Test #1 AMATYC Student Mathematics League Oct/Nov 2009

Leo Livshutz
Truman College, Chicago, Illinois

Solutions

1. Find the sum of the solutions to the equations $x^2 - 5x - 6 = 0$ and $x^2 + 4x + 3 = 0$

A -2 B -1 C 1 D 2 E 3

Solution. The solutions of the first equation are $6, -1$. The solutions of the second equation are $-3, -1$. The required sum is $6 + (-3) = 3$

Answer - E

2. Four consecutive integers are substituted in every possible order for a, b, c and d.Find the difference between the maximum and minimum values of $ab + cd$

A 1 B 2 C 3 D 4 E 5

Solution. There are only three cases of the sum of the products $ab + cd$, $ac + bd$, and $ad + bc$. Suppose, that $a \le b \le c \le d$.

Then $ac + bd = ab + cd - (c - b)(d - a) \le ab + cd$ and $ad + bc = ab + cd - (d - b)(c - a) \le ab + cd$.That is $ab + cd = \max\{ab + cd, ac + bd, ad + bc\}$

Also $ab + cd = ad + bc + (c - a)(d - b) \ge ad + bc$ and $ac + bd = ad + bc + (b - a)(d - c) \ge ad + bc$. That is $ad + bc = \min\{ab + cd, ac + bd, ad + bc\}$

Thus, $\max\{ab + cd, ac + bd, ad + bc\} - \min\{ab + cd, ac + bd, ad + bc\} = (ab + cd) - (ad + bc) = a(b - d) + c(d - b) = a(b - d) - c(b - d) = (c - a)(d - b)$

By the problem condition, for some integer m, $a = m, b = m + 1, c = m + 2, d = m+3$. So that $\max\{ab+cd, ac+bd, ad+bc\} - \min\{ab+cd, ac+bd, ad+bc\} = ((m + 2) - m)((m + 4) - (m + 2)) = 2 \cdot 2 = 4$

Answer - D

3. The product of a number and b more than its reciprocal is y $(b > 0)$. Express the number in terms of b and y

A $\frac{y-1}{b}$ B $\frac{y+1}{b}$ C $\frac{y}{b}-1$ D $\frac{y}{b}+1$ E $1-\frac{y}{b}$

Solution. Denote the number by x. Then $x(b + \frac{1}{x}) = y \Leftrightarrow xb + 1 = y \Leftrightarrow xb = y - 1 \Leftrightarrow x = \frac{y-1}{b}$

Answer - A

4. If $f(x) = x^2 - x + 2$, find the sum of all x values satisfying $f(x - 2) = 22$

A -5 B -1 C 1 D 3 E 5

Solution. Denote $u = x - 2$. Then $f(u) = 22$, or $u^2 - u + 2 = 22 \Leftrightarrow u^2 - u - 20 = 0$. The sum of this equation solutions is $u_1 + u_2 = 1$. Replacing u with $x - 2$, we will receive $(x_1 - 2) + (x_2 - 2) = 1$, or $x_1 + x_2 = 5$

Answer - E

5. Sue bikes 2.5 times as fast as Joe runs, and in 1 hr. they cover a total of 42 miles. What is their combined distance if Sue bikes for 0.5 hr. and Joe runs for 1.5 hr. ?

A 27 B 30 C 33 D 36 E 39

Solution. Let us denote the speed of Sue biking by u, and the speed of Joe running by v. Then $u = 2.5v$ and $u \cdot 1 + v \cdot 1 = 42$

Solving this system of equations we get $u = 30$, $v = 12$

The required to find distance $s = u \cdot 0.5 + v \cdot 1.5 = 30 \cdot 0.5 + 12 \cdot 1.5 = 33$

Answer - C

6. The equation $a^4 + b^3 + c^2 = 2009$ (a, b, c positive integers) has a solution in which a and b are both perfect squares. Find $a + b + c$

A 20 B 21 C 32 D 45 E 50

Solution. $a^4 = 2009 - b^3 - c^2 < 2009$, so that $a < (2009)^{\frac{1}{4}} < 6.7$. Since a is a positive perfect square, than $a = 1$ or $a = 4$. Similarly, $b < \sqrt[3]{2009} < 13$, and $b = 1$ or $b = 4$, or $b = 9$.Now find a positive integer $c = \sqrt{2009 - a^4 - b^3}$ for above values of a and b. The results are shown in the following table.

a	1	1	1	4	4	4
b	1	4	9	1	4	9
$c = \sqrt{2009 - a^4 - b^3}$	Not int.	Not int.	Not int.	Not int.	Not int.	32

The solution $a + b + c = 4 + 9 + 32 = 45$

Answer - *D*

7. How many 3-digit numbers have one digit equal to the average of the other 2?

A 96 B 100 C 112 D 120 E 121

Solution. Denote a required 3-digit number by $ABC = A \cdot 10^2 + B \cdot 10 + C$

Case 1. All digits are equal. With notable exception of digit 0, there are 9 possibilities.

Case 2. Not all digits are equal. Let digits m, μ, M be such that $m \le \mu \le M$. Show that under condition that one digit equal to the average of the other 2, the only possible case is $m < \mu < M$ and $\mu = \frac{m+M}{2}$. Really $m = \frac{m+m}{2} < \frac{\mu+M}{2}, M = \frac{M+M}{2} > \frac{m+\mu}{2}$. The only possible case is $\mu = \frac{m+M}{2}$..Since $\mu < M$, then $\mu = \frac{m+M}{2} > \frac{m+m}{2} = m$. So that $m < \mu < M$

Denote $\delta = \mu - m, \Delta = M - m$, so that $\mu = m + \delta, M = m + \Delta$. Since $\mu = \frac{m+M}{2}$, then $m + \delta = \frac{m+(m+\Delta)}{2} = m + \frac{\Delta}{2}$, or $\Delta = 2\delta$

Possible values of $\delta = \Delta, m, \mu, M$ are given in the tables below.

$m = 0$

δ	1	2	3	4
$\mu = m + \delta$	1	2	3	4
$M = m + 2\delta = \mu + \delta$	2	4	6	8

Total number of possibilities is 4

$m = 1$

δ	1	2	3	4
$\mu = m + \delta$	2	3	4	5
$M = m + 2\delta = \mu + \delta$	3	5	7	9

Total number of possibilities is 4

$m=2$

δ	1	2	3
$\mu = m+\delta$	3	4	5
$M = m+2\delta = \mu+\delta$	4	6	8

Total number of possibilities is 3

$m=3$

δ	1	2	3
$\mu = m+\delta$	4	5	6
$M = m+2\delta = \mu+\delta$	5	7	9

Total number of possibilities is 3

$m=4$

δ	1	2
$\mu = m+\delta$	5	6
$M = m+2\delta = \mu+\delta$	6	8

Total number of possibilities is 2

$m=5$

δ	1	2
$\mu = m+\delta$	6	7
$M = m+2\delta = \mu+\delta$	7	9

Total number of possibilities is 2

$m=6$

δ	1
$\mu = m+\delta$	7
$M = m+2\delta = \mu+\delta$	8

Total number of possibilities is 1

$m=7$

δ	1
$\mu = m+\delta$	8
$M = m+2\delta = \mu+\delta$	9

Total number of possibilities is 1

Returning to the ABC notation for required 3-digit numbers, we have

a) All digits $A,.B,C$ are equal. From *case 1* above, the total number of possibilities is 9

b) From *case 2* above, all digits $A,.B,C$ are different.

If $m=A, \mu=B, M=C$ or $m=A, \mu=C, M=B$, cases $m=1,2,3,4,5,6,7$ are applicable. The total number of possibilities is $2(4+3+3+2+2+1+1)=32$

If $m=B, \mu=A, M=C$ or $m=B, \mu=C, M=A$, cases $m=0,1,2,3,4,5,6,7$ are applicable. The total number of possibilities is $2(4+4+3+3+2+2+1+1)=40$

If $m=C, \mu=A, M=B$ or $m=C, \mu=B, M=A$, cases $m=0,1,2,3,4,5,6,7$ are applicable. The total number of possibilities is $2(4+4+3+3+2+2+1+1)=40$

Total number of possibilities is $32+40+40=112$

Summing the number of possibilities from case a) and case b) we get the overall number of possibilities as $9+112=121$

Answer - E

8. A rectangular solid has integer dimensions with length $\geq$ width $\geq$ height and volume 60. How many such distinct solids are there ?

A 6 B 8 C 10 D 12 E 15

Solution. $60 = 1\cdot 2\cdot 2\cdot 3\cdot 5$. Since the volume $V = h\cdot b\cdot l = 60$, then the number of solids equals the number of combinations of product $1\cdot 2\cdot 2\cdot 3\cdot 5$ into the product of three factors $h\cdot b\cdot l$. To do so, we have to combine three of the factors $1,2,2,3,5$ into one factor. This number is $C_5^3 = \frac{5!}{3!\cdot 2!} = 10$

Answer - C

9. $\frac{2\sin x}{\cos x - \sin x\cdot\tan x} =$

A $\quad \tan 2x \quad$ B $\quad \cot 2x \quad$ C $\quad \tan x \quad$ D $\quad \cot x \quad$ E $\quad \sec x$

Solution $\frac{2\sin x}{\cos x - \sin x \cdot \tan x} = \left(\frac{2\sin x}{\cos x - \sin x \cdot \tan x}\right)\frac{\cos x}{\cos x} = \frac{2\sin x \cdot \cos x}{(\cos x - \sin x \cdot \tan x)\cos x} = \frac{\sin 2x}{\cos^x x - \sin^2 x} = \frac{\sin 2x}{\cos 2x} = \tan 2x$

Answer - A

10. If $x + \frac{1}{y} = 12$ and $y + \frac{1}{x} = \frac{3}{8}$, find the largest value of xy

A $\quad \frac{1}{4} \quad$ B $\quad \frac{1}{2} \quad$ C $\quad 1 \quad$ D $\quad 2 \quad$ E $\quad 4$

Solution. Multiplying side-by-side the equalities, we get $\left(x + \frac{1}{y}\right)\left(y + \frac{1}{x}\right) = 12 \cdot \frac{3}{8}$, or $xy + \frac{1}{xy} + 2 = \frac{9}{2}$, or $xy + \frac{1}{xy} = \frac{5}{2}$

Denoting $xy = u$,we get $u + \frac{1}{u} = \frac{5}{2}$, or $u^2 - \frac{5}{2}u + 1 = 0$,or $2u^2 - 65u + 2 = 0$.

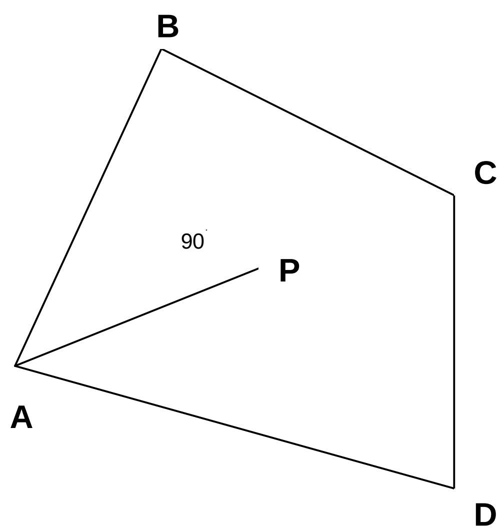
The solutions of this equations are $u = 2$ and $u = \frac{1}{2}$. since $u = xy$, then the largest value of xy is 2.

Answer - D

11. In quadrilateral $ABCD$, P is the point in its interior that $\angle DAP = \angle BAP, \angle CBP = \angle ABP$, and $\angle APB = 90°$. Quadrilateral $ABCD$ must be which of the following

A $\quad$ trapezoid $\quad$ B $\quad$ parallelogram $\quad$ C $\quad$ rectangle $\quad$ D $\quad$ B and C $\quad$ E $\quad$ none of these

Solution.

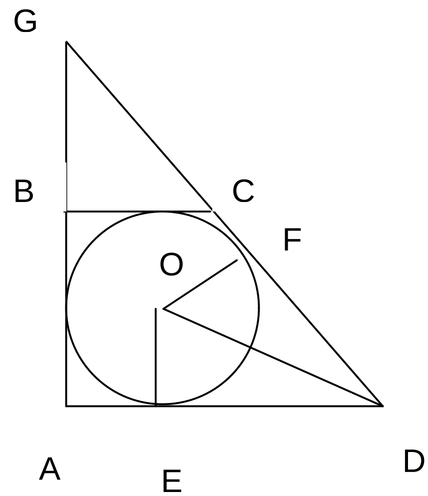


$\angle BAP + \angle ABP = 180° - \angle APB = 180° - 90° = 90°$. $\angle DAB + \angle ABC = (\angle DAP + \angle BAP) + (\angle CBP + \angle ABP) = 2\angle BAP + 2\angle ABP = 2(\angle BAP + \angle ABP) = 2 \circ 90° = 180°$. So the lines AD and BC are parallel.

Answer -A. The answer key indicates that the answer is E

12. The sum of the squares of the three roots of $P(x) = 2x^3 - 6x^2 + 3x + 5$ is

A $\quad 3 \quad$ B $\quad 6 \quad$ C $\quad 30 \quad$ D $\quad 33 \quad$ E $\quad 39$



Solution. Let a, b, c be the roots of $P(x)$. Then

$P(x) = 2x^3 - 6x^2 + 3x + 5 = 2(x-a)(x-b)(x-c) = 2(x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc)$

$= 2x^3 - 2(a+b+c)x^2 + 2(ab+ac+bc)x - 2abc$. So that

$a+b+c = 3, ab+ac+bc = \frac{3}{2}, abc = -\frac{5}{2}$. Squaring the first of these equations and using the second equation for substitution , we get $(a+b+c)^2 = 3^2$, $a^2 + b^2 + c^2 + 2(ab+ac+bc) = 9, a^2+b^2+c^2+2\left(\frac{3}{2}\right) = 9$. So that $a^2+b^2+c^2 = 9-3 = 6$

Answer - B

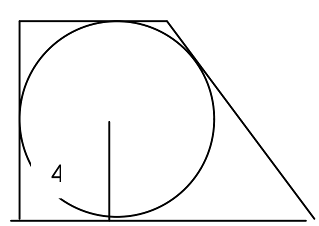
13. The value of $4^{\log_2\left(2^{1/4}2^{1/8}2^{1/16}...\right)}$ is

A 1 B $\sqrt{2}$ C 2 D $2\sqrt{2}$ E 4

Solution. $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+... = \frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{4}+...\right) = \frac{1}{4}\cdot\frac{1}{1-1/2} = \frac{1}{2}$, $2^{1/4}2^{1/8}2^{1/16}... = 2^{1/4+1/8+1/16+...} = 2^{1/2}$, $^{\log_2\left(2^{1/4}2^{1/8}2^{1/16}...\right)} = \log_2 2^{1/2} = \frac{1}{2}$ $4^{\log_2\left(2^{1/4}2^{1/8}2^{1/16}...\right)} = 4^{1/2} = 2$

Answer - C

14. The figure shows a circle of radius 4 inscribed in a trapezoid whose longer base is three times the radius of the circle. Find the area of the trapezoid.



A 72 B 74 C 76 D 78 E 60

Solution. Look at the advanced picture below.

$AD = 3OE = 3 \cdot 4 = 12$. $ED = AD - AE = AD - OE = 12 - 4 = 8$. $\tan(\angle EDO) = \frac{OE}{ED} = \frac{4}{8} = \frac{1}{2} \Leftrightarrow \angle EDO = \tan^{-1}\frac{1}{2}$. $\triangle OED \tilde{} \triangle OFD \Leftrightarrow \angle ODF = \angle EDO = \tan^{-1}\frac{1}{2}$. $\angle ADG = \angle EDO + \angle ODF = 2\tan^{-1}\frac{1}{2}$.

$AG = AD \cdot \tan(\angle ADG) = 12 \cdot \tan\left(2\tan^{-1}\frac{1}{2}\right) = 12\frac{2\cdot\tan\left(\tan^{-1}\frac{1}{2}\right)}{1-\left(\tan\left(\tan^{-1}\frac{1}{2}\right)\right)^2} = 12 \cdot \frac{2\cdot(1/2)}{1-(1/2)^2} = 16$. $AB = 2 \cdot OE = 2 \cdot 4 = 8$, $BG = AG - AB = 16 - 8 = 8$

$\triangle BGC \tilde{} \triangle AGD \Leftrightarrow \frac{BC}{BG} = \frac{AD}{AG} \Leftrightarrow \frac{BC}{8} = \frac{12}{16} \Leftrightarrow BC = 6$

Area of the trapezoid $Area = \frac{AD+BC}{2} \cdot AB = \frac{12+6}{2} \cdot 8 = 72$

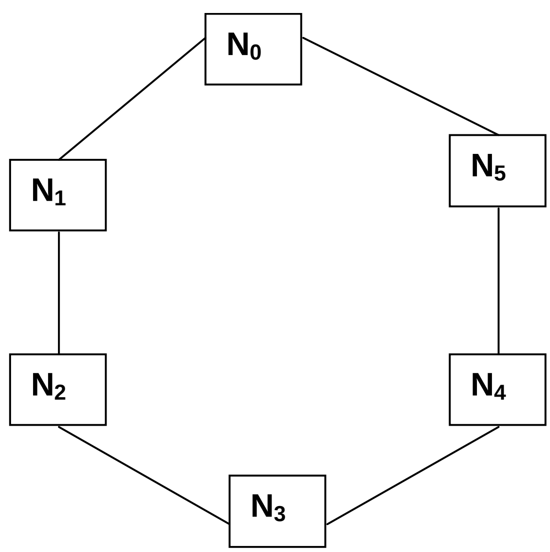
Answer - A

15. In how many ways can six computers be networked so that each computer is directly connected to exactly two other computers and all computers are connected directly or indirectly ?

A 24 B 36 C 48 D 60 E 120

Solution. We will call the computers in the network *nodes* and the connections between the computers *lines.* There only six nodes, any node has exactly two lines. So if we start with any node and move over the lines from one node to the neighboring node, we will finally come to the node we already passed. This shows that there are closed loops in the network of nodes. Note that if a node is in a loop, than because of exactly two possible lines it cannot be connected to the nodes outside the loop. This means that it impossible to have three, four, or five nodes loops, since there will nodes outside the loops, which contradicts to the fact that all computers are connected directly or indirectly.

The only possible loops consist of all six network nodes. For any 6-node loop, select and fix one of the nodes N_0, cut the loop and unfold it in counterclock direction to obtain the sequence of nodes $\{N_0, N_1, N_2, N_3, N_4, N_5, N_0\}$



For variable N_1, N_2, N_3, N_4, N_5, the total number of sequences is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Since symmetric sequences $\{N_0, N_1, N_2, N_3, N_4, N_5, N_0\}$ and

$\{N_0, N_5, N_4, N_3, N_2, N_1, N_0\}$ yield the same connection configurations, the number of different configurations is $\frac{120}{2} = 60$

Answer- D

16. The integer $r > 1$ is both the common ratio of an integer geometric sequence and the common difference of an integer arithmetic sequence. Summing corresponding terms of the sequences yields $7, 26, 90, \ldots$. The value of r is

A 2 B 4 C 6 D 8 E 12

Solution. Denote the geometric sequence as $a_0, a_0 + r, a_0 + 2r, \ldots$. Denote the arithmetic sequence as $b_0, b_0 r, b_0 r^2, \ldots$ Adding the terms of these sequences, we get

$$a_0 + b_0 = 7$$
$$a_0 + r + b_0 r = 26 \qquad (1)$$
$$a_0 + 2r + b_0 r^2 = 90.$$

Subtract the first equation sequentially from the second and the third equations to get

$$r + b_0(r - 1) = 19 \qquad (2)$$
$$2r + b_0(r^2 - 1) = 83$$

Multiplying the first equation of this system by $r + 1$ and subtracting the result from the second equation, we get $2r - r(r + 1) = 83 - 19(r + 1)$, or $r^2 - 20r + 64 = 0$. Solving this equation, we obtain $r_1 = 16$, $r_2 = 4$

Case $r = 16$. Substitute this value into the first two equations of system (1) to get

$$a_0 + b_0 = 7$$
$$a_0 + 16b_0 = 10$$

The solution of this system is $a_0 = \frac{34}{5}, b_0 = \frac{1}{5}$, which are not integers.

Case $r = 4$. Substitute this value into the first two equations of system (1) to get

$$a_0 + b_0 = 7$$
$$a_0 + 4b_0 = 22$$

The solution of this system is $a_0 = 2, b_0 = 5$, which are integers.

Answer - B

17. A hallway has 8 offices on one side and 5 offices on the other side. A worker randomly starts in one office and randomly goes to a second and then to a third office (all three different). Find the probability that the worker crosses the hallway at least once.

A $\frac{7}{13}$ B $\frac{8}{13}$ C $\frac{9}{13}$ D $\frac{10}{13}$ E $\frac{11}{13}$

Solution. The total number of offices is $8 + 5 = 13$. The total number of random selections of 3 offices out of 13 is $C_{13}^3 = 286$. The number of random selections of 3 offices out of 13 when there is no the hallway crossing is $C_8^3 = 56$ on one side and $C_5^3 = 10$ on the other side totaling to $56 + 10 = 66$. The probability P(no hallway crossed)$= \frac{66}{286} = \frac{3}{13}$. The probability P(the hallway crossed at least once)$= 1 - \frac{3}{13} = \frac{10}{13}$

Answer - D

18. Let $S = \{123, 124, ..., 987\}$ be the set of all three-digit numbers with distinct nonzero digits. For which number N below does S contain at least two different numbers with the same three digits, both divisible by N ?

A 31 B 37 C 39 D 41 E 43

Solution. We will show that $N = 37$ is a solution. We don't know the answer for the rest of answer numbers.

Let A and B are three-digit numbers that satisfy the conditions for some N. Then any linear combination of A and B is divided by N. Show that A and B cannot have the same digit at the same position.

Case $A = a10^2+b10+c$, $B = a10^2+c10+b$. Then $B-A = (c-b)10-(c-b) = (c-b)\cdot 9$. Neither $c-b$, nor 9 are divided by 37.

Case $A = a10^2+b10+c$, $B = c10^2+b10+a$. Then $B-A = (c-a)10^2-(c-a) = (c-a)\cdot 99$. Neither $c-a$, nor 99 are 37.

Case $A = a10^2+b10+c$, $B = b10^2+a10+c$. Then $B-A = (b-a)10^2-(b-a)10 = (b-a)\cdot 90$. Neither $b-a$, nor 90 are divided by 37.

We have shown that A and B cannot have the same digit at the same position. So digits in B are rotations of digits in A

Show that if $A = a10^2+b10+c$ is divisible by 37, then any number with digits that are rotations of digits of A is also divisible by 37. Let

$A = a10^2+b10+c$; $B = c10^2+a10+b$; $C = b10^2+c10+a$. Let $\beta = b-a, \gamma = c-a$, so that $b = a+\beta, c = a+\gamma$. Then $A = 111a+10\beta+\gamma$; $B = 111a+\beta+100\gamma$;

$C = 111a+100\beta+10\gamma$. Since $111 = 0 (\bmod 37)$, then $111a = 0 (\bmod 37)$

and $A = 10\beta+\gamma (\bmod 37), B = \beta+100\gamma (\bmod 37), C = 100\beta+10\gamma (\bmod 37)$

Assume that A is divisible by 37, so that $A = 10\beta+\gamma = 0 (\bmod 37)$. Then $10B = 10(\boldsymbol{\beta}+1000\gamma)(\bmod 37) = (10\boldsymbol{\beta}+\boldsymbol{\gamma})(\bmod 37)+(999\gamma)(\bmod 37) = 0(\bmod 37)+0(\bmod 37) = 0(\bmod 37) \Leftrightarrow B = 0(\bmod 37)$. $C = (100\beta+10\gamma)(\bmod 37) = 10(10\beta+\gamma)(\bmod 37) = (10\beta+\gamma)(\bmod 37) = 0(\bmod 37)$

Since $10\beta+\gamma(\bmod 37)$, then $10\beta+\gamma = 37k$. Since $1 \le |b-a| \le 8, 1 \le |c-a| \le 8$, then $1 \le |\beta| \le 8, 1 \le |\gamma| \le 8$.

Since $|37k| = |10\beta+\gamma| \ge |10\beta|-|\gamma| \ge 10\cdot 1-9 = 1$, then $|k| \ge 1$. Also $|k| = \left|\frac{10\beta+\gamma}{37}\right| \le |\frac{10\cdot 8+8}{37}| = 2.38$, that is $k = \pm 1, \pm 2$.

So the possible values of $10\beta+\gamma$ are $10\beta+\gamma = 37, 10\beta+\gamma = 74, 10\beta+\gamma = -37, 10\beta+\gamma = -74$.

The possible values of $A = 111a+10\beta+\gamma$ are $A = 111a+37, A = 111a+74, A = 111a-37, A = 111a-74$.

Since $111a-37 = 111(a-1)+74$ and $111a-74 = 111(a-1)+37$, then the independent cases are $A = 111a+37, A = 111a+74$

Selecting the appropriate values of a,we will find the related values of A.The results are collected into the following table

	$A = 111a+37$	$A = 111a+74$
(a, A)	$(1, 148)$	$(1, 185)$
(a, A)	$(2, 259)$	$(2, 296)$
(a, A)	$(4, 481,)$	$(4, 518)$
(a, A)	$(5, 592)$	$(5, 629)$
(a, A)	$(7, 814)$	$(7, 851)$
(a, A)	$(8, 925)$	$(8, 962)$

Looking at the table we see that the "base" numbers are $148, 185, 259, 296$. The rest are the rotations of these numbers

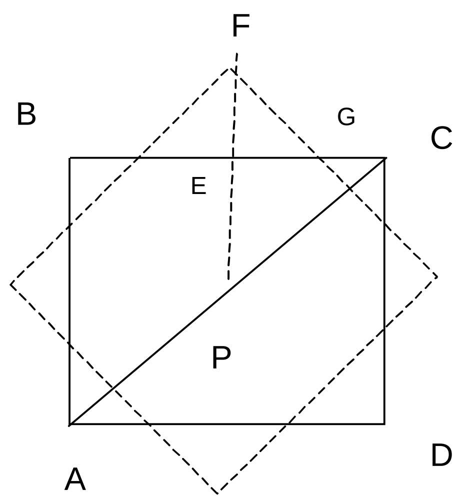
Answer - B

19. In square $ABCD$, $AB = 10$. The square is rotated $45°$ around point P, the intersection of $\overline{AC}$ and $\overline{BD}$

Find the area of the union of $ABCD$ and the rotated square to the nearest square unit.

A 117 B 119 C 121 D 123 E 125

Solution. All acute angles of interest in the figure below are $45°$.



$PE = \frac{AB}{2} = \frac{10}{2} = 5$. $PF = PB = \frac{BD}{2} = \frac{\sqrt{AB^2+AD^2}}{2} = \frac{\sqrt{10^2+10^2}}{2} = 5\sqrt{2}$. $EF = PF - PE = 5\sqrt{2} - 5 = 5(\sqrt{2} - 1)$.

$Area(\triangle FEG) = \frac{1}{2}EF^2 = \frac{1}{2}\left(5(\sqrt{2} - 1)\right)^2 = \frac{25}{2}(2 - 2\sqrt{2} + 1) = \frac{25}{2}(3 - 2\sqrt{2})$.

The area outside the square $ABCD$ is

$8Area(\triangle FEG) = 8\frac{25}{2}(3 - 2\sqrt{2}) = 100(3 - 2\sqrt{2})$.

The total ares is the sum of inside and outside area for the square $ABCD$, which is

$10^2 + 100(3 - 2\sqrt{2}) = 100(4 - 2\sqrt{2}) = 200(2 - \sqrt{2}) \approx 117$

Answer - A

20. The sum of the 100 consecutive perfect squares starting with a^2 $(a > 0)$ equals the sum of the next 99 consecutive perfect squares.

Find a

Solution. The problem conditions for a can be written in the form

$\sum\limits_{k=0}^{99}(a + k)^2 = \sum\limits_{k=0}^{98}(a + 100 + k)^2$.

Transforming we will get

$\sum\limits_{k=0}^{98}(a + k)^2 + (a + 99)^2 = \sum\limits_{k=0}^{98}(a + 100 + k)^2 \iff$

$(a+99)^2 = \sum_{k=0}^{98}(a+100+k)^2 - \sum_{k=0}^{98}(a+k)^2 \iff$
$(a+99)^2 = \sum_{k=0}^{98}\left[(a+100+k)^2 - (a+k)^2\right] \iff$
$(a+99)^2 = \sum_{k=0}^{98}[(a+100+k+a+k)(a+100+k-a-k)] \iff$
$(a+99)^2 = \sum_{k=0}^{98}[2(a+k+50)100)]$
$\iff (a+99)^2 = 200\sum_{k=0}^{98}(a+k+50).$
Using the formula for the sum of arithmetic sequence, obtain
$(a+99)^2 = 200\frac{(a+50)+(a+98+50)}{2}99 \iff$
$(a+99)^2 = 19800\,(a+99) \iff$
$a+99 = 19800 \iff$
$a = 19701$

Answer - 19701