Test #1 AMATYC Student Mathematics League Oct/Nov 2008

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Solutions

1. Line L has equation $y = 2x + 3$, and line M has the same y-intercept as L. Which of the points below must M contain to be perpendicular to L ?

A (-4, 5) B (-2, 5) C (-1, 5) D (1, 5) E (4, 5)

Solution. For line L, slope $m_1 = 2$ and y-intercept $b_1 = 3$. For line L, slope $m_2 = -\frac{1}{2}$ and y-intercept $b_2 = 3$. Thus, the equation of line L is $y = -\frac{1}{2}x + 3$. Direct substitution of the points A, B, C, D, E into the equation of line L shows that only point A = (-4, 5) satisfies the equation.

Answer - A

2. Sue just received a 5% raise. Now she earns \$1200 more than Lisa. Before Sue's raise, Lisa's salary was 1% higher than Sue's. What is Lisa's salary?

A \$28.000 B 29,400 C 30,000 D 30,300 E 31,200

Solution. Denote by S Sue's salary before raise, and by L Lisa's salary. Then variables S and L satisfy the equations

$1.05S = L + 1200$

$L = 1.01S$

Solving the system, we get $L = 30,300$

Answer - D

3. If $x = -1$ is one solution of $ax^2 + bx + c = 0$, what is the second solution?

A $x = -a/b$ B $x = -b/a$ C $x = b/a$ D $x = -c/a$ E $x = c/b$

Solution. It is known that $x_1 x_2 = \frac{c}{a}$. If $x_1 = -1$, then $x_2 = -\frac{c}{a}$

Another, less technical, solution. Plug $x = -1$ into the equation: $a - b + c = 0$, or $b = a + c$. Plug this expression for b into the equation. $ax^2 + (a+c)x + c = 0$. Factor the last equation. $ax^2 + ax + cx + c = 0$, $ax(x+1) + c(x+1) = 0$, $(x+1)(ax+c) = 0$. The solutions of this equation are $x = -1$ and $x = -\frac{c}{a}$

Answer - D

4. Ryan told Sam that he had 9 coins worth 45¢. Sam said, "There is more than one possibility. How many are pennies?" After Ryan answered truthfully, Sam said. Now I know what coins you have." How many nickels did Ryan have?

A 0 B 3 C 4 D 5 e 9

Solution. Let us denote the number of quarters, dimes, nickels, and pennies by respectively q, d, n, p Then

$25q + 10d + 5n + p = 45$ (1)

$q + d + n + p = 9$ (2)

From (1), $q = 0, 1$
Subtract (2) from (1),
$24q + 9d + 4n = 36 \quad (3)$
Since the right side of (3) is divided by 4, then the left side is also must be divided by 4. On the left, $24q$ and $4n$ are divided by 4, thus $9d$ is divided by 4. That means d is multiple of 4.
Thus, $d = 0, 4$
From (3), $n = \frac{36-24q-9d}{4}$
From (1), $p = 45 - 25q - 10d - 5n$
The following table of possible cases for q, d, n, p

number of quarters	0		1
number of dimes	0	4	0
number of nickels	9	0	3
number of pennies	0	5	5

Since Sam was able to uniquely identify the solution, then the number of pennies was 0, and consequently the number of nickels was 9
Answer - E

5. A point (a, b) is a lattice point if both a and b are integers. It is called *visible* if the line segment from $(0, 0)$ to it does NOT pass through any other lattice points. Which of the following lattice points is visible?
A (28,14) B (28,15) C (28,16) D (28,18) E (28,21)
Solution. If a, b have a common nontrivial factor, say $c, \frac{b}{a} = \frac{b/c}{a/c}$, then the visibility of (a, b) will be obstructed from the origin by point $(a/c, b/c)$.
Thus for a point (a, b) to be visible from the origin, the values of a, b should have no common nontrivial factors.
In the list above, only point (28,15) satisfies the condition.
Answer - B

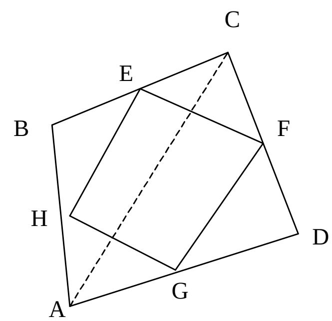
6. A flea jumps clockwise around a clock starting at 12. The flea first jumps one number to1, then two numbers to 3, then three to 6, then two to 8, then 1 to 9, then two, then three, etc. What number does the flea land at his 2008^{th} jump?
A 4 B 5 C 6 D 7 E 8
Solution. Every block of four jumps the flea covers a block of eight clock numbers. In 2008 jumps the flea covers $\frac{2008}{4}8 = 4016$ clock numbers. Since $4016 = 334 \cdot 12 + 8$, then after 2008^{th} jump the flea will be at 8 o'clock.
Answer - E

7. In quadrilateral $ABCD$, E is a midpoint of $\overline{AB}$, F is a midpoint of $\overline{BC}$, G is a midpoint of $\overline{CD}$, and H is a midpoint of $\overline{DA}$.Which of the following must be true?
A $\angle FEH = \angle FGH$ B $\angle FEH = \angle EHG$ C $\angle FEH + \angle EHG = 180°$
D both A and B E both B and C

Solution. $\triangle ADC$ is similar to $\triangle FDG$.So, lines AC and FG are parallel.

Similarly, lines AC and HE are parallel. Thus lines FG and HE are parallel, as parallel to the same line AC.

In the same way we show that lines EF and HG are parallel.This means that



$HEFG$ is a parallelogram, from which $\angle FEH = \angle FGH$ and $\angle FEH + \angle EHG = 180°$

Answer - D

8. All non-empty subsets of $\{2, 4, 5, 7\}$ are selected. How many different sums do the elements of each of these subsets add up to.

A 10 B 11 C 12 D 14 E 15

Solution. Below are listed all non-empty (total of 15) subsets of the given set, and in the parentheses the sum of elements is shown.

2 (2)	4 (4)	5 (5)	7 (7)	2,4 (6)
2,5 (7)	2,7 (9)	4,5 (9)	4,7 (11)	5,7 (12)
2,4,5 (11)	2,4,7 (13)	2,5,7 (14)	4,5,7 (16)	2,4,5,7 (18)

The total number of different sums is 12.

Answer - C

9. Luis solved the equation $ax - b = c$, and Anh solved $bx - c = a$. If they get the same correct answer for x, and a, b, and c are distinct and nonzero, what must be true?

A $a + b + c = 0$ B $a + b + c = 1$ C $a + b = c$ D $b = a + c$ E $a = b + c$

Solution . From the system

$ax - b = c$

$bx - c = a$

eliminate x. We get $b^2 - ac = -bc + a^2$, which is $b^2 - a^2 = ac - bc$, or $(b-a)(b+a) = -c(b-a)$. Since $a \neq b$, cancel and get $b+a = -c$,or $a+b+c = 0$

Answer - A

10. How many asymptotes does the function $f(x) = \frac{x^2-22x+40}{x^2+13x-30}$ has?

A 0 B 1 C 2 D 3 E 4

Solution. Factoring the numerator and the denominator, we have for $x \neq 2$

$f(x)=\frac{(x-20)(x-2)}{(x+15)(x-2)}=\frac{x-20}{x+15}$.The function has one vertical asymptote at $x=-15$

Also the function has the horizontal asymptote $y=1$

Totally, the function has two asymptotes

Answer - C

11. Replace each letter of $AMATYC$ with a digit 0 through 9 (equal letters replaced by equal digits, different letters replaced by different digits). If the resulting number is the largest such number divisible by 55, find $A+M+A+T+Y+C$

A 36 B 38 C 40 D 42 E 44

Solution. The resulting number is divisible by 5 and 11.

Since it is divisible by 5, $C=0$ or 5

Represent $AMATYC$ as $A\cdot 10^5+M\cdot 10^4+A\cdot 10^3+T\cdot 10^2+Y\cdot 10+C$
$=A(11-1)^5+M(11-1)^4+A(11-1)^3+T(11-1)^2+Y(11-1)+C$

If this umber is divisible by 11, then the number $A(-1)^5+M(-1)^4+A(-1)^3+T(-1)^2+Y(-1)+C$, which is $-A+M-A+T-Y+C$, must be divisible by 11. Consequently, the number $A-M+A-T+Y-C$ is divided by 11.

To get the largest resulting number we will assign to letters the largest possible digits under condition that the alternating sum above is divisible by 11.

Assign $A=9, M=8$

Case of $C=0$

The alternating sum $A-M+A-C=9-8+9-0=10$. The largest possible conditional assignments for T and Y are $T=5, Y=6$. The total alternating sum is $A-M+A-T+Y-C=9-8+9-5+6-0=11$.Then $AMATYC=989560$ and $A+M+A+T+Y+C=9+8+9+5+6+0=36$

Case of $C=5$

The alternating sum $A-M+A-C=9-8+9-5=5$. The largest possible conditional assignments for T and Y are $T=7, Y=2$. The total alternating sum is $A-M+A-T+Y-C=9-8+9-7+2-5=0$.Then $AMATYC=989725$ and $A+M+A+T+Y+C=9+8+9+7+2+5=40$

Comparing cases, we get that the resulting largest number $AMATYC$ is 989725 and $A+M+A+T+Y+C=40$

Answer - C

12. The equation $a^6+b^2+c^2=2009$ has a solution in positive integers a, b, and c in which exactly two of a, b, and c are powers of 2. Find $a+b+c$.

A 43 B 45 C 47 D 49 E 51

Solution.

Case of $a=2^n$.

In this case $2009>a^6=2^{6n}=64^n$. Then $\log 2009>n\log 64 \Longleftrightarrow n<1.83$. Therefore, $n=0,1$

If $n=0$, then $b^2+c^2=2009-a^6=2009-1=2008$

Assume that $b=2^m$. Then $2008>b^2=2^{2m}=4^m$. Thus $\log 2008>m\log 4 \Longleftrightarrow m<5.49$

Then we manually check that $c=\sqrt{2008-b^2}=\sqrt{2008-2^{2m}}=\sqrt{2008-4^m}$ is not an integer for $m=0,1,2,3,4,5$

If $n=1$, then $b^2+c^2=2009-a^6=2009-2^6=1945$

Assume that $b=2^m$. Then $1945>b^2=2^{2m}=4^m$. Thus $\log 1945 > m\log 4 \iff m<3.45$

Then we manually check that $c=\sqrt{1945-b^2}=\sqrt{1945-2^{2m}}=\sqrt{1945-4^m}$ is not an integer for $m=0,1,2,3$

Case of $b=2^m$ and $c=2^n$

Assume that $b\geqslant c$. Then $2009>b^2=(2^m)^2=4^m$, and as above, $m<5.49$

So $a=\sqrt[6]{2009-b^2-c^2}=\sqrt[6]{2009-4^m-4^n}$, where $0\le n\le m\le 5$

The results are shown in the following table

m	n	$a=\sqrt[6]{2009-4^m-4^n}$
5	5	Not Integer
	4	3
	3	Not Integer
	2	Not Integer
	1	Not Integer
	0	Not Integer
4	4	Not Integer
	3	Not Integer
	2	Not Integer
	1	Not Integer
	0	Not Integer
3	3	Not Integer
	2	Not Integer
	1	Not Integer
	0	Not Integer
2	2	Not Integer
	1	Not Integer
	0	Not Integer
1	1	Not Integer
	0	Not Integer
0	0	Not Integer

The table shows that the only solution of the equation is $a=3, b=2^5, c=2^4$

$a+b+c=3+2^5+2^4=51$

Answer - E

13. ACME Widget employees are paid every other Friday (i.e., on Fridays in alternate weeks). The year 2008 was unusual in that ACME had 3 paydays in February. What is the units digit of the next year in which ACME has 3 February paydays?

A 0 B 2 C 4 D 6 E 8

Solution. Since $365=7\cdot 52+1$, then with the passage of any no leap year Friday slides one day backward in February calendar. Similarly, as $366=7\cdot 52+2$, then with the passage of any leap year Friday slides two days backward in February calendar. Then with the passage of four years, Friday slides five days backward in February calendar.

Construct the following table that shows the numbers of 4-year package and the day of Friday in February closest to the month end

4-year block number	0	1	2	3	4
The last Friday date	29	29-5=24	24-5+7=26	26-5+7=28	28-5=23

4-year block number	5	6	7
The last Friday date	23-5+7=25	25-5+7=27	27-5+7=29

So with the passage of $7 \cdot 4 = 28$ years after year of 2008, February 29 will be Friday. This will be in year of $2008 + 28 = 2036$. The next years with the above property are $2064, 2092, ...$

Another, number theoretical, solution.

Denote by p the number of 4-year blocks. This corresponds to the number of 5-day shifts to the left.

Denote by r the number of 7-day shifts to the right.

Then l is a solution of the following equation

$-5p + 7r = 0$

The equation is equivalent to

$7r = 5p$

Solve the last equation. Since the left side is divisible by 7, then p is divisible by 7, or $p = 7m$. Since the right side is divisible by 5, then r is divisible by 5, or $r = 5n$.Substituting the last expressions for l and r to the equation, we will have

$7(5n) = 5(7m)$, or $n = m$.The solution of the last equation is any integer. So $p = 7m$, where m is any integer. In particular, the next, after 2008, year is $2008 + (7 \cdot 1)4 = 2036$

Answer - D, but not C as indicated in the answer keys list.

14. Five murder suspects, including the murderer, are being interrogated by the police. Results of a polygraph indicate two of them are lying and three are telling the truth. if polygraph results are correct, who is the murderer?

Suspect A: "D is the murderer", Suspect B: "I am innocent", Suspect C: "It wasn't E", Suspect D: "A is lying", Suspect E: "B is telling the truth".

A A, B B, C C, D D, E E.

Solution. We denote by A, B, C, D, E the claims of correspondingly suspects A,B,C,D,E. Then the truth values of D and A are opposite: if D is true, then A is false, and vice versa. So one of A, D is lying, and the other is telling the truth.

Therefore, among B, C, E there are one false and two true claims.

If E is false, then *not* E="B is not telling the truth" is true, that is B is false. In this case both E, B are false, which contradicts that among B, C, E there is only one false statement.

Thus E is true, and consequently B is true. Then E is false, and *not* E = "It was E" is true.

So E is the murderer.

Answer - E

15. Two arithmetic sequences are multiplied together to produce the sequence $468, 462, 384, ...$ What is the next next term of this sequence?

A 250, B 286, C 300, D 324, E 336

Solution. Let the first sequence be $a, a+d, a+2d, a+3d, ...$

Let the second sequence be $\alpha, \alpha+\delta, \alpha+2\delta, \alpha+3\delta, , ...$

Then the product sequence will be $a\alpha, (a+d)(\alpha+\delta), (a+2d)(\alpha+2\delta), (a+3d)(\alpha+3\delta), ...$

By the condition

$a\alpha = 468$

$(a+d)(\alpha+\delta) = 462$

$(a+2d)(\alpha+2\delta) = 384$

$(a+3d)(\alpha+3\delta) = X$, which should be found

Expend the left sides of the last two equations above.

$a\alpha = 468$

$a\alpha + (a\delta + \alpha d) + d\delta = 462 \qquad (1)$

$a\alpha + 2(a\delta + \alpha d) + 4d\delta = 384$

$a\alpha + 3(a\delta + \alpha d) + 9d\delta = X$

Adding in (1) the 1st and the 4th equations, and subtracting from the sum the 2nd and the 3rd equations, we get $4d\delta = X - 378$, or

$X = 378 + 4d\delta \quad (2)$

To find $d\delta$, in (1) add the 1st and the 3rd equations, and subtract from the sum twice the 2nd equation. We get

$2d\delta = -72 \quad (3)$

Then from (2) and (3),

$X = 378 + 4d\delta = 378 + 2(2d\delta) = 378 + 2(-72) = 234$

Answer- The next term of the sequence is 234. The answer does not correspond to listed choices

16. In $\triangle ABC, AB = 5, BC = 9,$ and $AC = 7$, Find the value of $(\tan\frac{A-B}{2})/(\tan\frac{A+B}{2})$

A $\frac{1}{8}$, B $\frac{7}{9}$, C $\frac{3}{2}$, D $\frac{9}{7}$ E 8

Solution. Let $a = BC, b = AC, c = AB$.

By the Cosine Rule, $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{7^2+5^2-9^2}{2\cdot7\cdot5} = -\frac{1}{10}$

By the Cosine Rule, $\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{9^2+5^2-7^2}{2\cdot9\cdot5} = \frac{19}{30}$

Quick Calculator Solution

$A = \arccos(-\frac{1}{10}),\ B = \arccos\frac{19}{30}$

Applying graphing calculator to calculate the expression $\tan\frac{A-B}{2}/\tan\frac{A+B}{2}$, where A, B are found above values,

we get that the value of the expression is $0.125 = \frac{1}{8}$

(Long)Algebraic Solution

Simplify the expression $\tan\frac{A-B}{2}/\tan\frac{A+B}{2}$.

By the formulas $\tan\frac{t}{2} = \frac{1-\cos t}{\sin t}$ and $2\sin x\cos y = \sin(x+y) + \sin(x-y)$,

$\tan\frac{A-B}{2}/\tan\frac{A+B}{2} = \frac{(1-\cos(A-B))/\sin(A-B)}{(1-\cos(A+B))/\sin(A+B)} = \frac{(1-\cos(A-B))\sin(A+B)}{(1-\cos(A+B))\sin(A-B)}$

$= \frac{\sin(A+B)-\cos(A-B)\sin(A+B)}{\sin(A-B)-\cos(A+B)\sin(A-B)} = \frac{\sin(A+B)-\cos(A-B)\sin(A+B)}{\sin(A-B)-\cos(A+B)\sin(A-B)} = \frac{\sin(A+B)-\sin(2A)/2-\sin(2B)/2}{\sin(A-B)-\sin(2A)/2+\sin(2B)/2}$

$= \frac{2\sin(A+B)-\sin(2A)-\sin(2B)}{2\sin(A-B)-\sin(2A)+\sin(2B)} = \frac{2(\sin A\cos B+\cos A\sin B))-2\sin A\cos A-2\sin B\cos B}{2(\sin A\cos B-\cos A\sin B))-2\sin A\cos B+2\sin B\cos B}$
$= \frac{\sin A\cos B+\cos A\sin B-\sin A\cos A-\sin B\cos B}{\sin A\cos B-\cos A\sin B-\sin A\cos A+\sin B\cos B} = \frac{\sin A(\cos B-\cos A)+\sin B(\cos A-\cos B)}{\sin A(\cos B-\cos A)-\sin B(\cos A-\cos B)}$
$= \frac{(\cos B-\cos A)(\sin A-\sin B)}{(\cos B-\cos A)(\sin A+\sin B)} = \frac{\sin A-\sin B}{\sin A+\sin B}$

Since $\cos A = -\frac{1}{10}, \cos B = \frac{19}{30}$, then $\sin A = \sqrt{1-\cos^2 A} = \sqrt{1-\left(-\frac{1}{10}\right)^2} = \frac{3}{10}\sqrt{11}$,

and $\sin B = \sqrt{1-\cos^2 B} = \sqrt{1-\left(\frac{19}{30}\right)^2} = \frac{7}{30}\sqrt{11}$

Thus, $\tan\frac{A-B}{2} / \tan\frac{A+B}{2} = \frac{\sin A-\sin B}{\sin A+\sin B} = \frac{\frac{3}{10}\sqrt{11}-\frac{7}{30}\sqrt{11}}{\frac{3}{10}\sqrt{11}+\frac{7}{30}\sqrt{11}} = \frac{1}{8}$

Answer - A

17. A pyramid has a square base 6 m on a side and a height of 9 m. Find the volume of the portion of the pyramid which lies above the base and below a plane parallel to the base and 3 m above the base.

A 32 m^3 B 36 m^3 C 64 m^3 D 72 m^3 E 76 m^3

Solution. Let us denote by a the length of the base square side, and by h the length of the pyramid height. The volume of the pyramid is $V = \frac{1}{3}a^2h = \frac{1}{3}6^2 9 = 108\ m^3$. The smaller pyramid above the parallel plane is similar to the initial pyramid and has the height of $9-3=6\ m$. Then the volume of the smaller pyramid is $\left(\frac{6}{9}\right)^3$ times less that the volume of the initial pyramid. That is, the volume of the smaller pyramid is $108\left(\frac{6}{9}\right)^3 = 32m^3$. Thus the volume of the portion of the of the pyramid which lies above the base and below the parallel plane is $108\ m^3 - 32m^3 = 76m^3$

Answer - E

18. In $\triangle ABC$, $AB = AC$ and in $\triangle DEF$, $DE = DF$. If AB is twice DE and $\measuredangle D$ is twice $\measuredangle A$, then the ratio of the areas of $\triangle ABC$ to the area of $\triangle DEF$ is

A $\tan A$ B $2\sec A$ C $\csc 2A$ D $\sec A\tan A$ E $\cot 2A$

Solution. The area $S_1$of $\triangle ABC$ is $S_1 = \frac{1}{2}|AB||AC|\sin A = \frac{1}{2}2|DE|\cdot 2|DF|\sin A = 2|DE||DF|\sin A$. The area $S_2$of $\triangle DEF$ is $S_2 = \frac{1}{2}|DE||DF|\sin D = \frac{1}{2}|DE||DF|\sin(2A) = \frac{1}{2}|DE||DF|2\sin A\cos A = |DE||DF|\sin A\cos A$.

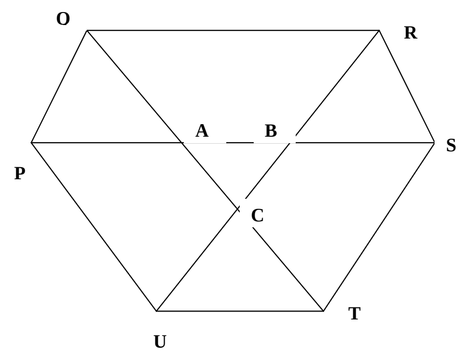
The ratio $\frac{S_1}{S_2} = \frac{2|DE||DF|\sin A}{|DE||DF|\sin A\cos A} = \frac{2}{\cos A} = 2\sec A$

Answer - B

19. In hexagon $PQRSTU$, all interior angles =120° . If $PQ = RS = TU = 50$, AND $QR = ST = UP = 100$, find the area of the triangle bounded by QT, RU, AND PS to the nearest tenth.

A 1082.5 B 1082.9 C 1083.3 D 1083.5 E 1083.9

Solution. Quadrilaterals $PQRS, QRST, RSTU, STUP, TUPQ, UPQR$ are trapezoids



Show this for $PQRS$. Separate scratch drawing might help.

$\triangle PQR$ is congruent to $\triangle QRS$. So $\measuredangle QPR = \measuredangle QSR$. $\triangle PRS$ is congruent to $\triangle PQS$. So $\measuredangle RPS = \measuredangle QSP$.

Therefore, $\measuredangle QPS = \measuredangle QPR + \measuredangle RPS = \measuredangle QSR + \measuredangle QSP = \measuredangle RSP$. As $\measuredangle QPS + \measuredangle Q + \measuredangle R + \measuredangle RSP = 360°$, then $\measuredangle QPS = \frac{1}{2}(360 - 2 \cdot 120) = 60°$, and $\measuredangle QPS + \measuredangle Q = 180°$. This means that PS is parallel to QR, and quadrilateral $PQRS$ is a trapezoid. Similarly, $QRST, RSTU, STUP, TUPQ, UPQR$ are trapezoids with base angles of 60°.

From parallelogram $PQRS$, $|PB| = |QR| = 100$. $\triangle PQA$ is equilateral, and $|PA| = |PQ| = 50$. Therefore, $|AB| = |PB| - |PA| = 100 - 50 = 50$. similarly, $|AC| = |BC| = 50$

So $\triangle ABC$ is equilateral and its area is $\frac{1}{2}50 \cdot 50 \cdot \sin 60° \approx 1082.53$

Answer - A

20. For all integers $k \geq 0, P(k) = \left(2^2 + 2^1 + 1\right)\left(2^2 - 2^1 + 1\right)\left(2^4 - 2^2 + 1\right)\ldots\left(2^{2^{k+1}} - 2^{2^k} + 1\right) - 1$ is always the product of two integers n and $n+1$. Find the smallest value of k for which $n + (n+1) \geq 10^{1000}$

A 9 B 10 C 11 D 12 E 13

Solution. We will use estimation technique to approximate the value of k. All numerical calculatins are done to the precision of TI-84 Plus calculator.

Since $n+(n+1) = 2n+1 \geq 10^{1000}$, then $n \geq \frac{1}{2}10^{1000} - \frac{1}{2}$. Since n is integer, $n \geq \frac{1}{2}10^{1000}$

$P(k) = n(n+1) \geq \frac{1}{2}10^{1000}\left(\frac{1}{2}10^{1000} + 1\right) = \frac{1}{4}10^{2000}\left(1 + \frac{1}{0.5 \cdot 10^{1000}}\right) \approx \frac{1}{4}10^{2000}$

The factors $\left(2^{2^{k+1}} - 2^{2^k} + 1\right) = 2^{2^{k+1}}(1 - \frac{1}{2^{2^k}} + \frac{1}{2^{2^{k+1}}}) \approx 2^{2^{k+1}}$ for large k, where we have approximation on overestimation side. Take $k \geq 5$, then $1 > (1 - \frac{1}{2^{2^k}} + \frac{1}{2^{2^{k+1}}}) > (1 - \frac{1}{2^{2^5}} + \frac{1}{2^{2^6}}) = 0.9999999998$

$P(k) = \left(2^2 + 2^1 + 1\right)\left(\prod_{j=0}^{k}\left(2^{2^{j+1}} - 2^{2^j} + 1\right) - 1 \approx \frac{1}{4}10^{2000}\right.$. So $\prod_{j=0}^{k}\left(2^{2^{j+1}} - 2^{2^j} + 1\right) \approx \frac{\frac{1}{4}10^{2000}+1}{7} \approx \frac{1}{28}10^{2000}$

$\log\left[\prod_{j=0}^{k}\left(2^{2^{j+1}} - 2^{2^j} + 1\right)\right] \approx \log\left(\frac{1}{28}10^{2000}\right) = 1998.552842$

$\log\left[\prod_{j=0}^{k}\left(2^{2^{j+1}}-2^{2^j}+1\right)\right]=\log\left[\prod_{j=0}^{k}2^{2^{j+1}}\left(1-2^{-2^j}+2^{-2^{j+1}}\right)\right]=\log\left(\prod_{j=0}^{k}2^{2^{j+1}}\right)+$
$\log\left[\prod_{j=0}^{k}\left(1-2^{-2^j}+2^{-2^{j+1}}\right)\right]$

$=\log 2\left(\sum_{j=0}^{k}2^{j+1}\right)+\sum_{j=0}^{k}\log\left(1-2^{-2^j}+2^{-2^{j+1}}\right)\approx\log 2\left(\sum_{j=0}^{k}2^{j+1}\right)+\sum_{j=0}^{4}\log\left(1-2^{-2^j}+2^{-2^{j+1}}\right)$

The last term is calculated manually as $\sum_{j=0}^{4}\log\left(1-2^{-2^j}+2^{-2^{j+1}}\right)=-.2413449165$

So $\log 2\left(\sum_{j=0}^{k}2^{j+1}\right)-.2413449165\approx 1998.552842$, or $\sum_{j=0}^{k}2^{j+1}\approx 6638.391206$

Geometric series $\sum_{j=0}^{k}2^{j+1}=2^{k+2}-2$, therefore $2^{k+2}-2\approx 6639.850565$, or $2^{k+2}\approx 6641.850565$

$\log 2^{k+2}\approx\log 6641.850565$, that is $(k+2)\log 2\approx 3.8222891$. Finally, $k\approx\frac{4.299227709}{\log 2}-2=10.69736955$

As k is an integer, then $k=11$

Answer - C