chop, chop.) We subtract the second equation from the first one.

$$\begin{array}{rl} & 74.444... = 10x \\ - & \underline{\ \ 7.444... = x\ \ \ } \\ & 67 \qquad\ = 9x \end{array}$$

Step 4. We solve the equation for x.

$$\begin{aligned} 67 &= 9x \qquad \text{divide by 9} \\ \frac{67}{9} &= x \end{aligned}$$

Thus the answer is $\frac{67}{9}$. We can check by long division. Indeed, $67 \div 9 = 7.444\,444\,44...$

Example 8. Convert the repeating decimal $0.\overline{405}$ to a fraction.

Solution: Step 1. We label our number x and write it without the bar notation. $0.405\,405\,405\,405\,405... = x$

Steps 2 and 3. This decimal has a three-digit long repeating block. To obtain proper alignment of the digits, we will move the decimal point by three digits, i.e. we will multiply by 1000. We multipliy both sides of this equation by 1000. We write these equations together, starting with the second one.

$$\begin{aligned} 405.405\,405\,405\,405... &= 1000x \\ 0.405\,405\,405\,405... &= x \end{aligned}$$

Step 3. (Chop, chop.) We subtract the second equation from the first one.

$$\begin{array}{rl} 405.405\,405\,405\,405... & = 1000x \\ -\quad 0.405\,405\,405\,405.... & = x \\ \hline 405 & = 999x \end{array}$$

Step 4. We solve the equation for x.

$$\begin{aligned} 405 &= 999x \qquad \text{divide by 999} \\ \frac{405}{999} &= x \end{aligned}$$

Please note that the fraction obtained is not reduced. However, the essential point in the problem is to find a fraction, not the reduced form of it. Thus the answer is $\frac{405}{999}$. We can check by long division. Indeed, $405 \div 999 = 0.405\,405\,405....$

Example 9. Convert the repeating decimal $18.29\overline{04}$ to a fraction.

Solution: Step 1. We label our number x and write it without the bar notation.

$$18.29\,04\,04\,04\,04... = x$$

Steps 2 and 3. This decimal has a two-digit long repeating block. To obtain proper alignment of the digits, we will move the decimal point by two digits, i.e. we will multiply by 100. We multipliy both sides of this equation by 100. We write the two equations together, starting with the second one.

$$\begin{aligned} 1829.04\,04\,04\,04\,04... &= 100x \\ 18.29\,04\,04\,04\,04... &= x \end{aligned}$$

Step 3. (Chop, chop.) We subtract the second equation from the first one.

$$\begin{array}{rl} 1829.0404040404\ldots &= 100x \\ -\quad 18.2904040404\ldots &= x \\ \hline 1810.75 \qquad\qquad &= 99x \end{array}$$

It appears that we have a problem: the right-hand side is not an integer after the subtraction. This is quite easy to fix: we just multipliy both sides by 100.

$$\begin{aligned} 1810.75 &= 99x \qquad \text{multiply by 100} \\ 181075 &= 9900x \end{aligned}$$

Step 4. We solve the equation for x.

$$\begin{aligned} 181075 &= 9900x \qquad \text{divide by 9900} \\ \frac{181075}{9900} &= x \end{aligned}$$

Thus the answer is $\frac{181075}{9900}$. We can check by long division. Indeed, $181075 \div 9900 = 18.29040404\ldots$

Practice Problems

1. Perform each of the following conversions.
   a) Convert the given fraction to decimals.
      i) $\frac{4}{5}$ ii) $\frac{26}{3}$ iii) $\frac{26}{25}$ iv) $\frac{26}{7}$
   How many digits long is the repeating block?
   b) Convert the given decimal to a fraction of integers. (You do not have to reduce them!)
      i) 2.18 ii) $2.\overline{9}$ iii) $6.\overline{47}$ iv) $1.8\overline{705}$
2. Based on your answer for 1b ii), what is a surprising new fact about decimal presentation of numbers?
3. Consider the fraction $\frac{1}{n}$. What numbers n will result in a terminating decimal?
4. The decimal presentation of $\frac{2}{13}$ does not appear to be repeating. Is it?
5. Find a fraction formed of two integers that will result in a non-terminating, non-repeating decimal.

Chapter 25

25.1 Square Root of 2 is Irrational

A very powerful proving technique is what we call **indirect proof**, or **proof by contradiction**.

The logic behind this proving technique is as follows. Suppose that we start with a true statement and arrive to other statements by making logically correct steps. Then these new statements must all be true.

Suppose we start with a statement and use logically correct steps to arrive to other statements, including one that is obviously false. Then we must have started with a false statement.

True statements only imply true statements. If our conclusion is false, we must have started with a false statement.

Suppose we want to prove a statement to be true. In case of a proof by contradiction, we formulate the exact opposite of our statement, and, using logically correct steps, we derive an obviously false statement. This proves that we started with a false statement. Therefore, the opposite of our statement is false, which means that our statement is true.

The fact that $\sqrt{2}$ is irrational can be proven by contradiction.

Definition: A number is **rational** if it can be written as a fraction of two integers.

Definition: A number is **irrational** if it is not rational, i.e. it can not be written as a fraction of two integers.

Theorem: $\sqrt{2}$ is an irrational number.

Proof. Suppose, for a contradiction, that $\sqrt{2}$ is rational, i.e. there exist two integers, a and b $(b \neq 0)$ such that

$$\sqrt{2} = \frac{a}{b}$$

We may also assume that the fraction $\frac{a}{b}$ is in lowest terms, otherwise we could reduce the fraction $\frac{a}{b}$ and replace it with the reduced equivalent. So, let us assume that $\frac{a}{b}$ is in lowest terms, which means that a and b do not share any divisor larger than 1. Now let us square both sides.

$$2 = \frac{a^2}{b^2}$$

Let us multiply both sides by b^2.

$$2b^2 = a^2$$

Since a^2 is twice another integer, it is even. This means that a itself must be even. Let us re-write $a = 2k$ where k

is some integer.

$$\begin{aligned} 2b^2 &= (2k)^2 \\ 2b^2 &= 4k^2 \end{aligned}$$

Let us divide both sides by 2. Then we have

$$b^2 = 2k^2$$

Since b^2 is twice another integer, it is even. This means that b itself must be even. We are now done, because the following statements cannot all be true.

1. a and b are two integers that do not share any divisors.
2. a is even.
3. b is even

This is a contradiciton, guaranteeing that there is at least one false statement among the three. This means that the assumption that $\sqrt{2}$ is rational must be false, its opposite therefore true. This completes our proof. ■

25.2 The Real Number System

Definition: The **set of all natural numbers**, denoted by $\mathbb{N}$, is the infinite set

$$\mathbb{N} = \{1, 2, 3, 4, ...\}$$

If we add two natural numbers, the sum is also a natural number. In other words, if x and y are natural numbers, then the sum $x+y$ is also a natural number. When this is true, we say that the set of all natural numbers is **closed under addition**. On the other hand, the set of all natural numbers is not closed under subtraction: while $10-3$ is a natural number, $3-10$ is not.

Theorem: The set of all natural numbers is closed under addition and multiplication, but not under subtraction and division.

Definition: The **set of all integers**, denoted by $\mathbb{Z}$, is the infinite set

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, ...\}$$

Notice that the set of all integers contains all natural numbers. When this happens, we say that the set of all natural numbers is a subset of the set of all integers. Notation: $\mathbb{N} \subseteq \mathbb{Z}$.

Theorem: The set of all integers is closed under addition, multiplication, and subtraction, but not under division.

Definition: A number is **rational** if it can be written as a quotient of two integers.

For example, $\frac{3}{8}$ is a rational number because both 3 and 8 are integers and so $\frac{3}{8}$ is a quotient of two integers.

Definition: The **set of all rational numbers**, denoted by $\mathbb{Q}$, is the infinite set

$$\mathbb{Q} = \left\{\frac{a}{b} : \ a \text{ and } b \text{ are integers, } b \neq 0\right\}$$

Notice that the set of all rational numbers entirely contains the set of all integers, i.e. $\mathbb{Z} \subseteq \mathbb{Q}$. In fact, the three sets are such that

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$$

For example, the number -5 is an integer and also a rational number, because we can write it as a quotient $\frac{-5}{1}$ where both -5 and 1 are integers. The number zero is also a rational number because it can be written as $\frac{0}{3}$.

Theorem: The set of all rational numbers is closed under addition, multiplication, subtraction, and division.

For some strange reason, mathematicians still needed additional types of numbers.

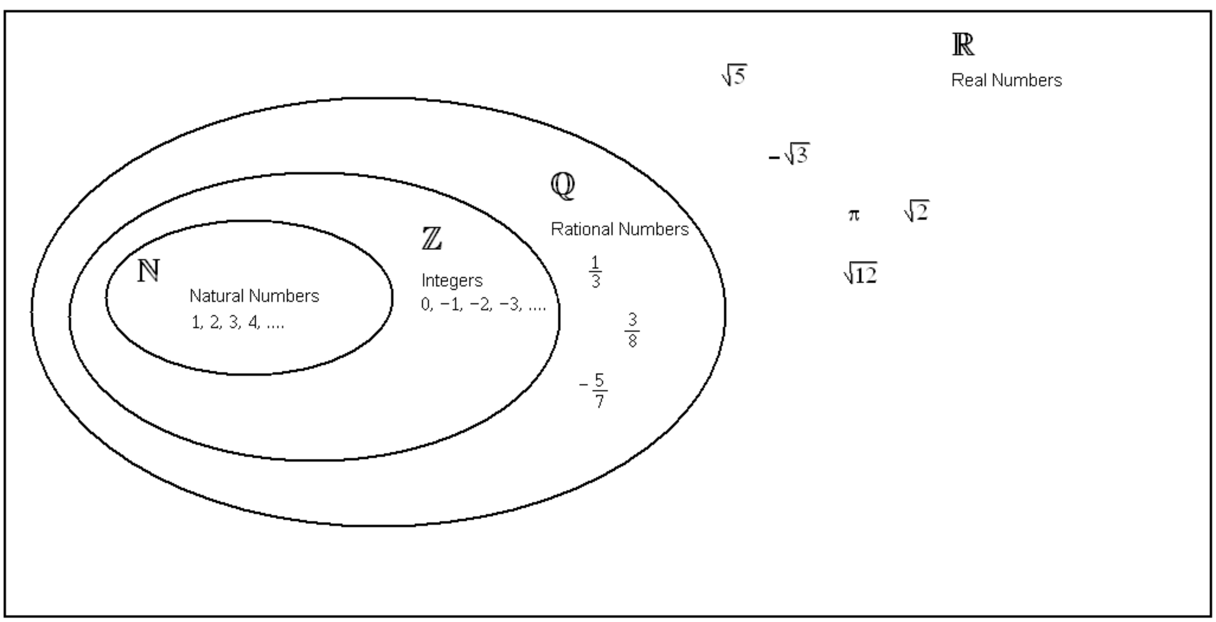
Definition: A number is **irrational** if it cannot be written as a quotient of two integers.

This is a very strange property because there are so many different integers from which to choose. However, irrational numbers exist. For example, π and $\sqrt{2}$ are irrational numbers. Surprisingly, in a sense, there are many more irrational numbers than rational numbers. (In a fascinating subject within mathematics called set theory, mathematicians have developed language to compare infinite sets. In that comparison, the set of irrational numbers proved to be much greater than the set of rational numbers.)

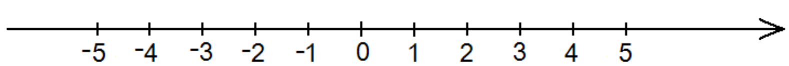
Definition: The set of all **real numbers**, denoted by $\mathbb{R}$, is the collection of all rational and irrational numbers.

The set of all real numbers contain all previous number sets as a subset. For example, every rational number is a real number.

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$



Mathematicians proved that there are exactly as many real nubers as many points there are on a straight line. Given a line, for every point on it, we can uniquely assign a real number to that point. Vice versa, for every real number, there is exactly one point on the line. This correspondance is expressed by the concept of the **number line**.



Theorem: Every terminating decimal represents a rational number.

We have seen this before. To convert a terminating decimal to a fraction of integers, see the previos section.

Theorem: Every non-terminating, repeating decimal represents a rational number.

We have seen this before. To convert a terminating decimal to a fraction of integers, see the previos section.

There is an important conclusion that can be drawn from these two facts. Consider $\sqrt{2}$, for example. If we accept the fact that $\sqrt{2}$ is irrational (which can be proved at this level) then it follows that its decimal presentation can not be terminating. (Why not?) And also, the decimal presentation of $\sqrt{2}$ can not be repeating. What is left for poor irrational numbers?

Theorem: The decimal presentation of irrational numbers is non-terminating and non-repeating.

By definition of irrational numbers, $\sqrt{2}$ can not be expressed as a fraction of two integers. We have just seen that $\sqrt{2}$ can not really be written as a decimal. If we attempted to write $\sqrt{2}$ as a decimal, we can only write approximations of the number, and never the exact value.

When we are prompted to give a number's *exact value*, in case of $\sqrt{2}$, the symbol $\sqrt{2}$ is our only option. Fractions formed from integers such as $\frac{141}{100}$ or decimals such as 1.41 are only *approximations*.

Problem Set 25

1. Label each of the following statements as true or false.
   a) 3 is an odd number or 10 is a prime number.
   b) 3 is an odd number and 10 is a prime number.
   c) -2 is a natural number or 5 is an even number.
   d) $-\frac{2}{3}$ is an even number or an odd number.

2. Suppose that $U = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{1,4,9\}$, and $B = \{2,4,6,7\}$. Find each of the following.
   a) $A \cap B$ b) $A \cup B$ c) $P = \{x \in U : x > 4 \text{ or } x \leq 7\}$ d) $Q = \{x \in U : x > 4 \text{ and } x \leq 7\}$

3. Suppose that S is the set of all squares and R is the set of all rectangles. Label each of the following statements as true or false.
   a) $S \subseteq R$ b) $R \subseteq S$ c) $R \subseteq R$ d) $\varnothing \subseteq S$ e) $R \cup S = S$ f) $R \cup S = R$ g) $R \cap S = S$ h) $R \cap S = R$ i) $R \cup \varnothing = R$

4. Suppose that F is the set of all integers divisible by four, S is the set of all integers divisible by six, and T is the set of all integers divisible by three. Label each of the following statements as true or false.
   a) $S \subseteq T$ b) $F \subseteq S$ c) $F \cap T = S$ d) $F \cap T \subseteq S$ e) $F \subseteq S \cup T$

5. Suppose that A and B are sets such that $A \cup B = \{1,3,4,5,6,7,8\}$ and $A \cap B = \{1,3,5,6,8\}$. How many different sets are possible for A?

6. Perform the division with remainder. $2020 \div 17$

7. List all factors of 84.

8. Consider the following numbers. 2011, 11 060 904, 321, 3106
   Select all the numbers from the list that are divisible a) by 2 b) by 3 c) by 6

9. Which of the following numbers is a prime? 2007, 143, 151, 91

10. Find the prime factorization of 720.

11. Find the prime factorization for x if a) $x = 12^{100}$ b) $x = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
    Note that there is a shorter notation for the product above: $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 10!$ (read as ten factorial)

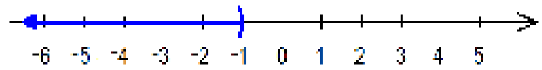
12. Find the smallest positive integer that is divisible by 2, 3, 4, 5, and 6.

13. Find the greatest common factor and least common multiple of 420 and 2400.

14. Label each of the following statements as true or false.
    a) Every integer is a rational number.
    b) Every prime number is odd.
    c) If a is divisible by 2 and b is divisible by 3, then the product ab is divisible by 6.
    d) The sum of two consecutive integers is always an odd number.
    e) The product of two consecutive integers is always an odd number.
    f) Suppose that a and b are two positive integers. Let us denote the greatest common factor of a and b by F and the least common multiple of a and b by M. Then $ab = FM$.
    g) If x and y are positive integers such that x and y are both divisible by 7, then $x - y$ is also divisible by 7.

h) If x and y are positive integers such that $x - y$ is divisible by 7, then both x and y are divisible by 7.

15. Re-write the set depicted on the picture in interval notation.



16. Find the multiplicative inverse of -0.75.

17. If $A = \left\{-\sqrt{9}, \pi, -\frac{3}{16}, 0.67, \sqrt{3}\right\}$, then A contains how many irrational numbers?

18. Perform the indicated operations and simplify.

a) $-3^2 - 2\left(5 - 3\left(2 \cdot 7 - 4^2\right)\right)$

b) $-6 - 7(-3)$

c) $\dfrac{5^2 - 4^2}{11 - 3 \cdot 2^2 + 1}$

d) $\dfrac{48 \div 8 \cdot 6}{-2^2 - 5}$

e) $12 - 2(5 - 2 + 1)$

f) 3^{-2}

g) $\dfrac{1}{(-2)^{-3}}$

h) $2^{-1} - 3^{-1}$

i) $\dfrac{(-2)^{-1} - (-3)^{-1}}{(-2)^{-2} + (-3)^{-2}}$

j) $\dfrac{5}{8} - \dfrac{2}{3} + \dfrac{1}{6}\left(-\dfrac{1}{2}\right)$

19. Simplify each of the following.

a) $|2 - 4|-3 - 7||$

b) $|2 - 4 - |3 - 7||$

c) $|2|-4 - 3 - 7||$

d) $|2 - 4||-3 - 7|$

e) $||2 - 4| - 3| - 7$

f) $||2 - 4 - 3| - 7|$

20. Prove that the given decimals represent rational numbers by re-writing each of them as a quotient of integers. You do not need to reduce the fraction.

a) $0.4\overline{93}$ b) $0.228\overline{5}$ c) $0.0\overline{652}$

21. Suppose that $A = 640\,000\,000\,000$ and $B = 0.000\,000\,000\,025$. Perform the indicated operations. Present your answer using scientific notation.

a) A b) B c) B^2 d) AB e) $\sqrt{A}$ f) $\dfrac{A}{B}$ g) A^2B^3

22. Evaluate each of the given epressions.

a) $b^2 - 4ac$ if $a = -1$, $b = -2$, and $c = -3$ b) $\dfrac{|2x - 8|}{2x + 1}$ if $x = -5$ c) $\dfrac{3a + b}{2a - b}$ if $a = -3$ and $b = -6$

23. Compute the value of the expression $-12p + 3$ when $p = \dfrac{5}{6}$.

24. Simplify each of the given expressions.

a) $8t - (-6t + 2)$ b) $2x - 4(y - x) - 3y$ c) $\dfrac{-10x + 24}{2}$ d) $2(x - 3y) - 5(2x - 4y)$

25. Simplify each of the given expressions.

a) $\dfrac{5x^5y^4z}{30x^3yz^2}$

b) $\left(x^4\right)^2\left(x^{-2}\right)^3$

c) $\left(\dfrac{-2ab^{-3}}{b^{-2}}\right)^{-4}$

d) $\dfrac{-2ab^5\left(3a^2b^{-3}\right)^4}{\left(-3a^{-1}b^0a^3\right)^5(3a)^{-2}}$

e) $-2x^3\left(-x^2\right)^4$

f) $\left(-2x^3\left(-x^2\right)\right)^4$

26. Consider the rule of exponents $(a^n)^m = a^{nm}$. Based on that, evaluate each of the given expressions.

a) $\sqrt{x^4}$ b) $\sqrt{x^{12}}$ c) $\sqrt{A^{100}}$

27. Suppose that $x = 2^{100}$. Express each of the given expression in terms of x.

a) 2^{101} b) $2^{103} - 2^{101}$ c) 2^{98} d) $3 \cdot 2^{102} + 5 \cdot 2^{101} - 2^{100}$ e) 8^{100} f) 2^{50}

28. Solve each of the given linear equations.

a) $5x - (x+4) = -8$

b) $-2(x-5) + 3x = 4x - 2$

c) $3(x-7) - 2(x-5) = x + 11$

d) $\frac{2}{3}(x-6) - \frac{3}{4}(x+8) = x - 10$

e) $x + \frac{4}{3} = \frac{5}{6}$

f) $x - \frac{2x-1}{3} = \frac{x+5}{2} - \frac{1}{6}$

g) $\frac{2x-3}{8} + \frac{x}{4} = \frac{10}{16}$

29. Solve each of the given linear inequalities. Present your answer using interval notation.

a) $\frac{3x-5}{-2} > 4$ b) $3(x-2) - 5(2x-1) \leq 4(3x-5)$ c) $\frac{x-6}{5} - \frac{2x+3}{3} \leq x + 11$

30. Solve each of the formulas for the indicated variable.

a) $A = \frac{(a+c)h}{2}$ for c b) $y = mx + b$ for x c) $V = \frac{1}{3}bh$ for h

31. Graph each of the following.

a) $y = 2x + 4$

b) $y = -\frac{2}{3}x + 1$

c) $x = -2$

d) $2x - 3y = -12$

e) $y = 3$

f) $x + 3y = -6$

32. a) Find the $x-$intercept of $x - 2y = 4$.

b) Find the $y-$intercept of $4x - 3y = 12$.

33. Find the slope of the straight line passing through the indicated points.

a) $(1, -2)$ and $(3, -4)$ b) $(-1, -3)$ and $(-4, 5)$.

34. Solve each of the given system of linear equations.

a) $\begin{cases} \frac{x}{3} + 6y = 4 \\ y = -5 - x \end{cases}$ b) $\begin{cases} 12x - 2y = 10 \\ y = 6x - 5 \end{cases}$ c) $\begin{cases} 2x + 3y = 11 \\ x - 4y = 0 \end{cases}$ d) $\begin{cases} 3x + 5y = 20 \\ 2x - 10y = 0 \end{cases}$

35. Expand or simplify each of the following.

a) $(2x-1)^2$ b) $(2x-1)^3$ c) $(x-3)^2 - (x-2)(3x+1)$ d) $(3y-2)(y-1) - (2y-1)^2$

36. Completely factor each of the following.

a) $p^2 - 4p - 32$ b) $3t^2 - 5t - 2$ c) $9x^2 - 25$ d) $2x^2 - 2x - 12$ e) $2x^4 - 32$

37. Completely factor $x^2 + x - 2$ and $x^2 - 4$. What is the factor they have in common?

38. Solve each of the given quadratic equations.

a) $5x + 2x^2 = 3$ b) $x^2 - 18 = -7x$ c) $6x^2 - 11x = 10$ d) $(x-5)(x+2) = x - 10$

39. Simplify each of the given expressions.

a) $\frac{2x-3}{3-2x}$

b) $\frac{x^2-36}{x^2-4x-12}$

c) $\frac{(x+1)^2}{x^2-1}$

d) $\frac{3ax - 6ay - bx + 2by}{3a - b}$

e) $\frac{x^2-2x-3}{x^2-1} \cdot \frac{x^2-x}{x^2+2x-15}$

f) $\frac{x}{x^2-2x} \div \frac{3x+6}{4x-8}$

40. 28 students in our class worked during the summer. 20 students in our class took a trip. Some students might have done both. 6 students did neither.

a) What is the smallest number possible for the size of our class?

b) What is the greatest number possible for the size of our class?

c) If 15 students traveled and worked during the summer, what is the size of the class?

41. There is an animal farm where chickens and cows live. All together, there are 73 heads and 204 legs. How many chickens and how many cows are there on the farm?

42. What is the area of the rectangle whose longer side is twice the length of the shorter side, and whose perimeter is 36 in?

43. If Jupiter is $483,800,000$ miles from the sun, what is its distance from the sun in inches? Round to the nearest hundredths and answer in scientific notation. (1 mile $= 5,280$ feet and 1 feet $= 12$ inches)

44. Most countries measure temperature in either Celsius or in Farenheit. The connection between these two measurements is $C = \frac{5}{9}(F - 32)$. Solve this formula for F.

45. You thought that dinner and a movie would cost \$35, but your estimate was \$17 less than the actual cost. If the movie was \$10, how much was dinner?

46. A coat is on a special sale at a 20% discount. If the sale price is \$96, what was the price of the coat before the discount?

47. A sweater originally cost \$65 and is now \$39. What is the percent change of the price of the shirt? Round your answers to the nearest percentage.

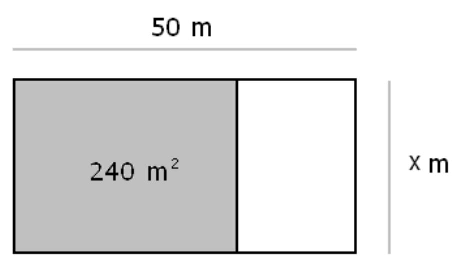
48. Find the $x-$coordinate of the point where the lines $x + 3y = -8$ and $4x - 3y = 23$ intersect.

49. We throw a small object upward from the top of a 1024 ft tall building. The height of the object, (measured in feet) t seconds after we threw it is

$$h = -16t^2 + 192t + 1024$$

a) Where is the object 2 seconds after we threw it?

b) How long does it take for the object to hit the ground?



50. A school purchases tickets to a show. A child ticket costs \$8 and an adult ticket costs \$14. The school has paid a total of \$610 for 65 tickets. How many of the 65 tickets were for adults?

51. The area of the shaded region of the rectangle shown on the picture is 240 square meters. Express the area of the unshaded region in terms of x.

52. One side of a rectangle is 2 inches shorter than three times another side. How long is the longer side in the rectangle if its area is 96 square inches?

53. Find an expression that describes the area in square meters of a rectangle that has width $4x^2y^2$ meters and length $3x^3y^3$ meters. Simplify your answer.

54. The formula for the volume of a rectangular box $V = lwh$; where l is the length, w is the width, and h is the height. If $V = 64$, $l = 8$, and $w = 4$, find the value of h.

55. Find the value of k so that the line connecting the points $(2,3)$ and $(5,k)$ has a slope of $\frac{1}{3}$.

56. Monico invests a total of \$12 500 in two accounts paying 4% and 3% annual interest, respectively. How much was invested in each account if, after one year, the total interest was \$455?

Appendix A

Answers

13.1 – Interval Notation

Practice Problems

1. a) $[3,\infty)$ b) $(-2,9]$ c) $(-\infty,\infty)$ d) $(-\infty,-5)\cup(4,\infty)$ e) $(-\infty,12)$ f) $[6,10)$ g) $(-\infty,-2)\cup(2,\infty)$ h) $(0,7)$ i) $[3,4]$

2. a) $(1,7)$ b) $(2,5)$ c) $[1,7]$ d) $[2,5]$ e) $[1,7)$ f) $(2,5]$ g) $(-1,10)$ h) $(3,8)$ i) $[-1,10]$ j) $[3,8]$ k) $(-1,10)$ l) $[3,8]$ m) $[-2,2]\cup(4,7)$ n) $\varnothing$ o) $[-2,8]$ p) $(0,1)$ q) $[5,11)$ r) $[7,10)$

3. a) $(-\infty,8)$ b) $(-\infty,4)$ c) $(-\infty,8]$ d) $(-\infty,4]$ e) $(-\infty,8)$ f) $(-\infty,4]$ g) $\mathbb{R}=(-\infty,\infty)$ h) $(3,5)$ i) $\mathbb{R}=(-\infty,\infty)$ j) $[3,5]$ k) $\mathbb{R}=(-\infty,\infty)$ l) $(3,5]$ m) $(-\infty,-2)\cup(1,\infty)$ n) $\varnothing$ o) $(-\infty,-2]\cup[1,\infty)$ p) $\varnothing$ q) $(-\infty,-2]\cup(1,\infty)$ r) $\varnothing$

13.2 – Graph of an Equation

Practice Problems

1. Consider $y+2=|2x|$

$A(-3,4)$ is on the graph, $6=6$ ✓
$B(0,-2)$ is on the graph, $0=0$ ✓
$C(7,0)$ is not on the graph, $2\neq 14$
Therefore, $y+2=|2x|$ is not the equation of the graph.

Consider $6-y=|8-|2x||$

$A(-3,4)$ is on the graph, $2=2$ ✓
$B(0,-2)$ is on the graph, $8=8$ ✓
$C(7,0)$ is not on the graph, $6=6$ ✓
Therefore, $6-y=|8-|2x||$ is the equation of the graph.

Consider $x^2+y^2=1+4(x+y+5)$

$A(-3,4)$ is on the graph, $25=25$ ✓
$B(0,-2)$ is not on the graph, $4\neq 13$
$C(7,0)$ is on the graph, $49=49$ ✓
Therefore, $x^2+y^2=1+4(x+y+5)$ is not the equation of the graph.

2. Consider $2y = x + 9$
$A(-3,3)$ is on the graph, $6 = 6$ ✓
$B(1,5)$ is on the graph, $10 = 10$ ✓
$C(4,-2)$ is not on the graph, $-4 \neq 13$
Therefore, $2y = x + 9$ is not the equation of the graph.

Consider $|x| + |y| = 6$
$A(-3,3)$ is on the graph, $6 = 6$ ✓
$B(1,5)$ is on the graph, $6 = 6$ ✓
$C(4,-2)$ is on the graph, $6 = 6$ ✓
Therefore, $|x| + |y| = 6$ is probably the equation of the graph.

Consider $y + 3 = 9 - |x|$
$A(-3,3)$ is on the graph, $6 = 6$ ✓
$B(1,5)$ is on the graph, $8 = 8$ ✓
$C(4,-2)$ is not on the graph, $1 \neq 5$ ✓
Therefore, $y + 3 = 9 - |x|$
is not the equation of the graph.

13.3 – The Zero Product Rule

Practice Problems

1. $-2, 5$ 2. $-1, 0, 3$ 3. $-7, 0, 10$ 4. $-2, 4$ 5. $-6, -1, 0, 1$ 6. $-8, -\frac{3}{7}, 0, \frac{1}{2}$

7. $(x-3)(x+6) = 0$ 8. $x(x-8)(x+4) = 0$

9. yes, for example $x^7 = 0$ has only one solution: $x = 0$

Problem Set 13

1. a) $(-\infty, 5)$ b) $(-\infty, 10]$ c) $[4, 8)$ d) $(3, 11]$ e) $(-\infty, \infty)$ f) $[2, 9)$ g) $\varnothing$ h) $(-\infty, 1] \cup (3, \infty)$

2. a) 16 b) -16 c) a^7 d) a^{12} e) $8a^5$ f) $\frac{1}{2a}$ g) $\frac{4}{a}$ h) $-8x^4$ i) $-\frac{x^2}{3}$ j) 2^{60} k) 2^{31}

3. $D < A < B < C$ 4. a) 48 b) $2x^5y$ c) x^6

5. a) $-5x^2 + 42x - 16$ b) $9x^2 - 6x + 1$ c) $27x^3 - 27x^2 + 9x - 1$ d) $8x^4$ e) $x^2 - 12x + 28$
f) $x^6 - 6x^4 + 3x^2 + 10$ g) $9x^{10} - 4$

6. a) -7 b) $\frac{1}{2}$ c) -11 d) $\frac{2}{3}$ e) -13 f) $-\frac{18}{5}$ g) -3 h) 1 i) $-\frac{3}{5}$ j) -1 k) no solution l) -17

7. a) $(-\infty, 0]$ b) $(6, \infty)$ c) $(-\infty, 4)$ d) $\left[\frac{1}{2}, \infty\right)$

8. a) $D(-3, 6)$ b) $M(1, 4)$ 9. 4 units 10. a) 34 b) 54 c) 39

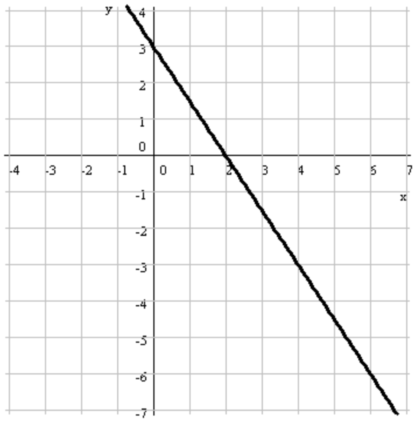
11. a) \$9360 b) 17% increase 12. 50 13. 7 bagels and 25 rolls

14. a) 203 b) 0 15. division by zero 16. 16

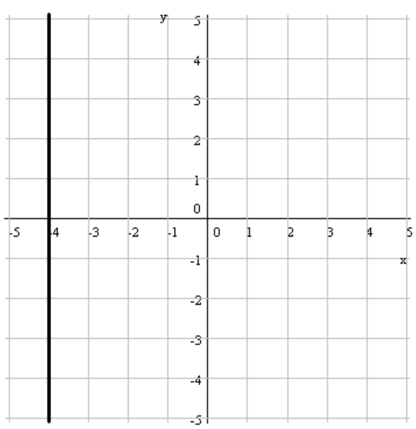
14.1 – Graphing a Line

Practice Problems

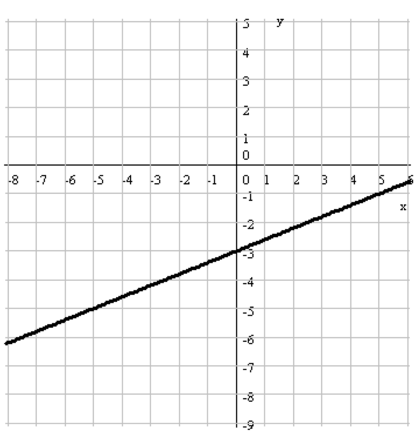
1. $3x+2y=6$



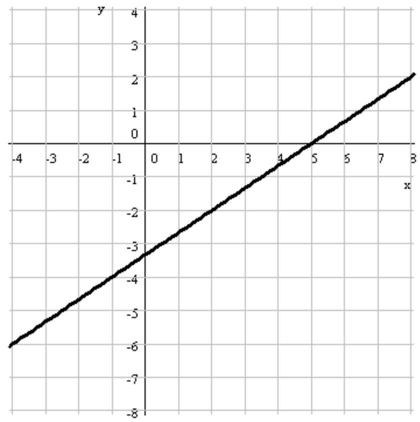
2. $x=-4$



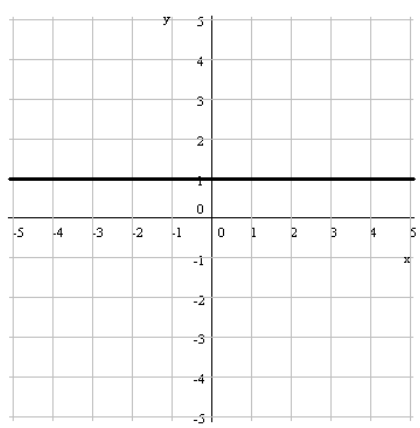
3. $y=\frac{2}{5}x-3$



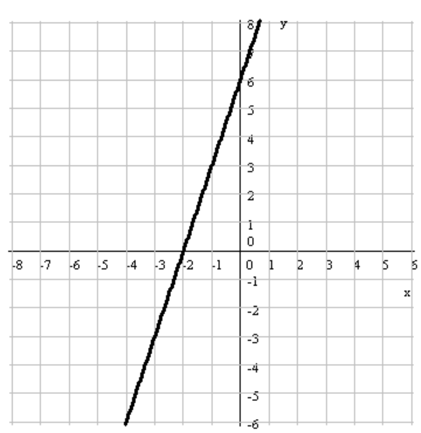
4. $2x-3y=10$



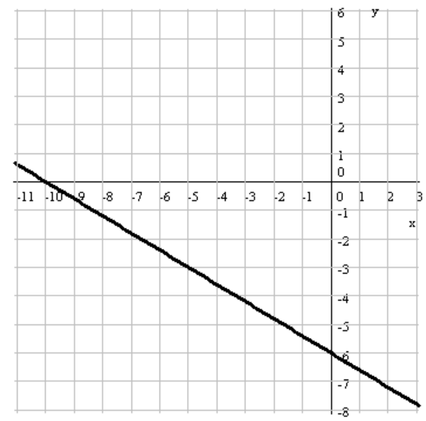
5. $y=1$



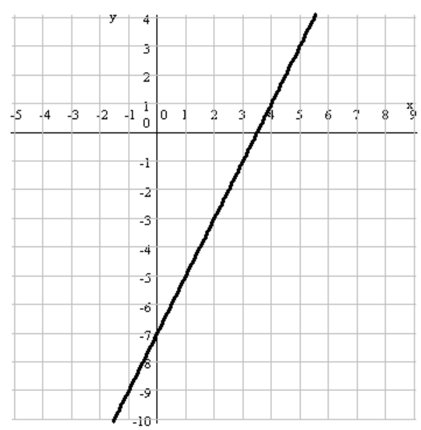
6. $y=3x+6$



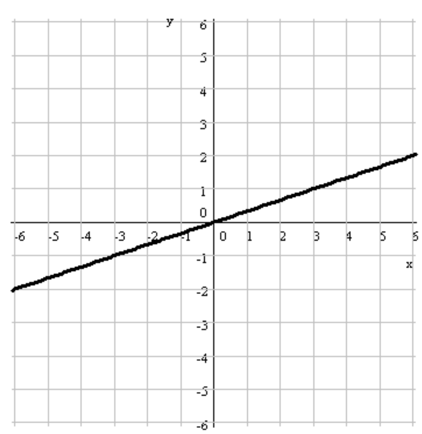
7. $3x+5y=-30$



8. $2x-y=7$



9. $y=\frac{1}{3}x$



14.2 – Factoring out the GCF and -1

Sample Problems

1. a) $3(x-4)$ b) $4a^2b(5a-3b+4)$ c) $3(a^2-4)$ d) $3a^2(a-4)$ e) $5x(x^2+4)$ f) $(x-2)(8x^3+3x-11)$

2. $-(5x^3-2x^2+x+8)$ 3. a) $2, -3$, and $-\frac{1}{2}$ b) 0 and -7 c) 0 and 9 d) 0 and $\frac{25}{4}$ 4. $0, 1$

Practice Problems

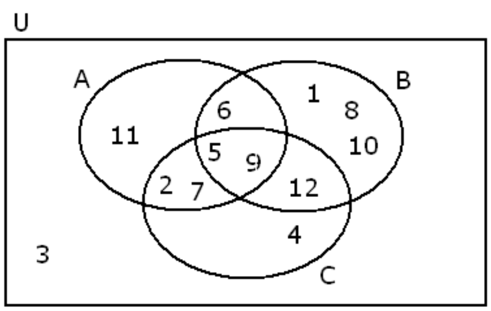
1. a) $5ab^2(2a-3b+5abc)$ b) $3x^2(2x-5x^2-1)$ c) $a^2(a^2-a+1)$ d) $6a^2b(2a-5ab+1)$
   e) $x^3(x^2-2x+4)$ f) $(a-3)(3xy+8t-200x^5)$

2. a) $-(-x^3+x^5-2)$ b) $-(x^2-3x+1)$ c) $-(x^2-3x+5)$

3. a) $-5, 1$ b) $0, 2, -3$ c) $2, -3$ d) $0, 4$ e) $0, -6$ f) $0, 25$ 4. $0, -1$

Problem Set 14

1. a) see below b) i) $\{2,5,6,7,9\}$ ii) $\{2,5,6,7,9\}$ iii) $\{2,5,6,7,9,11,12\}$ iv) $\{5,9\}$



2. a) $[-2,5)$ b) $(-\infty,\infty)$ c) $[3,12)$ d) $(7,10]$ e) $\varnothing$
   f) cannot be simplified g) $(2,\infty)$ h) $[5,\infty]$

3. a) true b) true c) true d) true e) false

4. a) $1,2,3,5,6,10,15,25,30,50,75,150$ b) $53,59,61,67$
   c) $119 = 7\cdot 17$

5. a) i) $180 = 2^2\cdot 3^2\cdot 5$ ii) $1575 = 3^2\cdot 5^2\cdot 7$ iii) $80^{100} = 2^{400}\cdot 5^{100}$ b) 45 and 6300
   c) Only 90 is possible d) Yes, but only if the two numbers are equal.

6. a) $-\frac{1}{4}$ b) -9 c) 4 d) -1 e) 8 f) 3 7. a) $1.024\cdot 10^{15}$ b) $4\cdot 10^{13}$ c) $2.56\cdot 10^{1}$

8. a) 3 b) 1 c) $\frac{8}{3}$ d) undefined e) $\frac{7}{2}$ f) 0 g) $\frac{3}{2}$ 9. a) $9\cdot 2^{100}$ b) $2\cdot 3^{100}$ c) 3^{101}

10. a) $64a^8$ b) $8x^{12}$ c) $-4x^5y^{11}$ d) $-\frac{3}{8}ab^2$ e) $-2x^4$ f) $\frac{9}{4}a^6b^{12}$ g) $5x^2-13x-6$ h) $25a^2-10a+1$
    i) a^4-b^4 j) $9a^{10}-1$ k) $8x^3-36x^2+54x-27$

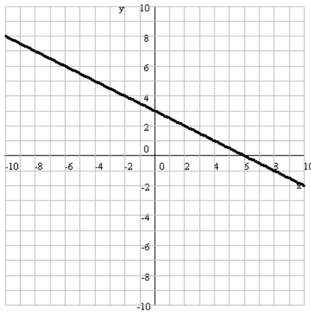
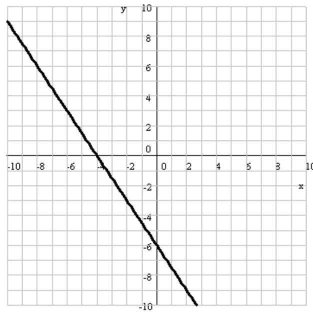
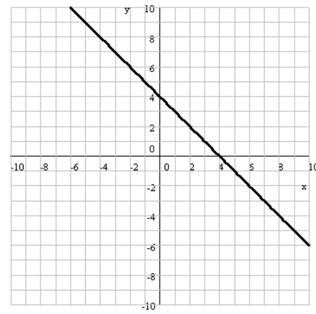
11. a) $-(2x^4-x^3-5x+1)$ b) $-(5y+1)$ c) $-(3m-5)$ d) $b-a$

12. a) $5ab^3(2a-bc+1)$ b) $2x^3(x^2-6x+3)$ c) $2abc(3a-b^2+6a^3b^2c)$ d) $pq(4p+9q-6)$
    e) $x^2(x^2-5x+1)$ f) $3(a-3)(x-2x^2+4)$ g) $(5x-2)(8x-1)$

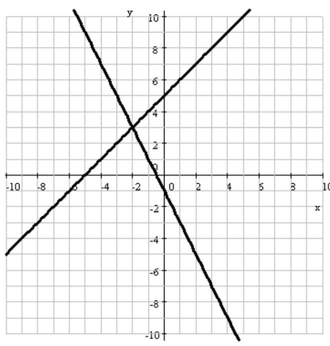
13. a) $2^{99}\cdot 3^{101}$ b) $2^{100}\cdot 5^{100}$ c) 2^{200}

14. a) -1 b) no solution c) $0,-1$ d) -13 e) 0 f) $0,-5$ g) $0,\frac{1}{9}$
h) identity, all numbers are solution i) 7 j) $0,3$ k) 12

15. a) $y=-\frac{1}{2}x+3$ b) $3x+2y=-12$ c) $x+y=4$

16. a) $(-2,3)$



b) Is the point $(-2,3)$ on the line $2x+y=-1$?

$$\begin{aligned}\text{LHS} &= 2x+y=2(-2)+3=-4+3=-1 \quad \text{and} \quad \text{RHS}=-1\\ \text{LHS} &= \text{RHS} \checkmark\end{aligned}$$

Thus $(-2,3)$ is on the line $2x+y=-1$.
Is the point $(-2,3)$ on the line $x-y=-5$?

$$\begin{aligned}\text{LHS} &= x-y=-2-3=-5 \quad \text{and} \quad \text{RHS}=-5\\ \text{RHS} &= \text{LHS} \checkmark\end{aligned}$$

Thus $(-2,3)$ is on the line $x-y=-5$. Since it is on both line, it is the intersection point.

17. \$400 18. 15 and 22 19. \$7750 in stocks and \$12 250 in bonds 20. 7 adult tickets and 25 children tickets

22. \$420 and \$530 23. 9 ft and 23 ft 24. $0,9$ 25. -3 26. -4

27. 48 unit^2 28. a) $A=17\,\text{m}^2$ b) 10 ft 29. a) 10 ft b) 28 m

15.1 – The Difference of Squares Theorem

Practice Problems

1. a) $5ab^2(2a-3b+5abc)$ b) $3x^2\left(2x-5x^2-1\right)$ c) $a^2\left(a^2-a+1\right)$ d) $6a^2b(2a-5ab+1)$
e) $x^3\left(x^2-2x+4\right)$ f) $(a-3)\left(3xy+8t-200x^5\right)$

2. a) $-\left(-x^3+x^5-2\right)$ b) $-\left(x^2-3x+1\right)$ c) $-\left(x^2-3x+5\right)$

3. a) $(x+7)(x-7)$ b) $(3a+5)(3a-5)$ c) $(x+1)(x-1)$ d) $\left(y^3+10\right)\left(y^3-10\right)$

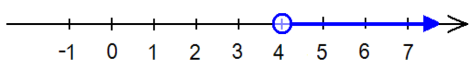
4. a) $5(a+3)(a-3)$ b) $-2(n-m)(m+n)(m^2+n^2)$ c) $2x^2(x+2)(x-2)$ d) $-3a(2b+1)(2b-1)$
   e) $x(x+1)(x-1)$ f) $5x^3(y-2)(y+2)(y^2+4)$ g) $(a-3)(a+3)(x-1)$ h) $2x^2(3a-5)(3a+5)$
   i) $(a+x-1)(a-x+1)$ j) $(a-2)(a+2)(a^2+4)$ k) $-6ab^2(b-10)(b+10)$ l) $4x^2y^3(x^2+9)$
   m) $-2(x^2+9)(x+3)(x-3)$ n) $5ab(a^2b-3)$

5. a) $-5, 1$ b) $0, 2, -3$ c) $2, -3$ d) $2, -2$ e) $0, -6$ f) $0, 25$ g) $-5, 0, 5$ h) $-\frac{2}{3}, 0, \frac{2}{3}$

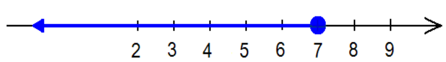
15.2 – Linear Inequalities

Sample Problems

1. set-builder notation: $\{x|x > 2\}$
   interval notation: $(2, \infty)$
   graph:

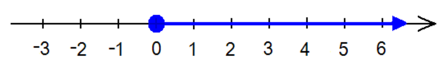


2. set-builder notation: $\{x|x \leq 7\}$
   interval notation: $(-\infty, 7]$
   graph:

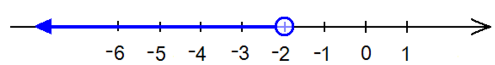


3. set-builder notation: $\{x|x \geq 0\}$

4. interval notation: $[0, \infty)$
   graph:



5. set-builder notation: $\{x|x < -2\}$
   interval notation: $(-\infty, -2)$
   graph:

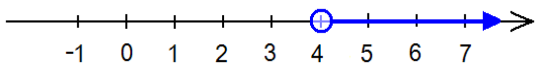


Practice Problems

1. set-builder notation: $\{x|x > 4\}$

   interval notation: $(4, \infty)$

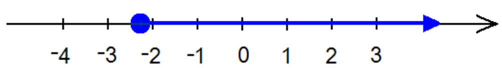
   graph:



2. set-builder notation: $\left\{x|x \geq -\frac{7}{3}\right\}$

   interval notation: $\left[-\frac{7}{3}, \infty\right)$

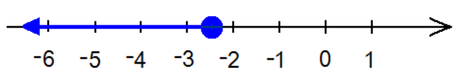
   graph:



3. set-builder notation: $\left\{x|x < -\frac{5}{2}\right\}$

   interval notation: $\left(-\infty, -\frac{5}{2}\right)$

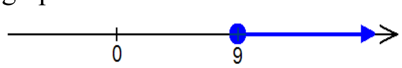
   graph:



4. set-builder notation: $\{x|x \geq 9\}$

   interval notation: $[9, \infty)$

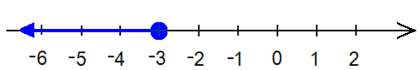
   graph:



5. set-builder notation: $\{x|x \leq -3\}$

   interval notation: $(-\infty, -3]$

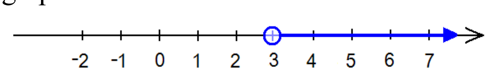
   graph:



6. set-builder notation: $\{x|x > 3\}$

   interval notation: $(3, \infty)$

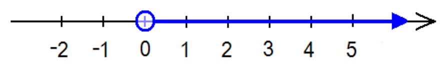
   graph:



7. set-builder notation: $\{x|x > 0\}$

   interval notation: $(0, \infty)$

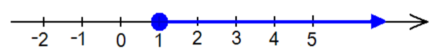
   graph:



8. set-builder notation: $\{x|x \geq 1\}$

   interval notation: $[1, \infty)$

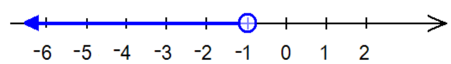
   graph:



9. set-builder notation: $\{x|x < -1\}$

   interval notation: $(-\infty, -1)$

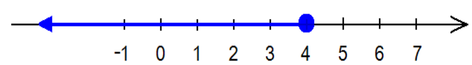
   graph:



10. set-builder notation: $\{x|x \leq 4\}$

    interval notation: $(-\infty, 4]$

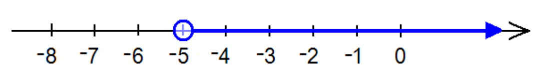
    graph:



11. set-builder notation: $\{x|x > -5\}$

    interval notation: $(-5, \infty)$

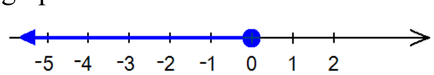
    graph:



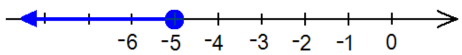
12. set-builder notation: $\{x|x \leq 0\}$

    interval notation: $(-\infty, 0]$

    graph:

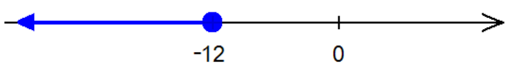


13. set-builder notation: $\{x|x \le -5\}$
interval notation: $(-\infty, -5]$
graph:

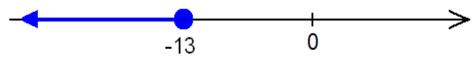


14. set-builder notation: $\{x|x > -8\}$
interval notation: $(-8, \infty)$
graph:

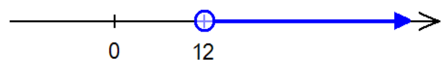
15. set-builder notation: $\{x|x \le -12\}$
interval notation: $(-\infty, -12]$
graph:



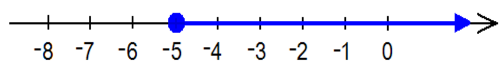
16. set-builder notation: $\{x|x \le -13\}$
interval notation: $(-\infty, -13]$
graph:



17. set-builder notation: $\{x|x > 12\}$
interval notation: $(12, \infty)$
graph:



18. set-builder notation: $\{x|x \ge -5\}$
interval notation: $[-5, \infty)$
graph:



Problem Set 15

1. a) $\frac{2}{7}$ b) 20   2. a) 4 b) undefined c) 0

3. a) $-4a^3b^{11}$ b) $4a^4b^{16}$ c) $8a^7b^8$ d) $4x+8$ e) 6^{400} or $2^{400}\cdot 3^{400}$ f) 54

4. a) $2(x+3)(x-3)$ b) $3x^2(2a-5)(2a+5)$ c) can not be factored d) $5a^3(a^2+1)(a+1)(a-1)$
   e) $-x(x-4)(x+4)$ f) $(x^8+5)(x^8-5)$

5. a) $8(2x+3)$ b) $8x(2x-1)(x^2+1)$ c) $4(a-2c)(2b-c)$

6. a) $0, 16$ b) $4, -4$ c) $2, -2$

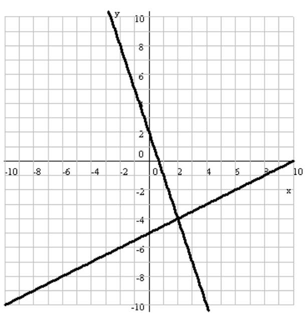
7. a) $0, 4$ b) $-2, 0, 2$ c) 7 d) 0 e) all numbers are solution f) $-\frac{3}{2}$ g) $-4, 0$
   h) -8 i) $-1, 0, 5$ j) $-1, 1$

8. a) $(-8,\infty)$ b) $[-3,\infty)$ c) $(-8,\infty)$ d) $[-3,\infty)$ 9. a) 61 miles b) \$12.65   10. 4

11. -3 or 3   12. Andy pays \$24 and Brett pays \$31

13. a) 4% decrease b) no change c) 10% decrease d) 12% decrease e) 87.5% decrease

14. a) $(2,-4)$



b) Is the point $(2,-4)$ on the line $3x=-y+2$?

$$\begin{aligned} LHS &= 3\cdot 2 = 6 \\ RHS &= -(-4)+2 = 6 \\ LHS &= RHS\ \checkmark \end{aligned}$$

Thus $(2,-4)$ is on the line $3x=-y+2$.

Is the point $(2,-4)$ on the line $y=\frac{1}{2}x-5$?

$$\begin{aligned} LHS &= -4 \\ RHS &= \frac{1}{2}\cdot 2-5 = -4 \\ RHS &= LHS\ \checkmark \end{aligned}$$

Thus $(2,-4)$ is on the line $y=\frac{1}{2}x-5$.

Since $(2,-4)$ is on both lines, it must be the intersection point.

c) 28 unit^2

15. a) $95F$ b) $50C$ c) $C=\frac{5}{9}(F-32)$ d) yes, -40 16. 200

17. a), b), c) the area will decrease by 9 unit^2 because $(s+3)(s-3)=s^2-9$

18. a) 4039 b) 1800 c) 34000 d) 1000 e) 1 f) -9 g) 1

16.1 – Integer Exponents

Practice Problems

1. $\frac{1}{9}$ 2. 8 3. $\frac{1}{m^4}$ 4. x^5 5. a^7 6. p^4 7. x^5 8. $5a^{15}$ 9. $\frac{1}{t^7}$ 10. 1

11. -1 12. 1 13. $\frac{1}{b^4}$ 14. b^4 15. m^3 16. $\frac{x^3z^4}{y^5}$ 17. $3q^6$ 18. $\frac{27}{8}$ 19. $\frac{2}{y^3}$

20. $\frac{1}{8y^3}$ 21. $\frac{25}{9}$ 22. $\frac{a^5}{b^8}$ 23. $\frac{1}{9m^6}$ 24. $-\frac{b^9}{8a^3}$ 25. k 26. $\frac{9}{4}a^3b$ 27. $-\frac{a^5}{8b^4}$

28. $18p^9q^{10}$ 29. $\frac{4a^{10}}{b^{12}}$ 30. $\frac{x^4}{y^6}$ 31. 1 32. $\frac{xy}{y-x}$ 33. $-\frac{b^8}{2}$ 34. $\frac{1}{b^8}$ 35. $\frac{2x^4}{y^3}$

36. a) $6.375\cdot 10^{-4}$ b) $6.1413\cdot 10^{-34}$ c) $4.7813\cdot 10^{4}$ d) $1.1333\cdot 10^{-19}$ e) $1.6903\cdot 10^{63}$

16.2 – Solving Linear Systems by Elimination

Practice Problems

1. a) $x=-5, y=-2$ b) $p=1, q=-7$ c) $x=3, y=-1$ d) $x=-6, y=8$
   e) $x=\frac{3}{2}, y=\frac{1}{2}$ f) $a=12, b=-7$ g) $x=\frac{16}{13}, y=\frac{11}{13}$ h) $x=5, y=0$
   i) $x=-0.5, y=0.8$

2. a) $(-3,7)$ b) $(3,0)$ 3. 38 chickens, 22 cows 4. \$3500 at 7% and \$2500 at 11%

5. 38 dimes and 13 quarters 6. \$2000 at 9% and \$5600 at 13%

Problem Set 16

1. a) -11 b) $\frac{13}{80}$ c) $\frac{5}{63}$ d) $\frac{13}{18}$

2. a) $\{8,10\}$ b) $\{2,4,6,8,9,10\}$ c) $\varnothing$ d) $\{1,2,8,9,10\}$ e) $\{7,8,9,10\}$ f) $\{4,5,6,7,8,9,10\}$

3. a) $2^2\cdot 5^4$ b) $2^{150}\cdot 3^{100}$ 4. 24 and 2880

5. a) false b) true c) true d) true e) false f) true g) false h) true

6. a) 8^5 b) 10^{100} c) 6^{20}

7. a) $\frac{1}{x}$ b) $-\frac{1}{x}$ c) $\frac{1}{x}$ d) $\frac{1}{x^{42}}$ e) $\frac{y^9}{8x^3}$ f) $\frac{1}{x^{19}}$ g) x^6 h) x^4 i) $\frac{25b^2}{a^4}$ j) $-\frac{125b^3}{a^6}$ k) $\frac{-2x^{13}}{y^8}$ l) $-\frac{a^2}{b^4}$

8. a) $2.5\cdot 10^9$ b) $4\cdot 10^{-6}$ c) $4\cdot 10^{-2}$ d) $4\cdot 10^2$ e) $2.5\cdot 10^5$

9. a) $2x^2+3x+2$ b) $-7x+4$ c) $-10x^2-36x+2$ d) $x^4+3x^3-8x^2+17x-3$ e) $8x^3-12x^2+6x-1$

10. a) a^2-b^2 b) a^3-b^3 c) a^4-b^4

11. a) $2ab(3a+5b)(3a-5b)$ b) $(x-2)(x+1)(x-1)$ c) $(x-1)\left(x^2+9\right)$ d) $-10x(x-10)$ e) $x(x+1)$
f) $3x(2x+1)(2x-1)(y+1)(y-1)$ g) $3p^2qy(y+4)(y-4)$ h) $-4(5x+1)(3x-4)$ i) $8b(a-3c)$
j) $\left(x^2+1\right)(x+1)(x-1)$

12. a) 2 b) $0,-6$ c) $0,-1,-5,\frac{7}{3}$ d) $0,2$ e) 0 f) There is no solution. g) All numbers are solution.
h) $0,36$ i) $-6,0,6$

13. a) $[5,\infty)$ b) $(-\infty,0]$ c) $\left(\frac{1}{6},\infty\right)$ 14. a) $(-8,3)$ b) $(4,7)$ c) $(-3,12)$ d) $(0,5)$

15. $(x+2)(x-6)=0$ (answers may vary) 16. 15 17. a) $(7,3)$ b) $\left(1,\frac{7}{2}\right)$ 18. 83 chickens and 37 cows

19. 6 ft by 20 ft 20. 49 nickels and 23 dimes 21. 2 units 22. \$3800 at 4% and 6200 at 3%

23. $0,-8$ 24. 45 chidren tickets and 10 adult tickets 25. 36 children and 4 adult tickets

17.1 – Solving Linear Systems by Substitution

Practice Problems

1. a) $x=-2, y=2$ b) $p=3, q=-5$ c) $a=-2, b=4$ d) $x=12, y=-4$ e) $x=-1, y=-3$
f) $a=1, b=-6$ g) $x=\frac{18}{19}, y=\frac{7}{19}$ h) $x=0, y=4$ i) $r=-0.5, s=1.4$
2. a) $(-3,7)$ b) $(-4,y-1)$ c) $(8,4)$ d) $(-6,5)$ e) $(24,-9)$ 3. 37 chickens, 15 cows
4. \$3200 at 7% and \$6500 at 12% 5. 23 dimes and 31 quarters 6. \$3100 at 9% and \$4700 at 10%

17.2 – Factoring by Trial and Error

Practice Problems

1. a) $(x-3)(x+5)$ b) $(x-4)(x-8)$ c) $(x+1)(x-3)$ d) $(x-5)^2$ e) $(x+4)(x+5)$
f) $(x+4)(x-5)$ g) $(x-2)(x-3)$ h) $(x-6)(x+1)$ i) $2(x-7)(x+3)$ j) $-3(x+2)(x-1)$
2. a) $-5,1$ b) $6,-10$ c) $7,-8$ d) $-3,0,5$ e) 1 f) $6,-2$ 3. a) $5,-4$ b) $-10,7$
4. a) 14 ft by 30 ft b) 8 ft by 36 ft

18.1 – Linear Equations 3

Sample Problems

1. -3 2. -5 3. $\frac{13}{5}$ 4. $\frac{25}{4}$ 5. 6 6. no solution 7. -5

8. identity, all numbers are solution 9. -11 10. -41 11. 18 12. 2 13. 0

14. 3 15. 11 units

Practice Problems

1. 3 2. 4 3. 2 4. -15 5. 4 6. 2 7. 22 8. 0 9. 1 10. 4 11. 3 12. 0

13. 9 14. -5 15. there is no solution 16. -8 17. -13 18. all real numbers are solution 19. -4

20. -5 21. $\frac{1}{2}$ 22. 6 23. 2 24. -5 25. 0 26. $\frac{1}{2}$ 27. 2 28. 0 29. 1 30. 3 31. 4

32. -3 33. there is no solution 34. -5 35. all real numbers are solution 36. 0 37. 4 38. 9 units

18.2 – Square Roots of Integers

Practice Problems

1. a) 10 b) undefined c) -10 d) -7 e) undefined d) 1 e) 0

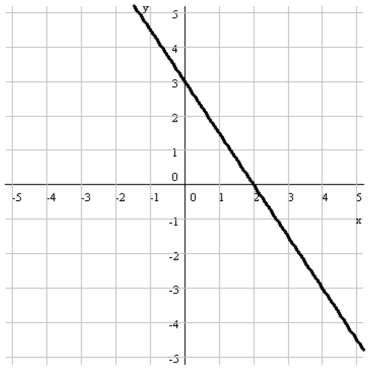
2. a) 4 b) 3 c) 2 d) 2 3. a) -7 b) 4 c) 2 4. a) 2 b) 4

19.1 – Slope of a Line

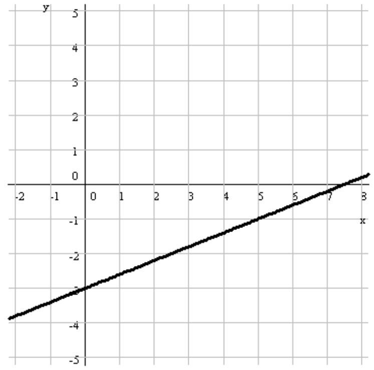
Practice Problems

1. a) -2 b) $\frac{3}{4}$ c) undefined d) 0

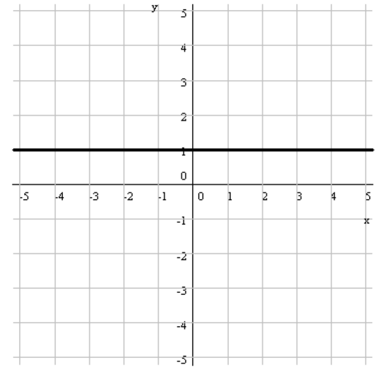
2. a) $3x+2y=6$



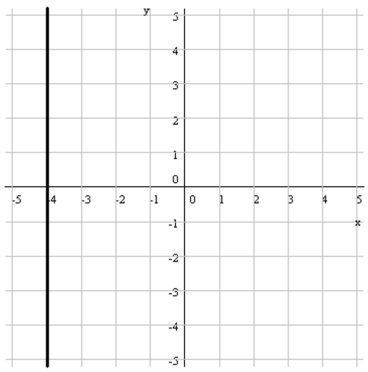
c) $y=\frac{2}{5}x-3$



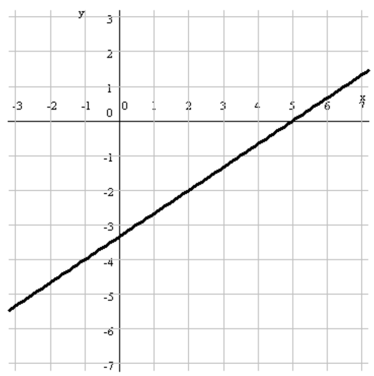
e) $y=1$



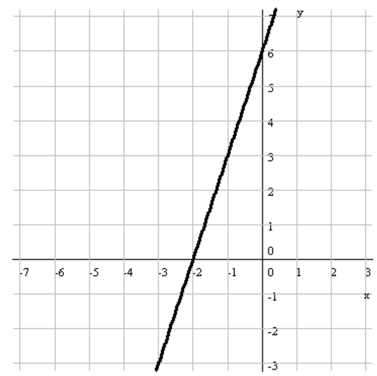
b) $x=-4$



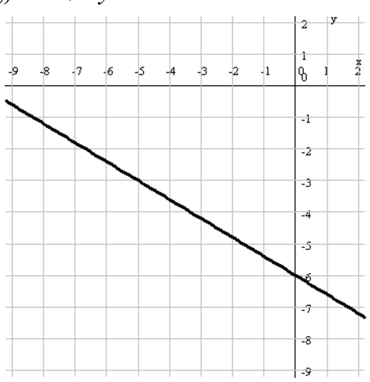
d) $2x-3y=10$



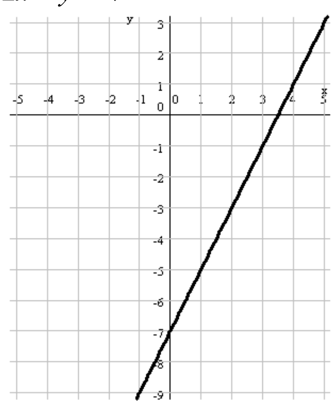
f) $y=3x+6$



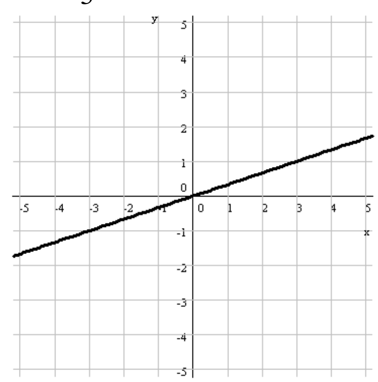
g) $3x+5y=-30$



h) $2x-y=7$



i) $y=\frac{1}{3}x$



3. a) $m=\frac{2}{3}$ b) $m=-\frac{3}{5}$ c) $m=1$ d) $m=-2$ e) $m=\frac{1}{2}$ f) $m=0$

20.1 – Factoring by Grouping

Practice Problems

1. $(x-3)(2a+5)$ 2. $2(a-2)(x^2+1)$ 3. $5x^2(a-b)(x+y)$ 4. $3a(x-1)(y-1)$

5. $2(x+3)(x-3)(5n-m)$ 6. $(x+y)(x-y)(p+q)(p-q)$ 7. $a^2b(x+2)(y-3t)$

8. $b(a+5)(a-5)(m^3+2)$ 9. $(3x^4+1)(2x-5)$ 10. $-2ax(3x^2-1)(2x^3+1)(2x^3-1)$

11. $(x-2)(x+2)^2$ 12. $(3x+8)(2x-1)$ 13. $(2x+5)(x-3)$ 14. $(x-2)(x-4)$

15. $(3x+2)(2x-1)$ 16. $(5x-2)(2x-5)$

20.2 – Rational Expressions

Sample Problems

1. a) -1 b) x^2-x c) $\dfrac{1}{2x-1}$ d) $\dfrac{x-1}{x+5}$ e) $\dfrac{1}{4}$

2. a) $\dfrac{a}{b}$ b) 3 c) $\dfrac{x-15}{x-5}$ d) $\dfrac{3}{4}$ e) $\dfrac{x+2}{x-6}$ f) $\dfrac{-2px^2}{y-6}$

Practice Problems

1. a) $-\dfrac{1}{2}$ b) $x-1$ c) $\dfrac{2t+3}{2}$ d) $\dfrac{p}{p+1}$ e) $\dfrac{m-2}{m+6}$ f) $\dfrac{x+6}{x-8}$ g) $-\dfrac{5}{4}$

2. a) $\dfrac{x^2}{2}$ b) $-a^2(a-4)$ c) 1 d) $\dfrac{x-5}{x+5}$ e) $\dfrac{3}{y-1}$ f) $4x-4$

Discussion: If $a=3$, then the expression is undefined. For all other numbers, the expression is -1.

21.1 – More on Linear Systems

Practice Problems

1. a) There is no solution. b) All points $\left(x, \dfrac{3}{2}x+\dfrac{5}{2}\right)$ are solution

   c) All points $\left(x, 3x-\dfrac{3}{2}\right)$ are solution d) There is no solution.

2. a) $A=-24$ b) A can be any number except for -24

21.2 – The Pythagorean Theorem

Practice Problems

1. a) There is a right angle opposite the 10 cm long side. b) not a right triangle

2. 17 m 3. 35 inches 4. 13 units 5. 20 units 6. 32 cm and 60 cm 7. 24 m

Sample Problems

1. a) not a right triangle b) There is a right angle opposite the 37 ft long side. c) not a right triangle

2. 13 mi 3. 16 cm 4. 24 m 5. a) 10 units b) 25 units 6. 9 cm, 40 cm, and 41 cm

7. 14 in, 48 in, and 50 in 8. 16, 30, and 34 units 9. a) 20 b) 7 c) 10

Factoring by The AC-Method

Practice Problems

1. a) $3(a+5)(2x-y)$ b) $(x-1)(y-1)$ c) $2a^2(b+c)(3b-5c)$ d) $(a-b)(a+b)(m+2n)$
   e) $(x-2y)(x+2y)\left(m^2+1\right)$ f) $(b-1)(b+1)(a+1)$ g) $-2\left(3n^2-m\right)\left(3n^2+m\right)\left(p^2+1\right)$
   h) $(x-y)(x+y)\left(a^2+b^2\right)$ i) $(3x+1)(x-1)$ j) $(3x-1)(2x-1)$
   k) $-2(2x+1)(3x-5)$ l) $(5m-2n)(m-n)$ m) $(5x-3p)(2x+7p)$
   n) $(x-y)(x+y)\left(2x^2+3y^2\right)$
2. a) $-\frac{1}{2}, 0, \frac{2}{3}$ b) $\frac{1}{5}, 5$ c) $-\frac{3}{5}, \frac{2}{7}$
3. 11 in by 29 in

Factoring the Difference and Sum of Cubes

Sample Problems

1. $(x-2y)\left(x^2+2xy+4y^2\right)$ 2. $-\left(3a^4-5\right)\left(9a^8+15a^4+25\right)$ 3. $\left(x^2+10\right)\left(x^4-10x^2+100\right)$

4. $(x-2)\left(x^2+5x+13\right)$ 5. $-2a^4\left(a+b^3\right)\left(a^2-ab^3+b^6\right)$ 6. $2a\left(a^2+12\right)$

7. $(a+b)(a-b)\left(a^2+ab+b^2\right)\left(a^2-ab+b^2\right)$

Practice Problems

1. $8(x+5)\left(x^2-5x+25\right)$ 2. $2(q+5)\left(q^2+10q+100\right)$ 3. $(m-y-1)\left(m^2+my+m+y^2+2y+1\right)$

4. $a^2b\left(a-x^2\right)\left(a^2+ax^2+x^4\right)$ 5. $2(a-b)\left(a^2+ab+b^2\right)(2m+n)$

6. $-3(x+y)\left(x^2-xy+y^2\right)(2m-n)(2m+n)$ 7. $2a^2(a-1)\left(a^2+a+1\right)\left(b^2+1\right)$

8. $4(a-1)\left(7a^2-2a+7\right)$ 9. $2(a+1)\left(13a^2-22a+13\right)$ 10. $-2(x-1)(x+1)\left(x+x^2+1\right)\left(x^2-x+1\right)$

Fractions and Decimals

Practice Problems

1. a) i) 0.8 ii) $8.\overline{6}$ iii) 1.04 iv) $3.\overline{714285}$, the repeating block is 6 digits long
   b) i) $\frac{218}{100}$ ii) 3 iii) $\frac{641}{99}$ iv) $\frac{18687}{9990}$

2. The decimal presentation of real numbers is not unique. 3. n can only have 2 and 5 in its prime-factorization

4. It is repeating, only the repeating block is 6 digits long, $0.\overline{153\,846}$ 5. It is impossible

Problem Set 25

1. a) true b) false c) false d) false 2. a) $\{4\}$ b) $\{1,2,4,6,7,9\}$ c) $\{1,2,3,4,5,6,7,8,9,10\}$ d) $\{5,6,7\}$

3. a) true b) false c) true d) true e) false f) true g) true h) false i) true

4. a) true b) false c) false d) true e) false 5. 4 6. 118 R 14 7. $1,2,3,4,6,7,12,14,21,28,42,84$

8. a) $11\,060\,904$, 3106 b) $11\,060\,904$, 321 c) $11\,060\,904$ 9. 151 10. $720 = 2^4 \cdot 3^2 \cdot 5$

11. a) $2^{200} \cdot 3^{100}$ b) $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$ 12. 60 13. $\text{lcm} = 16\,800$ $\gcd = 60$

14. a) true b) false c) true d) true e) false f) true g) true h) false 15. $(-\infty, -1)$ 16. $-\dfrac{4}{3}$

17. two 18. a) -31 b) 15 c) undefined d) -4 e) 4 f) $\dfrac{1}{9}$ g) -8 h) $\dfrac{1}{6}$ i) $-\dfrac{6}{13}$ j) $-\dfrac{1}{8}$

19. a) 38 b) 6 c) 28 d) 20 e) -6 f) 2 20. a) $\dfrac{489}{990}$ b) $\dfrac{2057}{9000}$ c) $\dfrac{652}{9990}$

21. a) $6.4 \cdot 10^{11}$ b) $2.5 \cdot 10^{-11}$ c) $6.25 \cdot 10^{-22}$ d) $1.6 \cdot 10^{1}$ e) $8 \cdot 10^{5}$ f) $2.56 \cdot 10^{22}$ g) $6.4 \cdot 10^{-9}$

22. a) -8 b) -2 c) undefined 23. -7 24. a) $14t - 2$ b) $6x - 7y$ c) $-5x + 12$ d) $-8x + 14y$

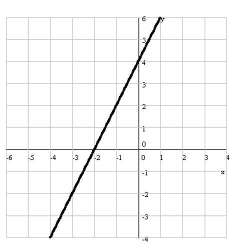
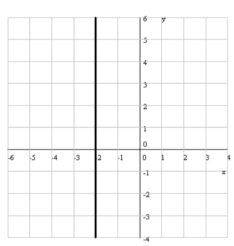
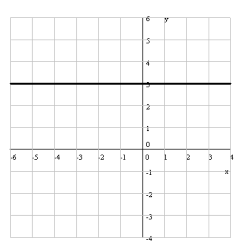
25. a) $\dfrac{x^2y^3}{6z}$ b) x^2 c) $\dfrac{b^4}{16a^4}$ d) $\dfrac{6a}{b^7}$ e) $-2x^{11}$ f) $16x^{20}$ 26. a) x^2 b) x^6 c) A^{50}

27. a) $2x$ b) $5x$ c) $\dfrac{x}{4}$ d) $21x$ e) x^3 f) $\sqrt{x}$

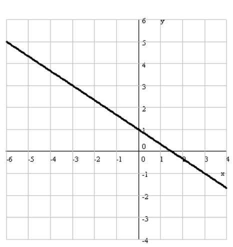
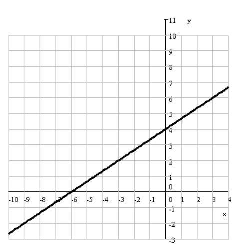
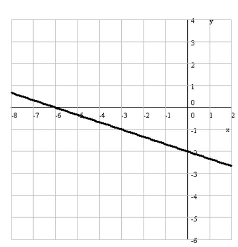
28. a) -1 b) 4 c) there is no solution d) 0 e) $-\dfrac{1}{2}$ f) -12 g) 2

29. a) $(-\infty, -1)$ b) $[1, \infty)$ c) $[-9, \infty)$ 30. a) $c = \dfrac{2A - ah}{h}$ b) $x = \dfrac{y - b}{m}$ c) $h = \dfrac{3V}{b}$

31. a) $y = 2x + 4$ c) $x = -2$ e) $y = 3$

b) $y = -\dfrac{2}{3}x + 1$ d) $2x - 3y = -12$ f) $x + 3y = -6$

32. a) $(4,0)$ b) $(0,-4)$ 33. a) -1 b) $-\frac{8}{3}$

34. a) $(-6,1)$ b) There are infinitely many solutions, in the form of $(x,6x-5)$ c) $(4,1)$ d) $(5,1)$

35. a) $4x^2-4x+1$ b) $8x^3-12x^2+6x-1$ c) $-2x^2-x+11$ d) $-y^2-y+1$

36. a) $(p+4)(p-8)$ b) $(3t+1)(t-2)$ c) $(3x-5)(3x+5)$ d) $2(x+2)(x-3)$ e) $2(x-2)(x+2)\left(x^2+4\right)$

37. $x+2$ 38. a) $x=-3$ or $x=\frac{1}{2}$ b) $x=-9$ or $x=2$ c) $x_1=-\frac{2}{3}$ and $x_2=\frac{5}{2}$ d) $x_1=0$ and $x_2=4$

39. a) -1 b) $\frac{x+6}{x+2}$ c) $\frac{x+1}{x-1}$ d) $x-2y$ e) $\frac{x}{x+5}$ f) $\frac{4}{3(x+2)}$ 40. a) 34 b) 54 c) 39

41. 44 chickens and 29 cows 42. $72\,\text{in}^2$ 43. $3.07\cdot 10^{13}$ in 44. $F=\frac{9}{5}C+32$ 45. \$42 46. \$120

47. 40% decrease 48. 3 49. a) 1344 ft b) 16 seconds 50. 15 51. $50x-240$ or $(50x-240)\,\text{m}^2$

52. 16 in 53. $12x^5y^5$ or $12x^5y^5\,\text{m}^2$ 54. 2 55. 4 56. \$8000 at 4% and \$4500 at 3%

Appendix B

Solutions

Solutions for 14.2 – Factoring out the GCF

1. Completely factor each of the following.

   a) $3x-12$

   Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 3.

$$3x-12=\boxed{3(x-4)}$$

   What is in the parentheses, $x-4$ can not be further factored. We can easily check our work by multiplication.

   b) $16a^2b+20a^3b-12a^2b^2$

   Solution: First we identify the GCF (greatest common factor). In this case, the GCF is $4a^2b$. So we have

$$4a^2b(\qquad\qquad)$$

   and need to figure out what to write into the parenthese so that the multiplication backwards works. What do we have to multiply $4a^2b$ by so the result is $16a^2b$? The answer is $\dfrac{16a^2b}{4a^2b}=4$, so we have so far

$$4a^2b(4\qquad\qquad)$$

   Next, what do we have to multiply $4a^2b$ by so the result is $20a^3b$? The answer is $\dfrac{20a^3b}{4a^2b}=5a$ and so now we have

$$4a^2b(4+5a\qquad)$$

   Next, what do we have to multiply $4a^2b$ by so the result is $-12a^2b^2$? The answer is $\dfrac{-12a^2b^2}{4a^2b}=-3b$ and so now we have

$$\boxed{4a^2b(4+5a-3b)}$$

   We check via multiplication backwards:

$$\begin{aligned}4a^2b(4+5a-3b) &= 4a^2b(4)+4a^2b(5a)+4a^2b(-3b)\\ &= 16a^2b+20a^3b-12a^2b^2\end{aligned}$$

   and so our solution is correct.

c) $3a^2 - 12$
Solution: The greatest common factor (or GCF) is 3.

$$3a^2 - 12 = 3\left(a^2 - 4\right)$$

We check our work by multiplication:

$$3\left(a^2 - 4\right) = 3a^2 - 12$$

and so our answer, $\boxed{3\left(a^2 - 4\right)}$ is correct.

d) $3a^3 - 12a^2$
Solution: The greatest common factor (or GCF) is $3a^2$.

$$3a^3 - 12a^2 = 3a^2\left(\qquad \right)$$

what do we have to multiply $3a^2$ by so the result is $3a^3$? The answer is $\dfrac{3a^3}{3a^2} = a$ and so now we have

$$3a^3 - 12a^2 = 3a^2\left(a \qquad \right)$$

Next, what do we have to multiply $3a^2$ by so the result is $-12a^2$? The answer is $\dfrac{-12a^2}{3a^2} = -4$ and so now we have

$$3a^3 - 12a^2 = 3a^2\left(a - 4\right)$$

We check our work by multiplication:

$$3a^2\left(a - 4\right) = 3a^2 \cdot a - 3a^2 \cdot 4 = 3a^3 - 12a^2$$

and so our answer, $\boxed{3a^2\left(a - 4\right)}$ is correct.

e) $20x + 5x^3$
Solution: We rearrange the terms by degree first and then factor out the GCF.

$$20x + 5x^3 = 5x^3 + 20x = 5x\left(x^2 + 4\right)$$

The final answer is $\boxed{5x\left(x^2 + 4\right)}$. We can easily check the result by mulitplication.

f) $3x\left(x - 2\right) + 8x^3\left(x - 2\right) - 11\left(x - 2\right)$

Solution: Now the GCF is the linear expression $x - 2$. So we factor it out:

$$\left(x - 2\right)\left(\qquad \right)$$

What do we have to multiply $x - 2$ by so the result is $3x\left(x - 2\right)$? The answer is $3x$ and so now we have

$$\left(x - 2\right)\left(3x \qquad \right)$$

Next, what do we have to multiply $x - 2$ by so the result is $8x^3\left(x - 2\right)$? The answer is $8x^3$ and so now we have

$$\left(x - 2\right)\left(3x + 8x^3 \qquad \right)$$

Next, what do we have to multiply $x - 2$ by so the result is $-11\left(x - 2\right)$? The answer is -11 and so now we have

$$\left(x - 2\right)\left(3x + 8x^3 - 11\right)$$

It is good practice to rearrange polynomials by degree and so our answer is $\boxed{\left(x - 2\right)\left(8x^3 + 3x - 11\right)}$.

2. Factor out -1 from the given expression.

$$-5x^3+2x^2-x-8$$

Solution: We start with a minus sign (short for -1) and a parentheses.

$$-5x^3+2x^2-x-8=-(\qquad\qquad)$$

Inside the parenthese, we write the original expression, but change all signs. Again, when we multiply back to check and apply the distributive law, we should get back the original expression.

$$-5x^3+2x^2-x-8=-\left(5x^3-2x^2+x+8\right)$$

So our answer is $\boxed{-\left(5x^3-2x^2+x+8\right)}$.

3. Solve each of the following equations. Make sure to check your solution.

a) $(x-2)(x+3)(2x+1)=0$
Solution: Since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule. Most of these were already done for us as the right-hand side is zero and the left-hand side is completely factored. All we need to do is apply the zero product rule. **A product can only be zero if one of its factors is zero.** $(x-2)(x+3)(2x+1)=0$ means that either $x-2=0$ or $x+3=0$ or $2x+1=0$. We solve these linear equations separately:

$$\begin{array}{ccccc} x-2=0 & \text{or} & x+3=0 & \text{or} & 2x+1=0 \\ x=2 & & x=-3 & & 2x=-1 \\ & & & & x=-\frac{1}{2} \end{array}$$

We check all three solutions. If $x=2$, then

$$(2-2)(2+3)(2(2)+1)=0\cdot 5\cdot 5=0$$

If $x=-3$, then

$$(-3-2)(-3+3)(2(-3)+1)=-5\cdot 0\cdot(-5)=0$$

and if $x=-\frac{1}{2}$, then

$$\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}+3\right)\left(2\left(-\frac{1}{2}\right)+1\right)=-\frac{3}{2}\cdot\frac{5}{2}\cdot 0=0$$

and so all three numbers, $\boxed{2,\ -3,\ \text{and}\ -\frac{1}{2}}$ are correct.

b) $m(m+7)=0$

Solution: We will apply the zero product rule. **A product can only be zero if one of its factors is zero.** $m(m+7)=0$ means that either $m=0$. We solve these linear equations separately and obtain $m=0$ and $m=-7$. We check: If $m=0$, then

$$0(0+7)=0\cdot 7=0$$

and if $m=-7$, then

$$-7(-7+7)=-7\cdot 0$$

and so both numbers, $\boxed{0 \text{ and } -7}$ are correct.

c) $x^2 = 9x$

Solution: Since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} x^2 &= 9x && \text{subtract } 9x \\ x^2 - 9x &= 0 && \text{factor out the GCF} \\ x(x-9) &= 0 \end{aligned}$$

A product can only be zero if one of its factors is zero. $x(x-9) = 0$ means that either $x = 0$ or $x - 9 = 0$. We solve these linear equations separately and obtain $\boxed{0 \text{ and } 9}$. We check: $0^2 = 9 \cdot 0$ and $9^2 = 9 \cdot 9$ and so our solution is correct.

d) $8x^3 = 50x^2$

Solution: since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} 8x^3 &= 50x^2 && \text{subtract } 50x^2 \\ 8x^3 - 50x^2 &= 0 && \text{the GCF is } 2x^2 \\ 2x^2(4x-25) &= 0 \end{aligned}$$

We now apply the zero product rule. If this product is zero, then either $2x^2 = 0$ or $4x - 25 = 0$. We solve these equations for x.

$$\begin{aligned} 2x^2 &= 0 && \text{or} && 4x - 25 = 0 \\ 2 \cdot x \cdot x &= 0 && \text{or} && 4x = 25 \\ x &= 0 && \text{or} && x = \frac{25}{4} \end{aligned}$$

We check both solutions. If $x = 0$, then LHS$= 8 \cdot 0^3 = 8 \cdot 0 = 0$ and RHS$= 50 \cdot 0^2 = 50 \cdot 0 = 0$

If $x = \frac{25}{4}$, then

$$\text{LHS} = 8\left(\frac{25}{4}\right)^3 = \frac{8}{1} \cdot \frac{15\,625}{64} = \frac{15\,625}{8} \quad \text{and} \quad \text{RHS} = 50\left(\frac{25}{4}\right)^2 = \frac{50}{1} \cdot \frac{625}{16} = \frac{15\,625}{8}$$

Thus both solutions, $\boxed{0 \text{ and } \frac{25}{4}}$ are correct.

4. Find all numbers that satisfy the following condition: if we square the number, we get back the same number.

Solution: Let us denote the number by x. The equation is

$$\begin{aligned} x^2 &= x && \text{reduce one side to zero} \\ x^2 - x &= 0 && \text{factor} \\ x(x-1) &= 0 && \text{apply the zero property} \\ x = 0 \quad &\text{or} \quad x - 1 = 0 \\ x = 0 \quad &\text{or} \quad x = 1 \end{aligned}$$

Thus there are two numbers, 0 and 1, satisfying the property. We check: $0^2 = 0$ and $1^2 = 1$. Thus our answer is: $\boxed{0 \text{ and } 1}$.

Solutions for 15.2 – Linear Inequalities

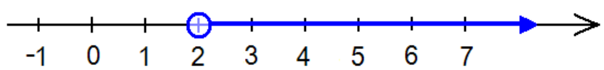
1. $-7 > -5x+3$

Solution: Solving linear inequalities requires almost the same techniques as solving linear equations. There is only one difference: **when multiplying or dividing an inequality by a negative number, the inequality sign must be reversed**.

$$\begin{aligned} -7 &> -5x+3 && \text{subtract } 3 \\ -10 &> -5x && \text{divide by } -5 \\ 2 &< x \end{aligned}$$

When we divided both sides by -5, we reversed the inequality sign. The final answer is all real numbers greater than 2. This set of numbers can be presented in numerous ways:

1) set-builder notation: $\{x | x > 2\}$
2) interval notation: $(2, \infty)$
3) graphing the solution set on the number line:



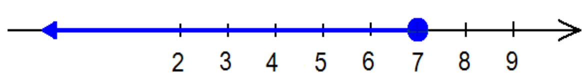
2. $3(x-2) \leq 2x+1$

Solution:

$$\begin{aligned} 3(x-2) &\leq 2x+1 && \text{distribute} \\ 3x-6 &\leq 2x+1 && \text{subtract } 2x \\ x-6 &\leq 1 && \text{add } 6 \\ x &\leq 7 \end{aligned}$$

The final answer is all real numbers less than or equal to 7. This set of numbers can be presented in numerous ways:

1) set-builder notation: $\{x | x \leq 7\}$
2) interval notation: $(-\infty, 7]$
3) graphing the solution set on the number line:

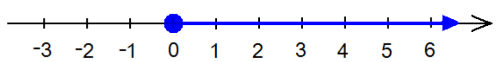


3. $5(4x-1)-(x-3) \geq -x-2$

Solution:

$$\begin{aligned}
5(4x-1)-(x-3) &\geq -x-2 && \text{distribute}\\
20x-5-x+3 &\geq -x-2 && \text{combine like terms}\\
19x-2 &\geq -x-2 && \text{add 2}\\
19x &\geq -x && \text{add } x\\
20x &\geq 0 && \text{divide by 20}\\
x &\geq 0
\end{aligned}$$

The final answer is all real numbers greater than or equal to 0. This set of numbers can be presented in numerous ways: in set-builder notation: $\{x|x \geq 0\}$, in interval notation: $[0,\infty)$, or by graphing the solution set on the number line:

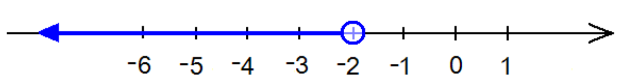


4. $\dfrac{m+4}{2}-\dfrac{4m+3}{5} > 2$

Solution:

$$\begin{aligned}
\frac{m+4}{2}-\frac{4m+3}{5} &> 2 && \text{make everything a fraction}\\
\frac{m+4}{2}-\frac{4m+3}{5} &> \frac{2}{1} && \text{bring to common denominator}\\
\frac{5(m+4)}{10}-\frac{2(4m+3)}{10} &> \frac{20}{10} && \text{multiply by 15}\\
5(m+4)-2(4m+3) &> 20 && \text{distribute}\\
5m+20-8m-6 &> 20 && \text{combine like terms}\\
-3m+14 &> 20 && \text{subtract 14}\\
-3m &> 6 && \text{divide by } -3\\
m &< -2
\end{aligned}$$

When we divided both sides by -3, we reversed the inequality sign. The final answer is all real numbers less than -2. This set of numbers can be presented in numerous ways: in set-builder notation: $\{x|x < -2\}$, in interval notation: $(-\infty,-2)$, or by graphing the solution set on the number line:



Solutions for 16.1 – Integer Exponents

Simplify each of the following. Assume that all variables represent positive numbers. Present your answer without negative exponents.

1. 3^{-2}

   Solution: We just apply the rule $a^{-n} = \frac{1}{a^n}$. $\quad 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

2. $\frac{1}{2^{-3}}$

   Solution: We apply the rule $a^{-n} = \frac{1}{a^n}$. $\quad \frac{1}{2^{-3}} = \frac{1}{\frac{1}{2^3}} = \frac{1}{\frac{1}{8}}$

   To divide is to multiply by the reciprocal: $\quad \frac{1}{\frac{1}{8}} = 1 \cdot \frac{8}{1} = 8$

   This is true in general: $\frac{1}{a^{-n}} = a^n$

   $$\frac{1}{a^{-n}} = \frac{1}{\frac{1}{a^n}} = 1 \cdot \frac{a^n}{1} = a^n$$

3. m^{-4}

   Solution: We apply the rule $a^{-n} = \frac{1}{a^n}$.

   $$m^{-4} = \frac{1}{m^4}$$

4. $\frac{1}{x^{-5}}$

   Solution: We have already proven that $\frac{1}{a^{-n}} = a^n$

   $$\frac{1}{x^{-5}} = x^5$$

5. $a^8 \cdot a^{-1}$

   Solution 1: We can apply the rule $a^n \cdot a^m = a^{n+m}$

   $$a^8 \cdot a^{-1} = a^{8+(-1)} = a^7$$

   Solution 2: We can apply the rule $a^{-n} = \frac{1}{a^n}$ and then the rule $\frac{a^n}{a^m} = a^{n-m}$.

   $$a^8 \cdot a^{-1} = a^8 \cdot \frac{1}{a^1} = \frac{a^8}{1} \cdot \frac{1}{a} = \frac{a^8}{a} = \frac{a^8}{a^1} = a^{8-1} = a^7$$

6. $p^3\left(p^{-7}\right)p^8$

Solution 1: We can apply the rule $a^n \cdot a^m = a^{n+m}$

$$p^3\left(p^{-7}\right)p^8 = p^{3+(-7)+8} = p^4$$

Solution 2: We can apply the rules $a^{-n} = \dfrac{1}{a^n}$ and $a^n \cdot a^m = a^{n+m}$ and $\dfrac{a^n}{a^m} = a^{n-m}$.

$$p^3\left(p^{-7}\right)p^8 = p^3 \cdot \frac{1}{p^7} \cdot p^8 = \frac{p^3}{1} \cdot \frac{1}{p^7} \cdot \frac{p^8}{1} = \frac{p^3 \cdot p^8}{p^7} = \frac{p^{3+8}}{p^7} = \frac{p^{11}}{p^7} = p^{11-7} = p^4$$

7. $\dfrac{x^{-4}}{x^{-9}}$

Solution 1: We can apply the rule $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{x^{-4}}{x^{-9}} = x^{-4-(-9)} = x^{-4+9} = x^5$$

Solution 2: We can apply the rules $a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{x^{-4}}{x^{-9}} = \frac{x^9}{x^4} = x^{9-4} = x^5$$

8. $\dfrac{50a^{12}}{10a^{-3}}$

Solution 1: We can apply the rule $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{50a^{12}}{10a^{-3}} = 5a^{12-(-3)} = 5a^{12+3} = 5a^{15}$$

Solution 2: We can apply the rules $a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{50a^{12}}{10a^{-3}} = \frac{50a^{12}a^3}{10} = 5a^{12+3} = 5a^{15}$$

9. $\dfrac{t^{-3}}{t^4}$

Solution 1: We can apply the rules $\dfrac{a^n}{a^m} = a^{n-m}$ and then $a^{-n} = \dfrac{1}{a^n}$.

$$\frac{t^{-3}}{t^4} = t^{-3-4} = t^{-7} = \frac{1}{t^7}$$

Solution 2: We can apply the rule $a^{-n} = \dfrac{1}{a^n}$ and then $a^n \cdot a^m = a^{n+m}$.

$$\frac{t^{-3}}{t^4} = \frac{1}{t^4 \cdot t^3} = \frac{1}{t^7}$$

10. x^0

Solution: There is a separate rule stating that as long as x is not zero, then $x^0 = 1$. So the answer is 1.

11. $-x^0$

Solution: This is the opposite of x^0 and so the answer is -1.

$$-x^0 = -1 \cdot x^0 = -1 \cdot 1 = -1$$

12. $(-x)^0$

Solution: This is again 1 because any non-zero riased to the power zero is 1.

13. $\left(b^{-5}\right)\left(b^{2}\right)\left(b^{-1}\right)$

Solution 1: We can apply the rules $a^n \cdot a^m = a^{n+m}$ and then $a^{-n} = \dfrac{1}{a^n}$.

$$\left(b^{-5}\right)\left(b^{2}\right)\left(b^{-1}\right) = b^{-5+2+(-1)} = b^{-4} = \frac{1}{b^4}$$

Solution 2: We can apply the rule $a^{-n} = \dfrac{1}{a^n}$ and then just cancel.

$$\left(b^{-5}\right)\left(b^{2}\right)\left(b^{-1}\right) = \frac{1}{b^5} \cdot b^2 \cdot \frac{1}{b^1} = \frac{1}{b^5} \cdot \frac{b^2}{1} \cdot \frac{1}{b^1} = \frac{b^2}{b^6} = \frac{\not{b} \cdot \not{b}}{\not{b} \cdot \not{b} \cdot b \cdot b \cdot b \cdot b} = \frac{1}{b^4}$$

14. $\dfrac{1}{\left(b^{-5}\right)\left(b^{2}\right)\left(b^{-1}\right)}$

Solution 1: We can apply the rules $a^n \cdot a^m = a^{n+m}$ and then $a^{-n} = \dfrac{1}{a^n}$.

$$\frac{1}{\left(b^{-5}\right)\left(b^{2}\right)\left(b^{-1}\right)} = \frac{1}{b^{-5+2+(-1)}} = \frac{1}{b^{-4}} = \frac{1}{\frac{1}{b^4}} = 1 \cdot \frac{b^4}{1} = b^4$$

Solution 2: We can apply the rule $a^{-n} = \dfrac{1}{a^n}$ and then $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{1}{\left(b^{-5}\right)\left(b^{2}\right)\left(b^{-1}\right)} = \frac{b^5 \cdot b^1}{b^2} = \frac{b^6}{b^2} = b^{6-2} = b^4$$

15. $\dfrac{m^{-2}}{m^{-5}}$

Solution 1: We can apply the rules $\dfrac{a^n}{a^m} = a^{n-m}$ and then $a^{-n} = \dfrac{1}{a^n}$.

$$\frac{m^{-2}}{m^{-5}} = m^{-2-(-5)} = m^{-2+5} = m^3$$

Solution 2: We can apply the rule $a^{-n} = \dfrac{1}{a^n}$ and then $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{m^{-2}}{m^{-5}} = \frac{m^5}{m^2} = m^{5-2} = m^3$$

16. $\dfrac{x^3y^{-5}}{z^{-4}}$

Solution: Each variable occurs only once and so this problem is just about bringing it to the form required. We can apply the rule $a^{-n} = \dfrac{1}{a^n}$. We hve alread shown that $\dfrac{1}{a^{-n}} = a^n$.

$$\frac{x^3y^{-5}}{z^{-4}} = \frac{x^3z^4}{y^5}$$

17. $\dfrac{18q^3}{6q^{-3}}$

Solution 1: We can apply the rule $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{18q^3}{6q^{-3}} = \frac{\not{6}\cdot 3q^{3-(-3)}}{\not{6}\cdot 1} = \frac{3q^{3+3}}{1} = 3q^6$$

Solution 2: We can apply the rules $a^{-n} = \dfrac{1}{a^n}$ and then $a^n \cdot a^m = a^{n+m}$.

$$\frac{18q^3}{6q^{-3}} = \frac{\not{6}\cdot 3q^3q^3}{\not{6}\cdot 1} = 3q^6$$

18. $\left(\dfrac{2}{3}\right)^{-3}$

Solution: We can apply the rule $a^{-n} = \dfrac{1}{a^n}$.

$$\left(\frac{2}{3}\right)^{-3} = \frac{1}{\left(\frac{2}{3}\right)^3} = \frac{1}{\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}} = \frac{1}{\frac{8}{27}} = 1\cdot\frac{27}{8} = \frac{27}{8} = \frac{3^3}{2^2} = \left(\frac{3}{2}\right)^3$$

Note that we basically proved here that $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$.

19. $2y^{-3}$

Solution: We can apply the rule $a^{-n} = \dfrac{1}{a^n}$. It is important to note that the base of exponentiation is y and not $2y$.

$$2y^{-3} = 2\cdot\frac{1}{y^3} = \frac{2}{1}\cdot\frac{1}{y^3} = \frac{2}{y^3}$$

20. $(2y)^{-3}$

Solution: We can apply the rule $a^{-n} = \dfrac{1}{a^n}$. This time the base of exponentiation is $2y$. So we will apply the rule $(ab)^n = a^nb^n$.

$$(2y)^{-3} = \frac{1}{(2y)^3} = \frac{1}{2^3y^3} = \frac{1}{8y^3}$$

21. $\left(-\frac{3}{5}\right)^{-2}$

Solution 1: We can apply the rule $a^{-n} = \frac{1}{a^n}$.

$$\left(-\frac{3}{5}\right)^{-2} = \frac{1}{\left(-\frac{3}{5}\right)^2} = \frac{1}{\left(-\frac{3}{5}\right)\left(-\frac{3}{5}\right)} = \frac{1}{\frac{-3}{5}\cdot\frac{-3}{5}} = \frac{1}{\frac{9}{25}} = 1\cdot\frac{25}{9} = \frac{25}{9}$$

Solution 2: We proved previously that $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$. Using that,

$$\left(-\frac{3}{5}\right)^{-2} = \left(-\frac{5}{3}\right)^2 = \left(-\frac{5}{3}\right)\left(-\frac{5}{3}\right) = \frac{25}{9}$$

22. $\frac{a^3b^{-5}}{a^{-2}b^3}$

Solution 1: We can apply the rule $\frac{a^n}{a^m} = a^{n-m}$ and then $a^{-n} = \frac{1}{a^n}$.

$$\frac{a^3b^{-5}}{a^{-2}b^3} = a^{3-(-2)}b^{-5-3} = a^{3+2}b^{-5-3} = a^5b^{-8} = a^5\cdot\frac{1}{b^8} = \frac{a^5}{1}\cdot\frac{1}{b^8} = \frac{a^5}{b^8}$$

Solution 2: We can apply the rules $a^{-n} = \frac{1}{a^n}$ and $a^n\cdot a^m = a^{n+m}$.

$$\frac{a^3b^{-5}}{a^{-2}b^3} = \frac{a^3a^2}{b^3b^5} = \frac{a^5}{b^8}$$

23. $\left(3m^3\right)^{-2}$

Solution: We can apply the rule $a^{-n} = \frac{1}{a^n}$ and then $(ab)^n = a^nb^n$ and also $(a^n)^m = a^{nm}$.

$$\left(3m^3\right)^{-2} = \frac{1}{\left(3m^3\right)^2} = \frac{1}{3^2\left(m^3\right)^2} = \frac{1}{9m^{3\cdot2}} = \frac{1}{9m^6}$$

24. $\left(-2ab^{-3}\right)^{-3}$

Solution: We can apply the rule $(ab)^n = a^nb^n$ and then $(a^n)^m = a^{nm}$.

$$\left(-2ab^{-3}\right)^{-3} = (-2)^{-3}a^{-3}\left(b^{-3}\right)^{-3} = (-2)^{-3}a^{-3}b^{-3(-3)} = (-2)^{-3}a^{-3}b^9$$

We now apply $a^{-n} = \frac{1}{a^n}$.

$$(-2)^{-3}a^{-3}b^9 = \frac{1}{(-2)^3}\cdot\frac{1}{a^3}\cdot b^9 = \frac{1}{-8}\cdot\frac{1}{a^3}\cdot\frac{b^9}{1} = \frac{b^9}{-8a^3} = -\frac{b^9}{8a^3}$$

25. $\dfrac{\left(k^3\right)^{-3}}{\left(k^{-5}\right)^2}$

Solution: We can apply the rule $\left(a^n\right)^m = a^{nm}$ and then $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{\left(k^3\right)^{-3}}{\left(k^{-5}\right)^2} = \frac{k^{3(-3)}}{k^{-5\cdot 2}} = \frac{k^{-9}}{k^{-10}} = k^{-9-(-10)} = k^{-9+10} = k^1 = k$$

26. $\left(\dfrac{2a^{-3}b^5}{-3a^3b^{-2}}\right)^{-2}\left(a^3b^{-5}\right)^{-3}$

Solution:

$$\begin{aligned}
E &= \left(\frac{2a^{-3}b^5}{-3a^3b^{-2}}\right)^{-2}\left(a^3b^{-5}\right)^{-3} = \left(\frac{2a^{-3-3}b^{5-(-2)}}{-3}\right)^{-2}\left(a^3b^{-5}\right)^{-3} && \text{apply } \frac{a^n}{a^m} = a^{n-m} \\
&= \left(\frac{2a^{-6}b^{5+2}}{-3}\right)^{-2}\left(a^3b^{-5}\right)^{-3} \\
&= \left(\frac{2a^{-6}b^7}{-3}\right)^{-2}\left(a^3b^{-5}\right)^{-3} && \text{apply } \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n \\
&= \left(\frac{-3}{2a^{-6}b^7}\right)^{2}\left(a^3b^{-5}\right)^{-3} && \text{apply } \left(\frac{a}{b}\right)^{n} = \frac{a^n}{b^n} \\
&= \frac{(-3)^2}{\left(2a^{-6}b^7\right)^2}\left(a^3b^{-5}\right)^{-3} && \text{apply } (ab)^n = a^nb^n \text{ and } a^{-n} = \frac{1}{a^n} \\
&= \frac{9}{2^2\left(a^{-6}\right)^2\left(b^7\right)^2}\cdot\frac{1}{\left(a^3b^{-5}\right)^3} && \text{apply } \left(a^n\right)^m = a^{nm} \text{ and } (ab)^n = a^nb^n \\
&= \frac{9}{4a^{-12}b^{14}}\cdot\frac{1}{\left(a^3\right)^3\left(b^{-5}\right)^3} && \text{apply } a^{-n} = \frac{1}{a^n} \text{ and } (ab)^n = a^nb^n \\
&= \frac{9a^{12}}{4b^{14}}\cdot\frac{1}{a^{3\cdot 3}b^{(-5)3}} \\
&= \frac{9a^{12}}{4b^{14}}\cdot\frac{1}{a^9b^{-15}} && \text{apply } a^{-n} = \frac{1}{a^n} \\
&= \frac{9a^{12}}{4b^{14}}\cdot\frac{b^{15}}{a^9} = \frac{9a^{12}b^{15}}{4b^{14}a^9} && \text{apply } \frac{a^n}{a^m} = a^{n-m} \\
&= \frac{9a^{12-9}b^{15-14}}{4} = \frac{9a^3b^1}{4} = \frac{9}{4}a^3b
\end{aligned}$$

27. $\left(-2a^{-3}\right)\left(-2a^{-2}b\right)^{-4}$

Solution:

$$\begin{aligned}
E &= \left(-2a^{-3}\right)\left(-2a^{-2}b\right)^{-4} && \text{apply } a^{-n}=\frac{1}{a^n}\\
&= \left(-2\cdot\frac{1}{a^3}\right)\frac{1}{\left(-2a^{-2}b\right)^4} && \text{apply } (ab)^n=a^nb^n\\
&= \left(\frac{-2}{1}\cdot\frac{1}{a^3}\right)\frac{1}{(-2)^4\left(a^{-2}\right)^4b^4} && \text{apply } (a^n)^m=a^{nm}\\
&= \frac{-2}{a^3}\cdot\frac{1}{16a^{-8}b^4} && \text{apply } a^{-n}=\frac{1}{a^n}\\
&= \frac{-2}{a^3}\cdot\frac{a^8}{16b^4}\\
&= \frac{-2a^8}{a^3\cdot 16b^4}=\frac{-2a^8}{16a^3b^4} && \text{apply } \frac{a^n}{a^m}=a^{n-m}\\
&= \frac{-1\cdot \cancel{2}a^{8-3}}{8\cdot\cancel{2}b^4}=\frac{-a^5}{8b^4}
\end{aligned}$$

28. $\dfrac{\left(-3p^3q^5\right)^2}{\left(2q^0p^3\right)^{-1}}$

Solution:

$$\begin{aligned}
E &= \frac{\left(-3p^3q^5\right)^2}{\left(2q^0p^3\right)^{-1}} && \text{apply } q^0=1 \text{ and } \frac{1}{a^{-n}}=a^n\\
&= \left(-3p^3q^5\right)^2\left(2\cdot 1p^3\right)^1\\
&= \left(-3p^3q^5\right)^2\cdot 2p^3 && \text{apply } (ab)^n=a^nb^n\\
&= (-3)^2\left(p^3\right)^2\left(q^5\right)^2\cdot 2p^3 && \text{apply } (a^n)^m=a^{nm}\\
&= 9p^{3\cdot 2}q^{5\cdot 2}\cdot 2p^3\\
&= 18p^6q^{10}p^3 && \text{apply } a^n\cdot a^m=a^{n+m}\\
&= 18p^{6+3}q^{10}=18p^9q^{10}
\end{aligned}$$

29. $\left(\dfrac{2a^{-2}b^3}{-2^2\left(a^{-1}b\right)^{-3}}\right)^{-2}$

Solution:

$$\begin{aligned}
E &= \left(\frac{2a^{-2}b^3}{-2^2\left(a^{-1}b\right)^{-3}}\right)^{-2} && \text{apply } \left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^n\\
&= \left(\frac{-2^2\left(a^{-1}b\right)^{-3}}{2a^{-2}b^3}\right)^2 && \text{apply } (ab)^n=a^nb^n\\
&= \left(\frac{-4\left(a^{-1}\right)^{-3}b^{-3}}{2a^{-2}b^3}\right)^2 && \text{apply } (a^n)^m=a^{nm}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{-4a^{-1(-3)}b^{-3}}{2a^{-2}b^{3}}\right)^{2} \\
&= \left(\frac{-2a^{3}b^{-3}}{a^{-2}b^{3}}\right)^{2} && \text{apply } \frac{a^n}{a^m} = a^{n-m} \\
&= \left(-2a^{3-(-2)}b^{-3-3}\right)^{2} \\
&= \left(-2a^{3+2}b^{-3-3}\right)^{2} \\
&= \left(-2a^{5}b^{-6}\right)^{2} && \text{apply } (ab)^n = a^n b^n \\
&= (-2)^2\left(a^5\right)^2\left(b^{-6}\right)^2 && \text{apply } (a^n)^m = a^{nm} \\
&= 4a^{5\cdot 2}b^{-6\cdot 2} = 4a^{10}b^{-12} && \text{apply } a^{-n} = \frac{1}{a^n} \\
&= 4a^{10}\cdot\frac{1}{b^{12}} \\
&= \frac{4a^{10}}{1}\cdot\frac{1}{b^{12}} = \frac{4a^{10}}{b^{12}}
\end{aligned}
$$

30. $\left(-\dfrac{x^3y^0x^{-5}}{y^{-3}}\right)^{-2}$

Solution:

$$
\begin{aligned}
E &= \left(-\frac{x^3y^0x^{-5}}{y^{-3}}\right)^{-2} && y^0 = 1 \text{ and } a^n\cdot a^m = a^{n+m} \\
&= \left(-\frac{x^{3+(-5)}}{y^{-3}}\right)^{-2} = \left(\frac{-1x^{-2}}{y^{-3}}\right)^{-2} && \text{apply } \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \\
&= \frac{\left(-1x^{-2}\right)^{-2}}{\left(y^{-3}\right)^{-2}} && \text{apply } (ab)^n = a^n b^n \\
&= \frac{(-1)^{-2}\left(x^{-2}\right)^{-2}}{\left(y^{-3}\right)^{-2}} && \text{apply } (a^n)^m = a^{nm} \text{ and } a^{-n} = \frac{1}{a^n} \\
&= \frac{x^{-2(-2)}}{(-1)^2y^{-3(-2)}} = \frac{x^4}{1y^6} = \frac{x^4}{y^6}
\end{aligned}
$$

31. $\left(-\dfrac{x^3y^7x^{-5}}{y^{-3}}\right)^{0}$

Solution: Any non-zero quantity raised to the power zero is 1. So the answer is 1.

32. $\dfrac{x^{-1}+y^{-1}}{x^{-2}-y^{-2}}$

Solution: This problem is very different because there are addition and subtraction involved. Because of that, we can not simply move the expressions with negative exponents. Instead, this will be a problem involving complex fractions.

$$E = \frac{x^{-1}+y^{-1}}{x^{-2}-y^{-2}} = \frac{\frac{1}{x^1}+\frac{1}{y^1}}{\frac{1}{x^2}-\frac{1}{y^2}} = \frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}} \qquad \text{bring fractions to the common denominator}$$

$$= \frac{\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}}{\frac{1\cdot y^2}{x^2\cdot y^2}-\frac{1\cdot x^2}{y^2\cdot x^2}} = \frac{\frac{y}{xy}+\frac{x}{xy}}{\frac{y^2}{x^2y^2}-\frac{x^2}{x^2y^2}} = \frac{\frac{y+x}{xy}}{\frac{y^2-x^2}{x^2y^2}} \qquad \text{to divide is to multiply by the reciprocal}$$

$$= \frac{y+x}{xy}\cdot\frac{x^2y^2}{y^2-x^2} \qquad \text{cancel out } xy$$

$$= \frac{y+x}{1}\cdot\frac{xy}{y^2-x^2} = \frac{xy\,(x+y)}{y^2-x^2} \qquad \text{factor } y^2-x^2 \text{ via the difference of squares theorem, cancel out } x+y$$

$$= \frac{xy\,(x+y)}{(y-x)\,(y+x)} = \frac{xy}{y-x}$$

33. $\dfrac{\left(-2a^{-2}\right)^{-2}b^3a^0\left(-aba^{-2}b^{-2}\right)^{-3}}{2a^2\left(-2a^{-2}b\right)^{-2}ab^0}$

Solution:

$$E = \frac{\left(-2a^{-2}\right)^{-2}b^3a^0\left(-aba^{-2}b^{-2}\right)^{-3}}{2a^2\left(-2a^{-2}b\right)^{-2}ab^0} \qquad a^0=b^0=1 \text{ and } x^nx^m=x^{n+m}$$

$$= \frac{\left(-2a^{-2}\right)^{-2}b^3\left(-a^{1+(-2)}b^{1+(-2)}\right)^{-3}}{2a^{2+1}\left(-2a^{-2}b\right)^{-2}} = \frac{\left(-2a^{-2}\right)^{-2}b^3\left(-1a^{-1}b^{-1}\right)^{-3}}{2a^3\left(-2a^{-2}b\right)^{-2}} \qquad \text{apply } (xy)^n=x^ny^n$$

$$= \frac{(-2)^{-2}\left(a^{-2}\right)^{-2}b^3(-1)^{-3}\left(a^{-1}\right)^{-3}\left(b^{-1}\right)^{-3}}{2a^3(-2)^{-2}\left(a^{-2}\right)^{-2}b^{-2}} \qquad \text{apply } (x^n)^m=x^{nm}$$

$$= \frac{(-2)^{-2}a^{-2(-2)}b^3(-1)^{-3}a^{-1(-3)}b^{-1(-3)}}{2a^3(-2)^{-2}a^{-2(-2)}b^{-2}} = \frac{(-2)^{-2}a^4b^3(-1)^{-3}a^3b^3}{2a^3(-2)^{-2}a^4b^{-2}} \qquad \text{cancel out } a^4 \text{ and } a^3 \text{ and } (-2)^{-2}$$

$$= \frac{b^3(-1)^{-3}b^3}{2b^{-2}} \qquad \text{apply } x^nx^m=x^{n+m}$$

$$= \frac{(-1)^{-3}b^{3+3}}{2b^{-2}} = \frac{(-1)^{-3}b^6}{2b^{-2}} \qquad \text{apply } x^{-n}=\frac{1}{x^n}$$

$$= \frac{b^6b^2}{(-1)^3\,2} \qquad \text{apply } x^nx^m=x^{n+m}$$

$$= \frac{b^{6+2}}{-1\cdot 2} = \frac{b^8}{-2} = -\frac{b^8}{2}$$

34. $\left(\dfrac{-a^2\left(b^{-1}a\right)^{-5}}{b^7\left(-ab^2\right)^{-3}}\right)^{-2}$

Solution:

$$
\begin{aligned}
E &= \left(\frac{-a^2\left(b^{-1}a\right)^{-5}}{b^7\left(-ab^2\right)^{-3}}\right)^{-2} && \text{apply } (xy)^n = x^n y^n \\
&= \left(\frac{-a^2\left(b^{-1}\right)^{-5}a^{-5}}{b^7\left(-1\right)^{-3}a^{-3}\left(b^2\right)^{-3}}\right)^{-2} && \text{apply } (x^n)^m = x^{nm} \\
&= \left(\frac{-a^2b^{-1(-5)}a^{-5}}{b^7\left(-1\right)^{-3}a^{-3}b^{2(-3)}}\right)^{-2} = \left(\frac{-a^2b^5a^{-5}}{b^7\left(-1\right)^{-3}a^{-3}b^{-6}}\right)^{-2} && \text{apply } x^n x^m = x^{n+m} \\
&= \left(\frac{-a^{2+(-5)}b^5}{\left(-1\right)^{-3}b^{7+(-6)}a^{-3}}\right)^{-2} = \left(\frac{-1\cdot a^{-3}b^5}{\left(-1\right)^{-3}b^1a^{-3}}\right)^{-2} && \text{cancel out } a^{-3} \\
&= \left(\frac{-1b^5}{\left(-1\right)^{-3}b^1}\right)^{-2} && \text{apply } a^{-n} = \frac{1}{a^n} \\
&= \left(\frac{-1\left(-1\right)^3b^5}{b^1}\right)^{-2} = \left(\frac{-1\left(-1\right)b^5}{b^1}\right)^{-2} = \left(\frac{1b^5}{b^1}\right)^{-2} && \text{apply } \frac{x^n}{x^m} = x^{n-m} \\
&= \left(b^{5-1}\right)^{-2} = \left(b^4\right)^{-2} && \text{apply } (x^n)^m = x^{nm} \\
&= b^{4(-2)} = b^{-8} = \frac{1}{b^8}
\end{aligned}
$$

35. $\dfrac{\left(x^{-2}\right)^{-2}y^3x^0\left(-2yx^0y^{-2}x^{-2}\right)^0}{yx^5\left(y^{-2}x\right)^{-3}\left(2x^{-1}yx^3\right)^{-1}}$

Solution:

$$
\begin{aligned}
E &= \frac{\left(x^{-2}\right)^{-2}y^3x^0\left(-2yx^0y^{-2}x^{-2}\right)^0}{yx^5\left(y^{-2}x\right)^{-3}\left(2x^{-1}yx^3\right)^{-1}} && \text{apply } a^0 = 1 \text{ and } a^n a^m = a^{n+m} \\
&= \frac{\left(x^{-2}\right)^{-2}y^3}{yx^5\left(y^{-2}x\right)^{-3}\left(2x^{-1+3}y\right)^{-1}} && \text{apply } (ab)^n = a^n b^n \\
&= \frac{\left(x^{-2}\right)^{-2}y^3}{yx^5\left(y^{-2}\right)^{-3}x^{-3}\left(2x^2y\right)^{-1}} && \text{apply } (ab)^n = a^n b^n \text{ and } a^n a^m = a^{n+m} \\
&= \frac{\left(x^{-2}\right)^{-2}y^3}{yx^{5+(-3)}\left(y^{-2}\right)^{-3}\left(2\right)^{-1}\left(x^2\right)^{-1}y^{-1}} && \text{apply } (a^n)^m = a^{nm} \\
&= \frac{x^{-2(-2)}y^3}{yx^2y^{-2(-3)}2^{-1}x^{2(-1)}y^{-1}} = \frac{x^4y^3}{2^{-1}yx^2y^6x^{-2}y^{-1}} && \text{apply } a^n a^m = a^{n+m} \\
&= \frac{x^4y^3}{2^{-1}y^{1+6+(-1)}x^{2+(-2)}} = \frac{x^4y^3}{2^{-1}y^6x^0} && x^0 = 1 \text{ and } \frac{a^n}{a^m} = a^{n-m} \\
&= \frac{x^4y^{3-6}}{2^{-1}} = \frac{x^4y^{-3}}{2^{-1}} && \text{apply } a^{-n} = \frac{1}{a^n} \\
&= \frac{2^1x^4}{y^3} = \frac{2x^4}{y^3}
\end{aligned}
$$

Solutions for 18.1 – Linear Equation 3

1. $2x+3=4x+9$

   Solution:

$$\begin{aligned} 2x+3 &= 4x+9 && \text{subtract } 2x \text{ from both sides} \\ 3 &= 2x+9 && \text{subtract } 9 \text{ from both sides} \\ -6 &= 2x && \text{divide both sides by 2} \\ -3 &= x \end{aligned}$$

   We check: if $x=-3$, then

$$\begin{aligned} \text{LHS} &= 2(-3)+3=-6+3=-3 \\ \text{RHS} &= 4(-3)+9=-12+9=-3 \end{aligned}$$

   Thus our solution, $\boxed{x=-3}$ is correct. (Note: LHS is short for the left-hand side and RHS is short for the right-hand side.)

2. $3w-5=5(w+1)$

   Solution: we first apply the law of distributivity to simplify the right-hand side.

$$\begin{aligned} 3w-5 &= 5(w+1) \\ 3w-5 &= 5w+5 && \text{subtract } 3w \text{ from both sides} \\ -5 &= 2w+5 && \text{subtract } 5 \text{ from both sides} \\ -10 &= 2w && \text{divide both sides by 2} \\ -5 &= w \end{aligned}$$

   We check. If $w=-5$, then

$$\begin{aligned} \text{LHS} &= 3(-5)-5=-15-5=-20 \\ \text{RHS} &= 5((-5)+1)=5(-4)=-20 \end{aligned}$$

   Thus our solution, $\boxed{w=-5}$ is correct.

3. $3y-9=-2y+4$

   Solution:

$$\begin{aligned} 3y-9 &= -2y+4 && \text{add } 2y \text{ to both sides} \\ 5y-9 &= 4 && \text{add } 9 \text{ to both sides} \\ 5y &= 13 && \text{divide both sides by } 5 \\ y &= \frac{13}{5} \end{aligned}$$

We check. If $y = \frac{13}{5}$, then

$$\begin{aligned} \text{LHS} &= 3\left(\frac{13}{5}\right) - 9 = \frac{3}{1}\cdot\frac{13}{5} - 9 = \frac{39}{5} - \frac{9}{1} = \frac{39}{5} - \frac{45}{5} = \frac{-6}{5} = -\frac{6}{5} \\ \text{RHS} &= -2\left(\frac{13}{5}\right) + 4 = \frac{-2}{1}\cdot\frac{13}{5} + \frac{4}{1} = \frac{-26}{5} + \frac{20}{5} = \frac{-6}{5} = -\frac{6}{5} \end{aligned}$$

Thus $\boxed{y = \frac{13}{5}}$ is the correct solution.

4. $4 - x = 3(x - 7)$

   Solution: We first apply the law of distributivity to simplify the right-hand side.

$$\begin{aligned} 4 - x &= 3(x-7) && \text{distribute } 3 \\ 4 - x &= 3x - 21 && \text{add } x \text{ to both sides} \\ 4 &= 4x - 21 && \text{add } 21 \text{ to both sides} \\ 25 &= 4x && \text{divide both sides by } 4 \\ \frac{25}{4} &= x \end{aligned}$$

   We check. If $x = \frac{25}{4}$, then

$$\begin{aligned} \text{LHS} &= 4 - x = 4 - \frac{25}{4} = \frac{4}{1} - \frac{25}{4} = \frac{16}{4} - \frac{25}{4} = \frac{16-25}{4} = \frac{-9}{4} = -\frac{9}{4} \\ \text{RHS} &= 3(x-7) = 3\left(\frac{25}{4} - 7\right) = 3\left(\frac{25}{4} - \frac{7}{1}\right) = 3\left(\frac{25}{4} - \frac{28}{4}\right) = 3\left(\frac{25-28}{4}\right) \\ &= 3\left(\frac{-3}{4}\right) = \frac{3}{1}\cdot\frac{-3}{4} = \frac{-9}{4} = -\frac{9}{4} \end{aligned}$$

   Thus our solution, $\boxed{x = \frac{25}{4}}$ is correct.

5. $7(j-5) + 9 = 2(-2j+5) + 5j$

   Solution:

$$\begin{aligned} 7(j-5) + 9 &= 2(-2j+5) + 5j && \text{distribute on both sides} \\ 7j - 35 + 9 &= -4j + 10 + 5j && \text{combine like terms} \\ 7j - 26 &= j + 10 && \text{subtract } j \\ 6j - 26 &= 10 && \text{add 26} \\ 6j &= 36 && \text{divide by 6} \\ j &= 6 \end{aligned}$$

   We check: if $j = 6$, then

$$\begin{aligned} \text{LHS} &= 7(6-5) + 9 = 7\cdot 1 + 9 = 7 + 9 = 16 \\ \text{RHS} &= 2(-2\cdot 6 + 5) + 5\cdot 6 = 2(-12+5) + 30 = 2(-7) + 30 = -14 + 30 = 16 \end{aligned}$$

   Thus our solution, $\boxed{j = 6}$ is correct.

6. $3(x-5)-5(x-1)=-2x+1$

Solution:

$$\begin{aligned}3(x-5)-5(x-1) &= -2x+1 && \text{multiply out parentheses}\\ 3x-15-5x+5 &= -2x+1 && \text{combine like terms}\\ -2x-10 &= -2x+1 && \text{add } 2x\\ -10 &= 1\end{aligned}$$

Since x disappeared from the equation and we are left with an unconditionally false statement, there is no solution for this equation. This type of an equation is called a **contradiction**.

7. $\dfrac{3-x}{4}-\dfrac{10-3x}{5}=x+2$

Solution:

$$\begin{aligned}\frac{3-x}{4}-\frac{10-3x}{5} &= x+2 && \text{make everything a fraction}\\ \frac{3-x}{4}-\frac{10-3x}{5} &= \frac{x+2}{1} && \text{common denominator}\\ \frac{5(3-x)}{20}-\frac{4(10-3x)}{20} &= \frac{20(x+2)}{20} && \text{multiply by 20}\\ 5(3-x)-4(10-3x) &= 20(x+2) && \text{distribute}\\ 15-5x-40+12x &= 20x+40 && \text{combine like terms}\\ 7x-25 &= 20x+40 && \text{subtract } 7x\\ -25 &= 13x+40 && \text{subtract } 40\\ -65 &= 13x && \text{divide by 13}\\ -5 &= x\end{aligned}$$

We check:

$$\begin{aligned}\text{LHS} &= \frac{3-(-5)}{4}-\frac{10-3(-5)}{5}=\frac{8}{4}-\frac{25}{5}=2-5=-3\\ \text{RHS} &= -5+2=-3\end{aligned}$$

Thus our solution, $\boxed{x=-5}$ is correct.

8. $\dfrac{3x+17}{2}=x-1+\dfrac{x+19}{2}$

Solution:

$$\begin{aligned}\frac{3x+17}{2} &= x-1+\frac{x+19}{2} && \text{express everything as a fraction}\\ \frac{3x+17}{2} &= \frac{x-1}{1}+\frac{x+19}{2} && \text{bring everything to the common denominator}\\ \frac{3x+17}{2} &= \frac{2(x-1)}{2}+\frac{x+19}{2} && \text{add fractions on right hand side}\end{aligned}$$

$$\begin{aligned}
\frac{3x+17}{2} &= \frac{2(x-1)+x+19}{2} && \text{multiply out parentheses}\\
\frac{3x+17}{2} &= \frac{2x-2+x+19}{2} && \text{combine like terms}\\
\frac{3x+17}{2} &= \frac{3x+17}{2} && \text{multiply by 2}\\
3x+17 &= 3x+17
\end{aligned}$$

Because the left hand side is now identical to the right hand side, this equation is an identity, and $\boxed{\text{all real numbers}}$ are solution.

9. $\frac{2}{3}(x-1) = \frac{3}{5}(x-4)+1$

Solution: We re-write the expressions as fractions.

$$\begin{aligned}
\frac{2(x-1)}{3} &= \frac{3(x-4)}{5}+\frac{1}{1} && \text{common denominator is 15}\\
\frac{5\cdot 2(x-1)}{5\cdot 3} &= \frac{3\cdot 3(x-4)}{3\cdot 5}+\frac{15}{15} && \\
\frac{10(x-1)}{15} &= \frac{9(x-4)}{15}+\frac{15}{15} && \text{multiply by 15}\\
10(x-1) &= 9(x-4)+15 && \text{distribute}\\
10x-10 &= 9x-36+15 && \text{combine like terms}\\
10x-10 &= 9x-21 && \text{subtract } 9x\\
x-10 &= -21 && \text{add 10}\\
x &= -11
\end{aligned}$$

We check. If $x = -11$, then

$$\begin{aligned}
\text{LHS} &= \frac{2}{3}(-11-1) = \frac{2}{3}(-12) = -8\\
\text{RHS} &= \frac{3}{5}(-11-4)+1 = \frac{3}{5}(-15)+1 = -9+1 = -8
\end{aligned}$$

Thus our solution, $\boxed{x=-11}$ is correct.

10. $\frac{2}{3}(x-7) = \frac{4}{5}(x+1)$

Solution:

$$\begin{aligned}
\frac{2}{3}(x-7) &= \frac{4}{5}(x+1)\\
\frac{2}{3}\cdot\frac{x-7}{1} &= \frac{4}{5}\cdot\frac{x+1}{1}\\
\frac{2(x-7)}{3} &= \frac{4(x+1)}{5} && \text{bring fractions to common denominator}\\
\frac{5\cdot 2(x-7)}{15} &= \frac{3\cdot 4(x+1)}{15} && \text{multiply both sides by 15}
\end{aligned}$$

$$\begin{aligned} 10(x-7) &= 12(x+1) && \text{multiplty out parentheses} \\ 10x-70 &= 12x+12 && \text{subtract } 10x \\ -70 &= 2x+12 && \text{subtract } 12 \\ -82 &= 2x && \text{divide by } 2 \\ -41 &= x \end{aligned}$$

We check:

$$\begin{aligned} \text{LHS} &= \frac{2}{3}(-41-7) = \frac{2}{3}(-48) = -32 \\ \text{RHS} &= \frac{4}{5}(-41+1) = \frac{4}{5}(-40) = -32 \end{aligned}$$

Thus our solution, $\boxed{x=-41}$ is correct.

11. $\dfrac{x+2}{4} - \dfrac{x-3}{5} = 20 - x$

Solution:

$$\begin{aligned} \frac{x+2}{4} - \frac{x-3}{5} &= 20-x && \text{make everything a fraction} \\ \frac{x+2}{4} - \frac{x-3}{5} &= \frac{20-x}{1} && \text{common denominator is } 20 \\ \frac{5(x+2)}{20} - \frac{4(x-3)}{20} &= \frac{20(20-x)}{20} && \text{multiply by } 20 \\ 5(x+2)-4(x-3) &= 20(20-x) && \text{distribute} \\ 5x+10-4x+12 &= 400-20x && \text{combine like terms} \\ x+22 &= -20x+400 && \text{add } 20x \\ 21x+22 &= 400 && \text{subtract } 22 \\ 21x &= 378 && \text{divide by } 21 \\ x &= 18 \end{aligned}$$

We check. If $x = 18$, then

$$\begin{aligned} \text{LHS} &= \frac{18+2}{4} - \frac{18-3}{5} = \frac{20}{4} - \frac{15}{5} = 5-3 = 2 \\ \text{RHS} &= 20-18 = 2 \end{aligned}$$

Thus $\boxed{x=18}$ is indeed the solution.

12. $(x-3)^2-(2x-5)(x+1)=5-(x-1)^2$

Solution: We first multiply the polynomials as indicated. If the product is subtracted or further multiplied, we must keep the parentheses.

$$\begin{array}{rcll}
(x-3)^2-(2x-5)(x+1) & = & 5-(x-1)^2 & \\
x^2-3x-3x+9-\left(2x^2+2x-5x-5\right) & = & 5-\left(x^2-x-x+1\right) & \text{combine like terms} \\
x^2-6x+9-\left(2x^2-3x-5\right) & = & 5-\left(x^2-2x+1\right) & \text{distribute} \\
x^2-6x+9-2x^2+3x+5 & = & 5-x^2+2x-1 & \text{combine like terms} \\
-x^2-3x+14 & = & -x^2+2x+4 & \text{add } x^2 \\
-3x+14 & = & 2x+4 & \text{add } 3x \\
14 & = & 5x+4 & \text{subtract } 4 \\
10 & = & 5x & \text{divide by } 5 \\
2 & = & x &
\end{array}$$

We check. If $x=2$, then

$$\begin{array}{rcl}
\text{LHS} & = & (2-3)^2-(2\cdot 2-5)(2+1)=(-1)^2-(4-5)(2+1)=(-1)^2-(-1)\cdot 3 \\
& = & 1-(-3)=4 \\
\text{RHS} & = & 5-(2-1)^2=5-1^2=5-1=4
\end{array}$$

Thus $\boxed{x=2}$ is indeed the solution.

13. $(x+1)^2-(2x-1)^2+(3x)^2=6x(x-2)$

Solution: We first multiply the polynomials as indicated. If the product is subtracted or further multiplied, we must keep the parentheses.

$$\begin{array}{rcll}
(x+1)^2-(2x-1)^2+(3x)^2 & = & 6x(x-2) & \\
x^2+x+x+1-\left(4x^2-2x-2x+1\right)+9x^2 & = & 6x^2-12x & \\
x^2+2x+1-\left(4x^2-4x+1\right)+9x^2 & = & 6x^2-12x & \text{distribute} \\
x^2+2x+1-4x^2+4x-1+9x^2 & = & 6x^2-12x & \text{combine like terms} \\
6x^2+6x & = & 6x^2-12x & \text{subtract } 6x^2 \\
6x & = & -12x & \text{add } 12x \\
18x & = & 0 & \text{divide by } 18 \\
x & = & 0 &
\end{array}$$

We check. If $x=0$, then

$$\begin{array}{rcl}
\text{LHS} & = & (0+1)^2-(2\cdot 0-1)^2+(3\cdot 0)^2=1^2-(-1)^2+(0)^2 \\
& = & 1-1+0=0 \\
\text{RHS} & = & 6\cdot 0\cdot(0-2)=6\cdot 0\cdot(-2)=0
\end{array}$$

Thus $\boxed{x=0}$ is indeed the solution.

14. $12-(2p-1)(p+1)=-2(-p+5)^2$

Solution: We first multiply the polynomials as indicated. If the product is subtracted or further multiplied, we must keep the parentheses.

$$\begin{aligned}
12-(2p-1)(p+1) &= -2(-p+5)^2 \\
12-\left(2p^2+2p-p-1\right) &= -2\left(p^2-5p-5p+25\right) && \text{combine like terms} \\
12-\left(2p^2+p-1\right) &= -2\left(p^2-10p+25\right) && \text{distribute} \\
12-2p^2-p+1 &= -2p^2+20p-50 && \text{combine like terms} \\
-2p^2-p+13 &= -2p^2+20p-50 && \text{add } 2p^2 \\
-p+13 &= 20p-50 && \text{add } p \\
13 &= 21p-50 && \text{add } 50 \\
63 &= 21p && \text{divide by } 21 \\
3 &= p
\end{aligned}$$

We check. If $p=3$, then

$$\begin{aligned}
\text{LHS} &= 12-(2\cdot 3-1)(3+1)=12-(6-1)(3+1)=12-5\cdot 4=12-20=-8 \\
\text{RHS} &= -2(-3+5)^2=-2\cdot 2^2=-2\cdot 4=-8
\end{aligned}$$

Thus $\boxed{p=3}$ is indeed the solution.

15. If we increase all sides of a square by 3 units, the area of the square increase by 75 units. How long are the sides of the square?

Solution: Let us denote the side of the square by x. Then the increased side is $x+3$. The equation will express the difference between the areas.

$$\begin{aligned}
(x+3)^2 &= x^2+75 && \text{expand complete square} \\
x^2+6x+9 &= x^2+75 && \text{subtract } x^2 \\
6x+9 &= 75
\end{aligned}$$

This equation started out as a degree two equation but the quadratic terms cancelled out, and we are left with a linear equation.

$$\begin{aligned}
6x+9 &= 75 && \text{subtract } 9 \\
6x &= 66 && \text{divide by } 6 \\
x &= 11
\end{aligned}$$

We check: If a square has sides 11 units long, its area is 121 unit^2. After the increase, the sides are 14 units long, and so the area is 196 unit^2. The difference between 196 and 121 is indeed 75, so our solution is correct. The sides of the square were $\boxed{\text{11 units long}}$.

Solutions for 20.2 – Rational Expressions

1. Simplify each of the following.

a) $\dfrac{2a-5}{5-2a}$

Solution: We need to notice that the numerator and denominator are opposites of each other. Indeed, the opposite of $2a-5$ is $5-2a$ since

$$-1(2a-5) = -2a+5 = 5-2a$$

Thus

$$\frac{2a-5}{5-2a} = \frac{2a-5}{-1(2a-5)} = \frac{\cancel{(2a-5)}}{-1\cancel{(2a-5)}} = \frac{1}{-1} = \boxed{-1}$$

b) $\dfrac{x^3-x}{x+1}$

Solution: In general, we factor both numerator and denominator and then simplify. In this case we only factor the numerator, since the denominator is too small to factor. After we factor out the greatest common factor (or GCF) which is x, the expression factors via the difference of squares theorem.

$$x^3-x = x\left(x^2-1\right) = x(x+1)(x-1)$$

Then we simplify the fraction by canceling out the same factor from numerator and denominator.

$$\frac{x^3-x}{x+1} = \frac{x(x+1)(x-1)}{x+1} = \frac{x\cancel{(x+1)}(x-1)}{\cancel{x+1}} = \boxed{x(x-1) \text{ or } x^2-x}$$

c) $\dfrac{2x+1}{4x^2-1}$

Solution: We factor the denominator via the difference of squares theorem, and then cancel.

$$\frac{2x+1}{4x^2-1} = \frac{2x+1}{(2x+1)(2x-1)} = \frac{\cancel{2x+1}}{\cancel{(2x+1)}(2x-1)} = \boxed{\frac{1}{2x-1}}$$

d) $\dfrac{x^2-4x+3}{x^2+2x-15}$

Solution: We factor both numerator and denominator and then simplify. We can easily factor both of these polynomials by trial and error.

$$\frac{x^2-4x+3}{x^2+2x-15} = \frac{(x-3)(x-1)}{(x+5)(x-3)} = \frac{\cancel{(x-3)}(x-1)}{(x+5)\cancel{(x-3)}} = \boxed{\frac{x-1}{x+5}}$$

e) $\dfrac{(x+5)-2}{5(x+2)-(x-2)}$

Solution: We simplify both numerator and denominator, then if possible, factor these and then simplify the fraction by cancellation.

$$\frac{(x+5)-2}{5(x+2)-(x-2)} = \frac{x+5-2}{5x+10-x+2} = \frac{x+3}{4x+12} = \frac{x+3}{4(x+3)} = \frac{\cancel{x+3}}{4\cancel{(x+3)}} = \boxed{\frac{1}{4}}$$

2. Perform the indicated operations and simplify.

a) $\dfrac{c}{5a}\cdot\dfrac{15a^2b}{3b^2c}$

Solution: We perform the multiplication among fractions (top by top, bottom by bottom) and then simplify by canceling factors appearing in both the numerator and denominator.

$$\frac{c}{5a}\cdot\frac{15a^2b}{3b^2c}=\frac{15a^2bc}{15ab^2c}=\frac{15abc\,a}{15abc\,b}=\frac{\cancel{15abc}\,a}{\cancel{15abc}\,b}=\boxed{\frac{a}{b}}$$

b) $\dfrac{5x-30}{x^2-36}\cdot\dfrac{3x+18}{5}$

Solution: we will factor whatever we can and then simplify by canceling factors appearing in both the numerator and denominator.

$$\frac{5x-30}{x^2-36}\cdot\frac{3x+18}{5}=\frac{5(x-6)}{(x+6)(x-6)}\cdot\frac{3(x+6)}{5}=\frac{\cancel{5}\cancel{(x-6)}}{(x+6)\cancel{(x-6)}}\cdot\frac{3(x+6)}{\cancel{5}}=\frac{1}{\cancel{x+6}}\cdot\frac{3\cancel{(x+6)}}{1}=\boxed{3}$$

c) $\dfrac{x^2-3x}{x^2-8x+15}\cdot\dfrac{x^2-16x+15}{x^2-x}$

Solution: We factor whatever we can and then simplify by canceling factors appearing in both the numerator and denominator. We can factor all of these polynomials by completing the square or by factoring out the greatest common factor.

$$\begin{aligned}\frac{x^2-3x}{x^2-8x+15}\cdot\frac{x^2-16x+15}{x^2-x} &= \frac{x(x-3)}{(x-3)(x-5)}\cdot\frac{(x-1)(x-15)}{x(x-1)}=\frac{x\cancel{(x-3)}}{\cancel{(x-3)}(x-5)}\cdot\frac{\cancel{(x-1)}(x-15)}{x\cancel{(x-1)}}\\ &= \frac{\cancel{x}}{x-5}\cdot\frac{x-15}{\cancel{x}}=\boxed{\frac{x-15}{x-5}}\end{aligned}$$

d) $\dfrac{x^2-9}{x^2-4x-21}\div\dfrac{4x-12}{3x-21}$

Solution: We first re-write the division as multiplication by the reciprocal.

$$\frac{x^2-9}{x^2-4x-21}\div\frac{4x-12}{3x-21}=\frac{x^2-9}{x^2-4x-21}\cdot\frac{3x-21}{4x-12}$$

We now factor the polynomials appearing in the fractions

$$\begin{aligned}x^2-9 &= (x+3)(x-3) \qquad & 4x-12=4(x-3)\\ x^2-4x-21 &= (x+3)(x-7) \qquad & 3x-21=3(x-7)\end{aligned}$$

We now re-write the fractions using these factored forms, and cancel out factors appearing in both numerator and denominator of the product.

$$\begin{aligned}\frac{x^2-9}{x^2-4x-21}\cdot\frac{3x-21}{4x-12} &= \frac{(x+3)(x-3)}{(x+3)(x-7)}\cdot\frac{3(x-7)}{4(x-3)}=\frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}(x-7)}\cdot\frac{3(x-7)}{4(x-3)}=\frac{3(x-3)(x-7)}{4(x-3)(x-7)}\\ &= \frac{3\cancel{(x-3)}\cancel{(x-7)}}{4\cancel{(x-3)}\cancel{(x-7)}}=\boxed{\frac{3}{4}}\end{aligned}$$

e) $\dfrac{x^2-10x+25}{x^2-10x+24}\left(\dfrac{x^2-2x-8}{x^2-6x+5}\div\dfrac{x-5}{x-1}\right)$

Solution: We first re-write the division as multiplication by the reciprocal.

$$\frac{x^2-10x+25}{x^2-10x+24}\left(\frac{x^2-2x-8}{x^2-6x+5}\div\frac{x-5}{x-1}\right)=\frac{x^2-10x+25}{x^2-10x+24}\left(\frac{x^2-2x-8}{x^2-6x+5}\cdot\frac{x-1}{x-5}\right)$$

We now factor the polynomials appearing in each fraction.

$$\begin{aligned} x^2-10x+25 &= (x-5)^2 \qquad & x^2-2x-8=(x+2)(x-4)\\ x^2-10x+24 &= (x-4)(x-6) \qquad & x^2-6x+5=(x-1)(x-5)\end{aligned}$$

We now re-write the problem, using the factored form of polynomials.

$$\frac{x^2-10x+25}{x^2-10x+24}\left(\frac{x^2-2x-8}{x^2-6x+5}\cdot\frac{x-1}{x-5}\right)=\frac{(x-5)^2}{(x-4)(x-6)}\left(\frac{(x+2)(x-4)}{(x-1)(x-5)}\cdot\frac{x-1}{x-5}\right)$$

We now perform the multiplication within the parentheses. Notice that we can cancel out $x-1$.

$$\begin{aligned} &= \frac{(x-5)^2}{(x-4)(x-6)}\left(\frac{(x+2)(x-4)}{\cancel{(x-1)}(x-5)}\cdot\frac{\cancel{x-1}}{x-5}\right)=\frac{(x-5)^2}{(x-4)(x-6)}\cdot\frac{(x+2)(x-4)}{(x-5)^2}\\ &= \frac{\cancel{(x-5)^2}}{(x-4)(x-6)}\cdot\frac{(x+2)(x-4)}{\cancel{(x-5)^2}}=\frac{1}{\cancel{(x-4)}(x-6)}\cdot\frac{(x+2)\cancel{(x-4)}}{1}\\ &= \boxed{\frac{x+2}{x-6}}.\end{aligned}$$

f) Recall that to divide is to multiply by the reciprocal.

$$\frac{6x^2y^2-36x^2y+48x^2}{py^2-2py-8p}\div\frac{-3y^2+24y-36}{p^2y+2p^2}=\frac{6x^2y^2-36x^2y+48x^2}{py^2-2py-8p}\cdot\frac{p^2y+2p^2}{-3y^2+24y-36}$$

As before, we start with four factoring exercises. As always, we sart with the GCF.

$$6x^2y^2-36x^2y+48x^2=6x^2\left(y^2-6y+8\right)=6x^2(y-2)(y-4)$$

$$py^2-2py-8p=p\left(y^2-2y-8\right)=p(y+2)(y-4)$$

$$p^2y+2p^2=p^2(y+2)$$

$$-3y^2+24y-36=-3\left(y^2-8y+12\right)=-3(y-2)(y-6)$$

$$\frac{6x^2y^2-36x^2y+48x^2}{py^2-2py-8p}\cdot\frac{p^2y+2p^2}{-3y^2+24y-36}=\frac{6x^2(y-2)(y-4)}{p(y+2)(y-4)}\cdot\frac{p^2(y+2)}{-3(y-2)(y-6)}$$

$$=\frac{6x^2(y-2)\cancel{(y-4)}}{p(y+2)\cancel{(y-4)}}\cdot\frac{p^2(y+2)}{-3(y-2)(y-6)}=\frac{6x^2(y-2)}{p\cancel{(y+2)}}\cdot\frac{p^2\cancel{(y+2)}}{-3(y-2)(y-6)}$$

$$=\frac{6x^2\cancel{(y-2)}}{\cancel{p}}\cdot\frac{\cancel{p}\cdot p}{-3\cancel{(y-2)}(y-6)}=\frac{2\cdot\cancel{3}x^2}{1}\cdot\frac{p}{-\cancel{3}(y-6)}=\frac{2x^2}{1}\cdot\frac{p}{-(y-6)}=\boxed{\frac{-2px^2}{y-6}}$$

Solutions for 21.2 – The Pythagorean Theorem

1. Could the three line segments given be the three sides of a right triangle? Explain your answer.

   a) 6 cm, 10 cm, and 8 cm

   Solution: The longest side is 10 cm long. Thus, only this side can be the hypotenuse. We use the Pythagorean theorem to check for a right angle:

   $$6^2+8^2 \stackrel{?}{=} 10^2$$

   We get that the two quantities are equal, thus this triangle has a right angle.

   b) 4 m, 5 m, and 6 m

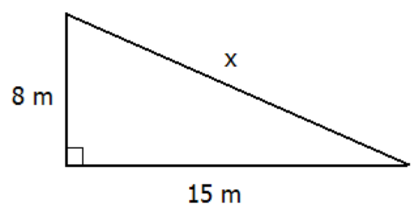
   Solution: The longest side is 6 m long. Thus, only this side can be the hypotenuse. We use the Pythagorean theorem to check for a right angle:

   $$4^2+5^2 \stackrel{?}{=} 6^2$$

   $$\begin{aligned} \text{LHS} &= 4^2+5^2=16+25=41 \\ \text{RHS} &= 6^2=36 \qquad\qquad \text{LHS} \neq \text{RHS} \end{aligned}$$

   We get that the two quantities are not equal, thus this triangle does not have a right angle.

2. Find the hypotenuse of the triangle shown on the figure.



   Solution: We apply the Pythagorean theorem. The longest side is always the one opposite the right angle.

   $$\begin{aligned} 8^2+15^2 &= x^2 \\ 289 &= x^2 \\ \pm 17 &= x \end{aligned}$$

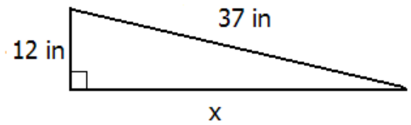
   Since distance can not be negative, -17 is ruled out. The answer is $17\,\text{m}$.

   Please note that the step taking us from $x^2=289$ to $x=\pm 17$ is a very nice shortcut. The tradiotional way of solving quadratic equations is to reduce one side to zero, factor, and apply the zero product rule.

   $$\begin{aligned} x^2 &= 289 \\ x^2-289 &= 0 \\ x^2-17^2 &= 0 \\ (x+17)(x-17) &= 0 \quad \Longrightarrow \quad x=-17 \text{ or } x=17 \end{aligned}$$

   Students are encouraged to use the shorter version, **as long as they don't make the serious algebraic error** of concluding from $x^2=289$ that $x=17$. While in the context of the geometry the negative solution is not possible, the equation $x^2=289$ has two solutions, 17 and -17.

3. Find the missing leg of the right triangle shown on the picture.



Solution: We apply the Pythagorean theorem. The longest side is always the one opposite the right angle.

$$\begin{aligned} (12\,\text{in})^2 + x^2 &= (37\,\text{in})^2 \\ x^2 + 144\,\text{in}^2 &= 1369\,\text{in}^2 && \text{subtract } 144\,\text{in}^2 \\ x^2 &= 1225\,\text{in}^2 && \sqrt{1225} = 35 \\ x &= \pm 35\,\text{in} \end{aligned}$$

Since distance can not be negative, $-35\,\text{in}$ is ruled out. The answer is 35 inches.

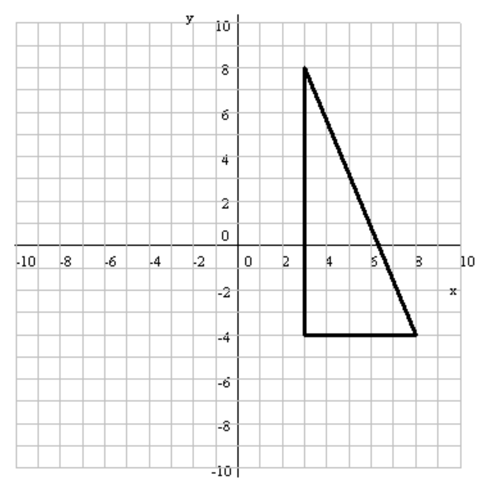
4. Find the distance between the points $(3,8)$ and $(8,-4)$.

Solution: We graph the points, and draw a horizontal and vertical line connecting the points as shown on the picture. We can compute the distance as the hypotenuse of the right triangle we created. How long are the shorter sides?

Algebraic approach: Subtract the coordinates. The length of the horizontal side is the difference between the $x-$coordinates: $8-3=5$ and the length of the vertical side is the difference between the $y-$coordinates: $8-(-4)=12$.

The difference will alwys work. Even if we get -5 instead of 5, it will not matter since we will square it in the Pythagorean theorem.

Geometric approach: From 3 to 8 we have to step 5 units up. From -4 to 8 : first we step 4 to get from -4 to 0. Then another 8 steps to 8, and so $4+8=12$ steps. The message here is that the algebra and geometry will always agree.



Now we know that the shorter sides are 5 and 12 units long, and we need to find the hypotenuse.

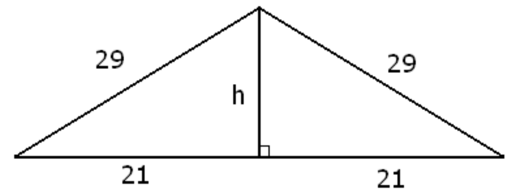
$$\begin{aligned} 5^2 + 12^2 &= x^2 \\ 25 + 144 &= x^2 \\ 169 &= x^2 \\ \pm 13 &= x \end{aligned}$$

Since distances are never negative, -13 is ruled out and so the answer is 13 units.

5. The sides of an isosceles triangle are 42 units, 29 units, and 29 units long. Find the length of the height drawn to the 42 units long side.

   Solution: In case of isosceles triangles, the height drawn to the base splits the triangle into two identical right triangles as shown on the picture. The height now can be easily computed via the Pythagorean theorem.

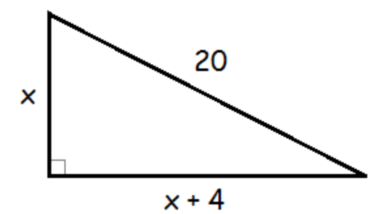
$$\begin{aligned} 21^2+h^2 &= 29^2 \\ 441+h^2 &= 841 \\ h^2 &= 400 \\ h &= \pm 20 \implies h=20 \end{aligned}$$



   Again, the negative solution of the equation is ruled out because distances cannot be negative. The height belonging to the base is 20 units long.

6. The hypotenuse of a right triangle is 20 cm. The difference between the other two sides is 4 cm. Find the sides of the triangle.

   Solution: Let x denote the shortest side. Then the other missing side is $x+4$ cm long. We state the Pythagorean theorem for the triangle and solve the quadratic equation for x.



$$\begin{aligned} x^2+(x+4)^2 &= 20^2 && \text{expand } (x+4)^2 \\ x^2+x^2+8x+16 &= 400 && \text{combine like terms} \\ 2x^2+8x+16 &= 400 && \text{subtract 400} \\ 2x^2+8x-384 &= 0 && \text{factor out 2} \\ 2\left(x^2+4x-192\right) &= 0 \end{aligned}$$

   We will factor $x^2+4x-192$ by trial and error. The negative sign in front of 192 indicates that one number is positive, the other is negative. Therefore, 4 is the difference of those two numbers. We list all pairs of factors for 192 until we find a pair with difference 4.

192		
1	192	difference is 191
2	96	difference is 94
3	64	difference is 61
4	48	difference is 44
6	32	difference is 26
8	24	difference is 16
12	16	This is the one!

   Thus $x^2+4x-192$ can be factored as $(x+16)(x-12)$. We apply the zero product rule.

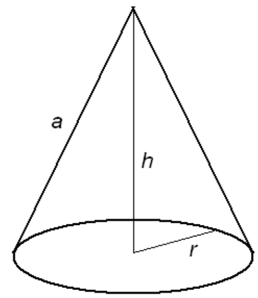
$$\begin{aligned} 2\left(x^2+4x-192\right) &= 0 \\ 2(x+16)(x-12) &= 0 \end{aligned}$$

$$x_1=-16,\ x_2=12$$

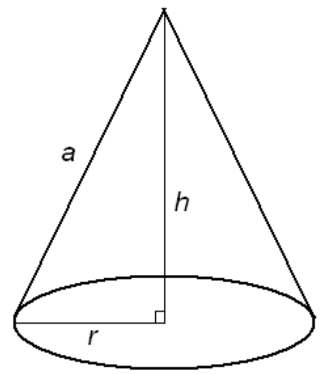
   Since distances are never negative, -16 is ruled out. If the shortest side is 12 cm, the other side is $12\,\text{cm}+4\,\text{cm}=16\,\text{cm}$. Thus the solution is 12 cm and 16 cm. We check:

$$16-12=4\ \checkmark \text{ and } 16^2+12^2=256+144=400=20^2\ \checkmark$$

7. Find the height h of the cone shown on the picture, if the base has a radius of $10\,\mathrm{m}$ and $a = 26\,\mathrm{m}$.



Solution: There is a right triangle formed by a, h, and r as the picture shows. We state the Pythagorean theorem for this triangle and solve for h.



$$\begin{aligned} r^2 + h^2 &= a^2 \\ 10^2 + h^2 &= 26^2 \\ h^2 + 100 &= 676 \\ h^2 &= 576 \\ h &= \pm 24 \end{aligned}$$

The negative value is ruled out, and so the height is $24\,\mathrm{m}$.

Solutions for 23.1 – Factoring the Difference and Sum of Cubes

Completely factor each of the following.

1. $x^3 - 8y^3$

   Solution: We will factor via the difference of cubes theorem,
   $A^3 - B^3 = (A-B)\left(A^2 + AB + B^2\right)$. In this case, $A = x$ and $B = 2y$.

$$\begin{aligned} x^3 - 8y^3 &= x^3 - (2y)^3 = (x - (2y))\left(x^2 + x(2y) + (2y)^2\right) \\ &= (x-2y)\left(x^2 + 2xy + 4y^2\right) \end{aligned}$$

   We check by multiplication:

$$\begin{aligned} (x-2y)\left(x^2+2xy+4y^2\right) &= x\left(x^2+2xy+4y^2\right) - 2y\left(x^2+2xy+4y^2\right) \\ &= x^3 + 2x^2y + 4xy^2 - 2x^2y - 4xy^2 - 8y^3 = x^3 - 8y^3 \end{aligned}$$

2. $125 - 27a^{12}$

   Solution: We first factor out -1.

$$125 - 27a^{12} = -\left(27a^{12} - 125\right)$$

   We will now factor via the difference of cubes theorem,
   $A^3 - B^3 = (A-B)\left(A^2 + AB + B^2\right)$. In this case, $A = 3a^4$ and $B = 5$.

$$\begin{aligned} 125 - 27a^{12} &= -\left(27a^{12} - 125\right) = -\left(\left(3a^4\right)^3 - 5^3\right) \\ &= -\left(\left(3a^4\right) - 5\right)\left(\left(3a^4\right)^2 + \left(3a^4\right)5 + 5^2\right) = -\left(3a^4 - 5\right)\left(9a^8 + 15a^4 + 25\right) \end{aligned}$$

   We check by multiplication:

$$\begin{aligned} -\left(3a^4-5\right)\left(9a^8+15a^4+25\right) &= -\left(3a^4\left(9a^8+15a^4+25\right) - 5\left(9a^8+15a^4+25\right)\right) \\ &= -\left(27a^{12} + 45a^8 + 75a^4 - 45a^8 - 75a^4 - 125\right) \\ &= -\left(27a^{12} - 125\right) = -27a^{12} + 125 = 125 - 27a^{12} \end{aligned}$$

3. $1000 + x^6$

   Solution: We will factor via the sum of cubes theorem,
   $A^3 + B^3 = (A+B)\left(A^2 - AB + B^2\right)$.

$$\begin{aligned} 1000 + x^6 &= x^6 + 1000 = \left(x^2\right)^3 + 10^3 \\ &= \left(x^2+10\right)\left(\left(x^2\right)^2 - 10x^2 + 10^2\right) = \left(x^2+10\right)\left(x^4 - 10x^2 + 100\right) \end{aligned}$$

   We check by multiplication:

$$\begin{aligned} \left(x^2+10\right)\left(x^4-10x^2+100\right) &= x^2\left(x^4-10x^2+100\right) + 10\left(x^4-10x^2+100\right) \\ &= x^6 - 10x^4 + 100x^2 + 10x^4 - 100x^2 + 1000 \\ &= x^6 + 1000 = 1000 + x^6 \end{aligned}$$

4. $(x+1)^3-27$

Solution: We will factor via the difference of cubes theorem,
$A^3-B^3=(A-B)\left(A^2+AB+B^2\right)$. In this case, $A=x+1$ and $B=3$.

$$\begin{aligned}(x+1)^3-27 &= (x+1)^3-3^3=((x+1)-3)\left((x+1)^2+3(x+1)+3^2\right)\\ &= (x-2)\left(x^2+2x+1+3x+3+9\right)=(x-2)\left(x^2+5x+13\right)\end{aligned}$$

5. $-2a^7-2a^4b^9$

Solution:

$$\begin{aligned}-2a^7-2a^4b^9 &= -2a^4\left(a^3+b^9\right)=-2a^4\left(a^3+\left(b^3\right)^3\right)\\ &= -2a^4\left(a+b^3\right)\left(a^2+ab^3+\left(b^3\right)^2\right)\\ &= -2a^4\left(a+b^3\right)\left(a^2-ab^3+b^6\right)\end{aligned}$$

6. $(a+2)^3+(a-2)^3$

Solution: We will factor via the sum of cubes theorem,
$X^3+Y^3=(X+Y)\left(X^2-XY+Y^2\right)$. In this case, $X=a+2$ and $Y=a-2$.

$$\begin{aligned}(a+2)^3+(a-2)^3 &= ((a+2)+(a-2))\left((a+2)^2+(a+2)(a-2)+(a-2)^2\right)\\ &= (a+2+a-2)\left(\left(a^2+4a+4\right)-\left(a^2-4\right)+\left(a^2-4a+4\right)\right)\\ &= (2a)\left(a^2+4a+4-a^2+4+a^2-4a+4\right)\\ &= 2a\left(a^2+12\right)=2a\left(a^2+12\right)\end{aligned}$$

7. a^6-b^6

Solution: We start by the difference of squares theorem.

$$a^6-b^6=\left(a^3+b^3\right)\left(a^3-b^3\right)$$

Now both factors will further factor, via the sum- and difference of cubes theorems. Since

$$a^3+b^3=(a+b)\left(a^2-ab+b^2\right) \quad \text{and} \quad a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

we have that

$$\begin{aligned}a^6-b^6 &= \left(a^3+b^3\right)\left(a^3-b^3\right)\\ &= (a+b)\left(a^2-ab+b^2\right)(a-b)\left(a^2+ab+b^2\right)\\ &= (a+b)(a-b)\left(a^2+ab+b^2\right)\left(a^2-ab+b^2\right)\end{aligned}$$