Test #1 AMATYC Student Mathematics League October/November 2007

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Solutions

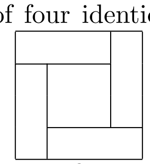
Contributors

1. One can of frozen juice concentrate, when mixed with $4\frac{1}{3}$ cans of water, makes 2 quarts (64 oz) of juice. Assuming no volume is gained or lost by mixing, how many oz does a can hold?

A 8, B 10, C 12, D 15, E 18

Solution. Let x(oz) denote the volume of 1 can. Then $x + 4\frac{1}{3} = 64 \Longrightarrow 5\frac{1}{3}x = 64 \Longrightarrow \frac{16}{3}x = 64$
$\Longrightarrow x = 12$
Answer - C

2. Define the operation Δ by $a\Delta b = ab + b$. Find $(3\Delta 2)\Delta(2\Delta 3)$
A 72, B 73, C 80, D 81, E 90
Solution. $a\Delta b = ab+b = (a+1)b \Longrightarrow 3\Delta 2 = (3+1)2 = 8$, $2\Delta 3 = (2+1)3 = 9$, $(3\Delta 2)\Delta(2\Delta 3) = 8\Delta 9 = (8+1)9 = 81$
Answer - D

3. A square is covered by a design made up of four identical rectangles surrounding a central square, as shown at the right.  If the area of the central square is 4/9 of the ares of the entire design, find the ratio of the length of a rectangle to the side of the central square.
A $\frac{5}{4}$, B $\frac{4}{3}$, C $\frac{7}{5}$, D $\frac{3}{2}$, E $\frac{8}{5}$
Solution. Let us denote by s the side of the central square, by l the length of a rectangle, and by w the width of the rectangle, and by x the side of the outside square. we have to find the ratio $l : s$. It is clear that $x = l + w$, $w = l - s$.
we have $s^2 = \frac{4}{9}x^2 \Longrightarrow s = \frac{2}{3}x = \frac{2}{3}(l + w) = \frac{2}{3}(l + (l - s)) = \frac{2}{3}(2l - s) \Longrightarrow 3s = 4l - 2s \Longrightarrow 5s = 4l$
$\Longrightarrow l : s = \frac{5}{4}$
Answer - A

4. A radio station advertises, "Traffic every 10 minutes, 24 hours a day; 1000 reports each week." What is the difference between the advertised number of reports and the exact number?
A 8, B 12, C 16, D 20, E 24
Solution. The number of minutes in a week is $60 \cdot 24 \cdot 7 = 10080$. As a report goes every 10 minutes, then the exact number of reports per week is 10080/10

$= 1008$. The difference between the advertised number of reports and the exact number is $1008 - 1000 = 8$

Answer - A

5. Trina has two dozen coins, all dimes and nickels, worth between \$1.72 and \$2.11. What is the least number of dimes she could have?

A 10, B 11, C 15, D 18, E 19

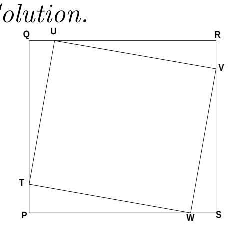
Solution. Let x be the number of dimes Trina has. Then the number of nickels is $(24 - x)$.The total money amount in cents in her possession is $A = 10x+(24-x)5 = 5x+120$. By the condition, $172 \le A \le 211$, or $172 \le 5x+120 \le 211$. After simplifications, $10.4 \le x \le 18.2$. As x is an integer, then $\min x = 11$

Answer - B

6. Square $PQRS$ has sides of length 10. Points T, U, V, and W are chosen on sides PQ, QR, RS, and SP respectively so that $PT = QU = RV = SW = 2$. Find the area of quadrilateral $TUVW$.

A 48, B 52, C 56, D 64, E 68

Solution.



Area of quadrilateral $TUVW$ = Area of $PQRS$ - (Area of PQU + Area of URV + Area of VSW + Area of WPT)$= 10^2 - 4 \cdot \frac{1}{2} \cdot 2 \cdot 8 = 68$

Answer - E

7. A bicycle travels at s feet/min. When its speed is expressed in inches/sec, the numerical vslue decreses by 16. Find s. (1 foot =12 inches)

A 12, B 16, C 18, D 20, E 24

Solution. s ft / min $= 12s$ inch / min $= \frac{12}{60}s$ inch / sec$\equiv \frac{1}{5}s$ inch / sec$\Longrightarrow$ $\frac{1}{5}s = s - 16 \Longrightarrow s = 20$

Answer - D

8. The average of A and $2B$ is 7, and the average of A and $2C$ is 8. What is the average of A, B, and C?

A 3, B 4, C 5, D 6, E 9

Solution. $(A + B + C)/3 = ((A/2 + B) + (A/2 + C))/3 = ((A + 2B)/2) + (A + 2C)/2)/3 = (7 + 8)/3 = 15/3 = 5$

Answer - C

9. Replace each letter of $AMATYC$ with a digit 0 through 9 to form a six-digit number (identical letters are replaced by identical digits, different letters are replaced by different digits). If the resulting number is the largest such number which is a perfect square, find the sum of its digits (that is, $A + M + A + T + Y + C$).

A 32, B 33, C 34, D 35, E 36

Solution.

First, I will theoretically show that the sum of its digits is either 36, or 34. I don't how to proceed with the unique choice.

Secondly, I will performe brute numerical calculations to show that the largest number is $898704 = 948^2$, and the sum of digits is 36

1. It is easy to show that $(a+b)(\text{mod}\, n) = a(\text{mod}\ n) + b(\text{mod}\ n)$

and $ab(\text{mod}\ n) = [a(\text{mod}\, n) \cdot b(\text{mod}\, n)]\,(\text{mod}\, n)$

For any number $\left[\sum\limits_{m=0}^{n} a_m 10^m\right](\text{mod}\,9) = \sum\limits_{m=0}^{n} a_m(\text{mod}\,9)10^m(\text{mod}\,9) = \sum\limits_{m=0}^{n} a_m(\text{mod}\,9)\,[10(\text{mod}\,9)]^m$

$= \sum\limits_{m=0}^{n} a_m(\text{mod}\,9)\,[1]^m = \sum\limits_{m=0}^{n} a_m(\text{mod}\,9)$

Since $AMATYC = (abc)^2$, then $(A + M + A + T + Y + C)(\text{mod}\,9) = (AMATYC)(\text{mod}\,9) = (abc)^2(\text{mod}\,9)$

$[(abc)(\text{mod}\,9)]^2\,(\text{mod}\,9)$

As $(abc)(\text{mod}\,9) \in \{0,1,2,3,4,5,6,7,8\}$,

then $(A+M+A+T+Y+C)(\text{mod}\,9) = [(abc)(\text{mod}\,9)]^2\,(\text{mod}\,9) \in \{0,1,4,7\}$

Of the list of solutions, $32(\text{mod}\,9) = 5$, $33(\text{mod}\,9) = 6$, $34(\text{mod}\,9) = 7$, $35(\text{mod}\,9) = 8$, $36(\text{mod}\,9) = 0$, so only 34 or 36 may be solutions.

2. Consider the following numerical cases

AMATYC	Upper Limit	$\sqrt{\text{Upper Limit}}$	$\lfloor\sqrt{\text{Upper Limit}}\rfloor$	$\lfloor\sqrt{\text{Upper Limit}}\rfloor^2$	Conclusion
989XXX	989999	994.99	994	988036	No Pattern
979XXX	979999	989.94	989	978121	No Pattern
969XXX	969999	984.89	984	968256	No Pattern
959XXX	959999	979.80	979	958441	No Pattern
949XXX	949999	974.68	974	948676	No Pattern
939XXX	939999	969.54	969	938961	No Pattern
929XXX	929999	964.36	964,963	929296,927369	No Pattern
919XXX	919999	959.17	959, 958	919681,917764	No Pattern
909XXX	909999	953.94	953	908209	No Pattern
898XXX	898999	948.16	948	898704	Pattern

As it follows from the table, the largest number is 898704, and the sum of the digits is $8 + 9 + 8 + 7 + 0 + 4 = 36$

Answer - E

A developed computer program found the following cases

$AMATYC$	$\sqrt{AMATYC}$	Sum of digits
898704	948	36
727609	853	31
717409	847	28
707281	841	25
636804	798	27
606841	779	25
535824	732	27
454276	674	28
434281	659	22
414736	644	25
323761	569	22
292681	541	28
181476	426	27
171396	414	27
141376	376	22
131769	363	27

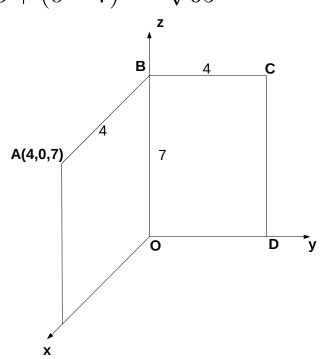
10. A door is 4 ft wide and 7 ft high. If the door is standing open at a 90° angle with the door frame, what is the greatest distance in feet from the outer top corner of the door to a point on the door frame?

A 8, B 9, C 9.5, D 10, E 11

Solution. The point at the outer top corner of the door $A = (4, 0, 7)$

1. If a point M on the door frame lies on the interval OB, then $M = (0, 0, z)$, where $0 \leq z \leq 7$

$d(A, M) = \sqrt{(0-4)^2 + (0-0)^2 + (z-7)^2} = \sqrt{16 + (z-7)^2}$. $\max d(A, M) = \sqrt{16 + (0-7)^2} = \sqrt{65}$



2. If a point M on the door frame lies on the interval BC, then $M = (0, y, 7)$, where $0 \leq y \leq 4$

$d(A, M) = \sqrt{(0-4)^2 + (y-0)^2 + (7-7)^2} = \sqrt{16 + y^2}$. $\max d(A, M) = \sqrt{16 + 4^2} = \sqrt{32}$

3. If a point M on the door frame lies on the interval CD, then $M = (0, 4, z)$, where $0 \leq z \leq 7$

$d(A,M)=\sqrt{(0-4)^2+(4-0)^2+(z-7)^2}=\sqrt{32+(z-7)^2}$. $\max d(A,M)=\sqrt{32+(0-7)^2}=\sqrt{81}$

4. If a point M on the door frame lies on the interval OD, then $M=(0,y,0)$, where $0\le y\le 4$

$d(A,M)=\sqrt{(0-4)^2+(y-0)^2+(0-7)^2}=\sqrt{65+y^2}$. $\max d(A,M)=\sqrt{65+4^2}=\sqrt{81}$

The greatest distance from the outer top corner of the door to a point on the door frame is $\sqrt{81}=9$

Answer - D

11. The class is exactly 40% female. When 3 male students are replaced by female students, ths class becomes exactly 44% female. How many more males than females are in the original class?

A 10, B 12, C 15, D 18, E 20

Solution. Detone by n the total number of the students in the class, by f the total number of the females in the original class, and by m the total number of the males in the original class. Then for the original class $f=0.4n$, and for the replaced class $f+3=0.44n$. Solving these equations together, we will get $n=75$, and $f=30$. Consequaently, $m=n-f,=75-30=45$, and $m-f=45-30=15$

Answer - C

12. A piece has 2 saxophone parts, 3 trumpet parts, and 3 trombone parts. If a band has 2 saxophonists, 3 trumpeters, and 3 trombonists, in how many ways can different parts be assigned to each player?

A 18, B 72, C 324, D 512, E 2916

Solution. $N=P_2P_3P_3=2!3!3!=72$

Answer - B

13. Add any integer N to the aquare of $2N$ to produce an integer M. For how many values of N is M prime?

A 0, B 1, C 2, D A finite number >2, E An infinite number

Solution. $M(N)=N+(2N)^2=N+4N^2=N(1+4N)$. If $M(N)$ is a prime, then $N=\pm1$. For $N=1$, $M(1)=1(1+4\cdot1)=5$. For $N=-1$, $M(-1)=-1(1+4\cdot(-1))=3$

Answer - C

14. Sixteen students in a dance have numbers 1 to 16. When they are paird up, they discover that each couple's numbers add to a perfect square. What is the largest difference between the two numbers for any couple?

A 13, B 17, C 29, D 43, E 47

Solution. We will call the pairs of numbers from 1,. . . ,16 connectible if their sum is a perfect square. The following table shows the connectible pairs.

Number	Connectible to the Numbers
1	3, 8,15
2	7,14
3	1,6,13
4	5,12
5	4,11
6	3,10
7	2,9
8	1
9	7,16
10	6,15
11	5,14
12	4,13
13	3,12
14	2,11
15	1,10
16	9

In this table number 8 is connectible only with 1, so the pair (8,1) must be formed. Difference 8-1=7.Also number 16 is connectible only with 9, so the pair (16,9) must be formed. Difference 16-9=7 Remove numbers 8,1,16,9 from the above table to obtain a new connectivity table.

Number	Connectible to the Numbers
2	7,14
3	6,13
4	5,12
5	4,11
6	3,10
7	2
10	6,15
11	5,14
12	4,13
13	3,12
14	2,11
15	10

In this table number 7 is connected only with 2 and number 15 is connectible only with 10, so the pairs (7,2) and (15,10) must be formed. Differences are 9-2=7 and 15-10=5. Remove numbers 2,7,10,15 from the above table to obtain a new connectivity table.

Number	Connectible to the Numbers
3	6,13
4	5,12
5	4,11
6	3
11	5,14
12	4,13
13	3,12
14	11

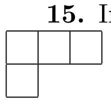
In this table number 6 is connectible only with 3 and number 14 is connectible only with 11. So the pair (6,3), (14,11) must be formed. Differences are 6–3=3, 14-11=3. Remove numbers 3,6,11,14 from the above table to obtain a new connectivity table.

Number	Connectible to the Numbers
4	5,12
5	4
12	4,13
13	12

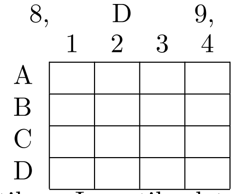
In this table number 5 is connectible only with 4 and number 13 is connectible only with 12. So the pair (5,4),(13,12) must be formed. Differences are 5-4=1, 13-12=1. Remove numbers 4,5,12,13 from the above table to obtain an empty table.

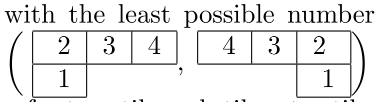
Max difference is 7.

Answer - B

15. In how many distinct ways can a 4x4 square be covered exactly by four  tiles? Assume that rotations and reflections are different coverings.

A 5, B 6, C 8, D 9, E 10

Solution. Consider 4x4 square 

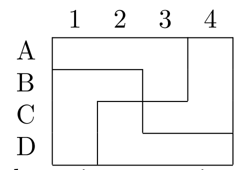
and find the covering with the least possible number of tiles. In a tile, let us refer to three cells 

lying on the same line as the tile base. We will refer to a tile as h-tile, ot v-tile depending on horizontal or vertical orientation of its base.

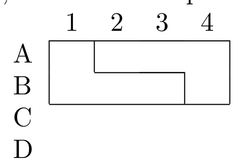
Consider possible coverings of cell (A,1) by h-tiles.

Cell (A,1) cannot be covered by cell 3 of an h-tile.

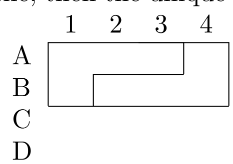
If cell (A,1) is covered by cell 4 of an h-tile, then it is easy to see that the only covering of the square is shown in the following picture



If cell (A,1) is covered by cell 1 of an h-tile, then the unique covering of rows A and B is shown in the following picture



If cell (A,1) is covered by cell 2 of an h-tile, then the unique covering of rows A and B is shown in the following picture



Similarly, considering the covering of cell (D,4) by cells 1 and 2 of an h-tiles, we will come to two unique coverings of rows C and D.

Combining possible coverings of rows A,B and rows C,D, we will come to four coverings of the square. So far, starting with h-tiles, we found five possible coverings of the square.

Repeating the above reasoning with v-tiles, we will find additional possible coverings of the square.

Thus, the total number of coverings of 4x4 square with given type of tiles is 10.

Answer - E

16. What is the smallest positive integer that cannot be the degree measure of an exterior angle of a regular polygon?

A 1, B 2, C 3, D 5, E 7

Solution. It is well known that for any regular $n-$gon the interior angle's degree measure is $\alpha = 360\frac{n-2}{n}$. Consequently, the exterior angle's degree measure is $\beta = 360 - \alpha = 360 - 360\frac{n-2}{n} = \frac{720}{n} = \frac{2^4 3^2 5}{n}$. It directly follows that $n = 7$ is the smallest integer value for which the degree measure of β is not an integer.

Answer - E

17. When certain proper fractions in simplest terms are added, the result is in simplest terms: $\frac{2}{15} + \frac{1}{21} = \frac{19}{105}$; in other cases, the result is not in simplest terms: $\frac{2}{15} + \frac{5}{21} = \frac{39}{105} = \frac{13}{35}$. Assume that $\frac{m}{15}$ and $\frac{n}{21}$ are positive proper fractions in simplest terms. For how many such fractions is $\frac{m}{15} + \frac{n}{21}$ not in simplest terms?

A 35, B 48, C 70, D 72, E 140

Solution. $\frac{m}{15} + \frac{n}{21} = \frac{7m+5n}{105}$ As $105 = 7 \cdot 5 \cdot 3$, then 7 is divided into $7m + 5n$, or 5 is divided into $7m + 5n$, or 3 is divided into $7m + 5n$

1) 7 is divided into $7m + 5n \Longrightarrow (7m + 5n)(\bmod 7) = 0$

$\Longrightarrow (7m)(\text{mod}\,7) + (5n)(\text{mod}\,7) = 0 \Longrightarrow 5 \cdot n(\text{mod}\,7) = 0$

$\Longrightarrow n(\text{mod}\,7) = 0 \Longrightarrow n = 7p \Longrightarrow \frac{n}{21} = \frac{7p}{21}$ is not in simplest terms. Contradiction

2) 5 is divided into $7m + 5n \Longrightarrow (7m + 5n)(\text{mod}\,5) = 0$

$\Longrightarrow (7m)(\text{mod}\,5) + (5n)(\text{mod}\,5) = 0 \Longrightarrow 7 \cdot m(\text{mod}\,5) = 0$

$\Longrightarrow m(\text{mod}\,5) = 0 \Longrightarrow m = 5p \Longrightarrow \frac{m}{15} = \frac{5p}{15}$ is not in simplest terms. Contradiction

3) 3 is divided into $7m + 5n \Longrightarrow (7m + 5n)(\text{mod}\,3) = 0$

$\Longrightarrow (7m)(\text{mod}\,3) + (5n)(\text{mod}\,3) = 0$

$\Longrightarrow m(\text{mod}\,3) + 2 \cdot n(\text{mod}\,3) = 0$

$\Longrightarrow m(\text{mod}\,3) = -2 \cdot n(\text{mod}\,3) \Longrightarrow m(\text{mod}\,3) = n(\text{mod}\,3)$.

Both $m(\text{mod}\,3) \neq 0$ and $n(\text{mod}\,3) \neq 0$, because otherwise $\frac{m}{15}$ or $\frac{n}{21}$ are not in simplest terms.

Thus $m(\text{mod}\,3) = n(\text{mod}\,3) = 1$ or $m(\text{mod}\,3) = n(\text{mod}\,3) = 2$

Since $1 \leq m < 15$ and $\frac{m}{15}$ is in simplest terms, then $m \in \{1, 2, 4, 7, 8, 11, 13, 14\}$.

For $m \in \{1, 4, 7, 13\}$, $m(\text{mod}\,3) = 1$. For $m \in \{2, 8, 11, 14\}$, $m(\text{mod}\,3) = 2$.

Since $1 \leq n < 21$ and $\frac{n}{21}$ is in simplest terms, then $n \in \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$.

For $n \in \{1, 4, 10, 13, 16, 19\}$, $m(\text{mod}\,3) = 1$.

For $m \in \{2, 5, 8, 11, 17, 20\}$, $m(\text{mod}\,3) = 2$.

So there are $4 \cdot 6 = 24$ pairs (m, n) for which $m(\text{mod}\,3) = n(\text{mod}\,3) = 1$

and there are $4 \cdot 6 = 24$ pairs (m, n) for which $m(\text{mod}\,3) = n(\text{mod}\,3) = 2$

The total number of pairs (m, n) for which $m(\text{mod}\,3) = n(\text{mod}\,3)$ is 48

Answer - B

18. Let r, s, and t be nonnegative integers. How many such triples (r, s, t) satisfy the system $\left\{ \begin{array}{ccccc} rs & + & t & = & 14 \\ r & + & st & = & 13 \end{array} \right.$?

A 2, B 3, C 4, D 5, E 6

Solution. Subtracting the second equation in the system from the first and simplifying the result, we will get $(s-1)(r-t) = 1$. Since all r, s, t are integers, then either

1) $s - 1 = -1$ and $r - t = -1 \Longrightarrow s = 0$ and $r = t - 1$. From the system we will find that $r = 13$, $t = 14$. The triple $(13, 0, 14)$ is a solution.

or

2) $s - 1 = 1$ and $r - t = 1 \Longrightarrow s = 2$ and $r = t + 1$.From the system we will find that $r = 5$, $t = 4$. The triple $(5, 2, 4)$ is a solution.

The total number of solutions is 2

Answer - A

19. The average of any 17 consecutive perfect squares is always k greater than a perfect square. If $k = 2^r m$, where m is odd, find r.

A 0, B 1, C 2, D 3, E 4

Solution. The sum S_p of 17 consecutive perfect squares starting with p^2 is

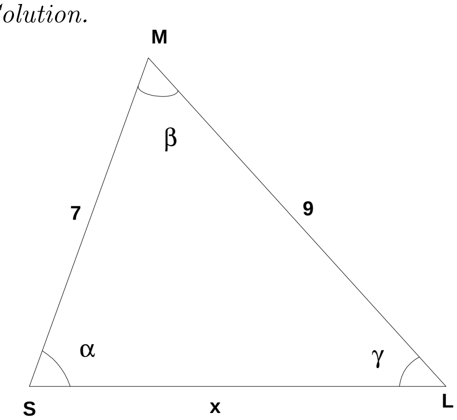
$$S_p = \sum_{m=0}^{16}(p+m)^2 = \sum_{m=0}^{16} p^2 + 2p\sum_{m=0}^{16} m + \sum_{m=0}^{16} m^2 = 17p^2 + 2p\sum_{m=1}^{16} m + \sum_{m=1}^{16} m^2$$

$= 17p^2 + 2p\frac{1+16}{2}16 + \frac{16(16+1)(2\cdot16+1)}{6} = 17(p^2+16p+88) = 17\left((p+8)^2+24\right)$
The average $A_P = \frac{S_P}{17} = (p+8)^2 + 24$
The constant $k = 24 = 3 \cdot 2^3$, so $r = 3$
Answer - A

20. In $\triangle SML$, $SM = 7$ and $ML = 9$. If $\angle M$ is exactly twice as large as $\angle S$, find SL.

A 10, B 11, C 12, D 13, E 14

Solution.



Let $\alpha = \angle S$, $\beta = \angle M$; $\gamma = \angle L$; $\beta = 2\alpha$; $\gamma = \pi - (\alpha+\beta) = \pi - (\alpha + 2\alpha) = \pi - 3\alpha$, $x = SL$

By De Moivre's formula,
$\sin 3\alpha = \text{Im}(\cos\alpha + i\sin\alpha)^3$
$= \text{Im}(\cos^3\alpha + 3i\cos^2\alpha\sin\alpha + 3(i\sin\alpha)^2\cos\alpha + (i\sin\alpha)^3)$
$= 3\cos^2\alpha\sin\alpha - \sin^3\alpha$
$= 3(1-\sin^2\alpha)\sin\alpha - \sin^3\alpha = 3\sin\alpha - 4\sin^3\alpha$
By the Law of Sines, $\frac{7}{9} = \sin\gamma/\sin\alpha$
$= \sin(\pi - 3\alpha)/\sin\alpha = \sin 3\alpha/\sin\alpha$
$= (3\sin\alpha - 4\sin^3\alpha)/\sin\alpha = 3 - 4\sin^2\alpha$
$\implies 4\sin^2\alpha = 3 - \frac{7}{9}$
$\implies \sin^2\alpha = \frac{5}{9}$
$\cos\beta = \cos 2\alpha = 1 - 2\sin^2\alpha = 1 - 2\cdot\frac{5}{9} = -\frac{1}{9}$
By the Law of Cosines,
$x^2 = 7^2 + 9^2 - 2\cdot7\cdot9\cos\beta$
$= 7^2 + 9^2 - 2\cdot7\cdot(-\frac{1}{9}) = 144$,
$x = 12$
Answer - C