Test #1 AMATYC Student Mathematics League Oct/Nov 2010

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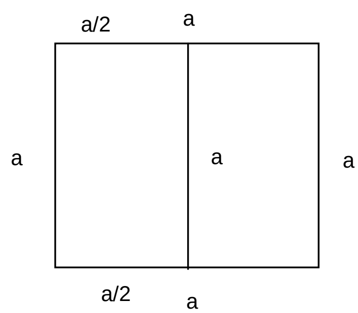
Solutions

1. A square is cut into two equal rectangles, each with perimeter 36. Find the area of the square.

A 36 B 64 C 100 D 144 E 196



Solution. The perimenter of a rectangle is $36 = 2a + 2\frac{a}{2} \equiv 3a$. So that $a = 12$, and the required area is $a^2 = 144$

Answer - D

2. Last year, the cost of milk was 150% of the cost of bread. If the cost of milk has risen by 20% and the cost of bread has risen by 25%, what percentage of the current cost of bread is the current cost of milk?

A 120 B 135 C 144 D 160 E 175

Solution. Denote by m_0, b_0 the last year costs of milk and bread. Then $m_0 = 1.5b_0$. This year the cost of milk is $m_1 = 1.2m_0$, the cost of bread is $b_1 = 1.25b_0$, and $\frac{m_1}{b_1} = \frac{1.2m_0}{1.25b_0} = \frac{1.2 \cdot 1.5b_0}{1.25b_0} = \frac{1.2 \cdot 1.5}{1.25} = 1.44 = 144\%$

Answer - C

3. Angles are complements if they add to 90°. Let $\angle A$ be nine times $\angle B$ and the complement of $\angle B$ is nine times the complement of $\angle A$. Find $\angle B$

A 6° B 8° C 9° D 10° E 12°

Solution. By the condition of the problem, $\angle A = 9\angle B$ and $90 - \angle B = 9(90 - \angle A)$. Substituting the expression for $\angle A$ from the first equation to the second, we receive $90 - \angle B = 9(90 - 9\angle B)$. Solving this equation, find that $\angle B = 9°$

Answer - C

4. Find the product of all values of x for which $f(x) = \frac{x-3}{x^2-10x-24}$ is undefined.

A -72 B -24 C 18 D 24 E 72

Solution. The function $f(x)$ is undefined for all values of x where the denominator is 0, that is at all solutions of the equation $x^2-10x-24=0$. The product of the zeros for any quadratic function with leading coefficient of 1 is the constant term, in our case -24

Answer - B

5. If you roll three fair dice, what is the probability that the product of the three numbers rolled is even.

A $\frac{1}{8}$ B $\frac{1}{6}$ C $\frac{1}{2}$ D $\frac{5}{6}$ E $\frac{7}{8}$

Solution. The total number of outcomes is $N=6\cdot 6\cdot 6=216$. Any unfavorable outcome will have all three odd rolled numbers. The number of unfavorable outcomes is $3\cdot 3\cdot 3=27$. The number of favorable outcomes is $216-27=189$. The probability of a favorable outcome is $\frac{189}{216}=\frac{7}{8}$

Answer - E

6. If $f(x)=ax^2+bx+c$, $f(-1)=10$, $f(0)=5$, and $f(1)=4$, find $f(2)$

A 7 B 8 C 9 D 10 E 11

Solution. Note that $c=f(0)=5$. Then $f(x)=ax^2+bx+5$, and $f(-x)=ax^2-bx+5$. Adding, and then subtracting these equalities, we get $a=\frac{f(x)+f(-x)-10}{2x^2}$, $b=\frac{f(x)-f(-x)}{2x}$. Taking here $x=1$, we will receive $a=\frac{4+10-10}{2\cdot 1^2}=2$, and $b=\frac{4-10}{2\cdot 1}=-3$. Then $f(2)=2(2)^2-3(2)+5=7$

Answer - A

7. A lattice point is a point with both coordinates integers. How many lattice points lie on or inside the triangle with vertices $(0,0),(10,0),(0,8)$?

A 51 B 52 C 53 D 54 E 55

Solution. An equation of the line through points $(10,0),(0,8)$ is $y=8-\frac{4}{5}x$. Thus, for a lattice points (x,y), $0\le y\le 8-\frac{4}{5}x$, with $x\in[0,10]$. Denote by $\lfloor u\rfloor$ the greatest integer less or equal to u. Then for any integer $x\in[0,10]$, the number of lattice points with fixed x is $n(x)=\left\lfloor 8-\frac{4}{5}x\right\rfloor+1$, or $n(x)=\left\lfloor 9-\frac{4}{5}x\right\rfloor$. Calculate

x	0	1	2	3	4	5	6	7	8	9	10
$n(x)$	9	8	7	6	5	5	4	3	2	1	1

. The total $\sum n(x)=51$

Answer - A

8. The perimeter of a rectangle is 52 and its diagonal is 20. Find its area.

A 134 B 138 C 142 D 144 E 148

Solution. Denote the sides of the rectangle by $a.b$, and its diagonal by $c=\sqrt{a^2+b^2}$. By the conditions, $2(a+b)=52, \sqrt{a^2+b^2}=20$, or $a+b=26, a^2+b^2=400$. Subtract the second equation from the squared first equation to obtain, $2ab=26^2-400$, or $ab=138$

Answer - B

9. The consecutive even numbers are written side-by-side to form the infinite decimal 0.24681023141618.... Find the digit in the 2010^{th} decimal place.

A 2 B 4 C 5 D 6 E 7

Solution. The smallest positive integer with m digits is 10^{m-1}. The largest positive integer with m digits is $10^m - 1$. The total number positive integers with m digits is $(10^m - 1) - 10^{m-1} + 1 = 9 \cdot 10^{m-1}$. The total number positive even integers with m digits is $9 \cdot 10^{m-1}/2 = 45 \cdot 10^{m-2}, m \geq 2$.

Denote by $N(m)$ the last decimal position taken by positive even integers with m digits in the decimal constructed in the problem. Then the following recurrence equality holds $N(m) = N(m-1) + m \cdot 45 \cdot 10^{m-2}$, $m \geq 2$. Find manually $N(1) = 4$. By the recurrence equation, $N(2) = N(1) + 2 \cdot 45 \cdot 10^0 = 94$, $N(3) = N(2) + 3 \cdot 45 \cdot 10^{3-2} = 1444$, $N(4) = N(3) + 4 \cdot 45 \cdot 10^{4-2} = 19444$.

Between 1445 position and 2010 position there are $2010 - 1444 = 566$ positions taken by four-digit positive even integers. Since $566 = 4 \cdot 141 + 2$, then 2010^{th} position is occupied by the 2^{nd} digit of 142^{th} four-digit positive even integer. This integer is $999 + 2(142 - 1) + 1 = 1282$. The digit at 2010^{th} place is 2.

Answer - A

10. When $AMATYC$ is transformed into a 6-digit number by replacing identical letters with identical digits and different letters with different digits, the result is divisible by 35. Find the final 2 digits for the least such number.

A 25 B 35 C 45 D 65 E 75

Solution. The factors of 35 are $5, 7$, so that $AMATYC$ is divisible by both $5, 7$. The divisibility by 5 indicates that $C = 0$, or 5. Find the conditions imposed on the rest of the digits by divisibility by 7. Write $AMATYC = A \cdot 10^5 + M \cdot 10^4 + A \cdot 10^3 + T \cdot 10^2 + Y \cdot 10 + C = A(10^5 + 10^3) + M \cdot 10^4 + T \cdot 10^2 + Y \cdot 10 + C$. Calculate $10^5 + 10^3 = 14428 \cdot 7 + 4$, $10^4 = 1428 \cdot 7 + 4$, $10^2 = 14 \cdot 7 + 2$, $10 = 7 + 3$. Then $AMATYC = 7(14428A + 1428M + 14T + Y) + (4A + 4M + 2T + 3Y + C)$. Since the first expression here on the right is divisible by 7, then $N_1(A, M, T, Y, C) = 4A + 4M + 2T + 3Y + C$ must also be divisible by 7.

We use trials and error to find the smallest number $AMATYC$ in which N_1 is divisible by 7. Since $A \neq 0$, put $A = 1$. Then set $M = 0$. This leaves $C = 5$. Thus, $N_1(1, 0, T, Y, 5) = 4 \cdot 1 + 4 \cdot 0 + 2T + 3Y + 5 = 9 + 2T + 3Y$. As $9 = 7 + 2$, then $N_1(1, 0, T, Y, 5) = 9 + 2T + 3Y = 7 + (2 + 2T + 3Y)$. It follows that $N_2(T, Y) = 2 + 2T + 3Y$ must be divisible by 7, where and T, Y are the digits from the set $\{2, 3, 4, 6, 7, 8, 9\}$. Tentatively, set $T = 2$. Thus, $N_2(2, Y) = 6 + 3Y$. Trying all $Y \in \{3, 4, 6, 7, 8, 9\}$, we find that $N_2(2, Y)$ is not divisible by 7. Tentatively, set $T = 3$, so that $N_2(3, Y) = 8 + 3Y$ and $Y \in \{2, 4, 6, 7, 8, 9\}$. We find that $N_2(3, 2) = 14$, being divisible by 7.

Concluding, the smallest possible value of $AMATYC = 101325$

Answer - A

11. Let $f(x) = \ln(x + \sqrt{1 + x^2})$. Find $f^{-1}(\ln 7)$

A $\frac{11}{7}$ B $\frac{24}{7}$ C $\frac{7}{2}$ D $\frac{7}{24}$ E $\frac{7}{11}$

Solution. Denote $f^{-1}(\ln 7) = x$. Then $f(x) \equiv \ln(x + \sqrt{1 + x^2}) = \ln 7$, and $x + \sqrt{1 + x^2} = 7$. Simplifying this equation by eliminating the radical, we will get $x = \frac{24}{7}$.

Answer - B

12. If $\sin(30^{\cdot} + \arctan x) = \frac{13}{14}$, and $0 < x < 1$, the value of x is $\frac{a}{b}\sqrt{3}$, where a and b are positive integers with no common prime factors. Find $a + b$.

Solution. $\sin(30^{\cdot} + \arctan x) = \frac{13}{14} \Leftrightarrow 30^{\cdot} + \arctan x = \arcsin\frac{13}{14} \Leftrightarrow \arctan x = \arcsin\frac{13}{14} - 30^{\cdot} \Leftrightarrow x = \tan(\arcsin\frac{13}{14} - 30^{\cdot}) \Leftrightarrow x = \frac{\tan\arcsin\frac{13}{14} - \tan 30^{\cdot}}{1+\tan\arcsin\frac{13}{14}\cdot\tan 30^{\cdot}}$. Since $\tan\arcsin t = \pm\frac{t}{\sqrt{1-t^2}}$, then $\tan\arcsin\frac{13}{14} = \pm\frac{13}{3\sqrt{3}}$. Also $\tan 30^\circ = \frac{1}{\sqrt{3}}$. Thus $x = \left(\pm\frac{13}{3\sqrt{3}} - \frac{1}{\sqrt{3}}\right) \div \left(1 \pm \frac{13}{3\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\right) = \frac{\pm 13-3}{3\sqrt{3}} \div \left(1 \pm \frac{13}{9}\right) = \frac{(\pm 13-3)9}{3\sqrt{3}(9\pm 13)} = \frac{\pm 13-3}{9\pm 13}\sqrt{3}$. Because $0 < x < 1$, in the last expression for x in $\pm$ only $+$ sign fits. So $x = \frac{13-3}{9+13}\sqrt{3} = \frac{10}{22}\sqrt{3} = \frac{5}{11}\sqrt{3}$, and $a = 5, b = 1$, $a + b = 16$

Answer - A

13. The equation $a^5 + b^2 + c^2 = 2010$ (a, b, c positive integers), has a solution in which b and c have a common factor $d > 1$. Find d.

A 2 B 3 C 5 D 7 E 11

Solution. From the equation, $a^5 = 2010 - (b^2 + c^2) < 2010$, or $a < \sqrt[5]{2010} < 4.6$. So that $a \in \{1, 2, 3, 4\}$. Consider $b^2 + c^2 = 2010 - a^5$. Because b, c have a common factor, denote by d their the greatest common factor. So $b = dp, c = dq$, where p and q have no common factors. Then the equality $b^2 + c^2 = 2010 - a^5$ transforms into $p^2 + q^2 = \frac{2010-a^5}{d^2}$

Case $a = 1$. $2010-a^5 = 2009$.Factoring $2009 = 41\cdot 7^2$, and we can take $d = 7$. Then $\frac{2010-a^5}{d^2} = 41$, and the equation $p^2 + q^2 = 41$ has a solution $(p, q) = (4, 5)$

Case $a = 2$. $2010 - a^5 = 1978$.Factoring $1978 = 2 \cdot 23 \cdot 43$ and among the factors there is no repeating ones.

Case $a = 3$. $2010 - a^5 = 1767$.Factoring $1767 = 3 \cdot 19 \cdot 31$ and among the factors there is no repeating ones.

Case $a = 4$. $2010-a^5 = 986$.Factoring $986 = 2\cdot 17\cdot 29$ and among the factors there is no repeating ones.

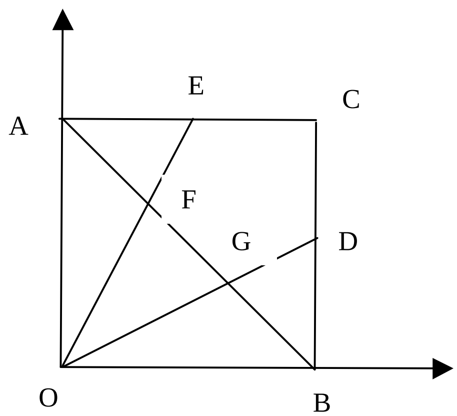
The only common factor $d = 7$

Answer - D

14. Two real numbers are chosen independently at random from the interval $[0, 1]$. Find the probability that their sum < 1 AND one is at least twice the other.

A $\frac{1}{5}$ B $\frac{1}{4}$ C $\frac{1}{3}$ D $\frac{2}{5}$ E $\frac{1}{2}$

Solution. The random space for the pair (x, y) of independent uniformly distributed on $[0, 1]$ random variables is the square $S = \{(x, y)|\ 0 \le x \le 1, 0 \le y \le 1\}$. The probability of any region $R \subset S$ is the area of R. The region defined in the problem is $R = \{(x, y)|x + y < 1 \text{ AND } (y \ge 2x \text{ OR } x \ge 2y)\}$



In the picture the sample square is represented by the square $OACB$. The line $x+y=1$ is represented by the segment AB, the line $y=2x$ is represented by the segment OE, and the line $x=2y$ is represented by the segment OD. The region $R=\triangle OAF \cup \triangle OGB$. The $Area(R) = Area(\triangle OAF) + Area(\triangle OGB) = 2Area(\triangle OGB)$.

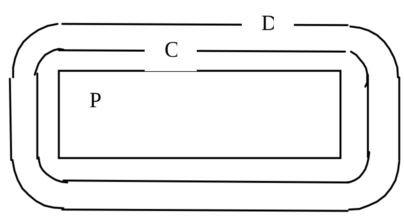
To find the y coordinate y_G of the point G solve the equations $x+y=1, x=2y$ to obtain $y_G = \frac{1}{3}$. $Area(\triangle OGB) = \frac{1}{2} y_G |OB| = \frac{1}{6}$. $Area(R) = 2 \cdot \frac{1}{6} = \frac{1}{3}$

Answer - C

15. A rectangle R has width 4 and length 6. The curve C consists of all points outside of R whose distance to the nearest point of R is 1, and D consists of all points of C whose distance to the nearest point of C is 1. Find the area enclosed by D, rounded to the nearest square unit.

A 71 B 73 C 75 D 77 E 79

Solution.



As follows from the picture, the area inside the curve D is the sum of the areas of several rectangles and the areas of four quarter-circles of radius 2. Meticulously calculating the total area of the rectangles, we obtain the value of 64. Calculating the total area of quarter-circles, we come to value of 12.57. The total area inside the curve D is $64 + 12.57 \approx 77$

Answer - C

16. Let $f(x) = \frac{\sqrt{x^2-1}}{x}$. Find the set of all real values of x for which $f(f(x))$ exists.

A $|x| > 1$ B $|x| \geq 1$ C $x = \pm 1$ D $|x| \leq 1$, $x \neq 0$ E no value of x

Solution. For a function $g(x)$, denote by $D(g)$ and $R(g)$ the domain and the range of $g(x)$. Then $D(f*f) = D(f) \cap R(f)$. $D(f) = \{x : |x| \geq 1\}$. For $x \in D(f)$, $|f(x)| = \frac{|x|\sqrt{1-1/x^2}}{|x|} = \sqrt{1-\frac{1}{x^2}} =< 1$, so that $R(f) \subset \{y : |y| < 1\}$. It follows that that $D(f) \cap R(f) \subset \{x : |x| \geq 1\} \cap \{y : |y| < 1\} = \varnothing$

Answer - E

17. The integer $r > 1$ is both the common ratio of an integer geometric sequence and the common difference of an integer arithmetic sequence. Summing resulting terms of the sequences yields $7, 26, 90, ...$ The value of r is

A 2 B 4 C 8 D 12 E 16

Solution. Denoting by g and a the terms of geometric and arithmetic sequances for which the sum is 7, we will have the system of three equations

$g + a = 7,$

$gr + a + r = 26,$

$gr^2 + a + 2r = 90.$

Subtracting the first equation from the second, and the second equation from the third, we will have

$g(r-1) + r = 19,$

$gr(r-1) + r = 64.$

From the first equation of the last system find $g(r-1) = 19 - r$ and plug this expression into the second system to obtain $r(19-r) + r = 64$. This is a quadratic equation with the solutions $r = 16$ and $r = 4$. From the first equation of the last system we get $g = \frac{19-r}{r-1}$

Case $r = 16$. Then $g = \frac{19-16}{16-1}$ which is not an integer. Value $r = 16$ is not a solution of the problem

Case $r = 4$. Then $g = \frac{19-4}{4-1} = 5$. From the first equation of the first system we find $a = 7 - g = 7 - 5 = 2$. Thus, $r = 4$ is the solution of the problem.

Answer - B

18. In a certain sequence, the first two terms are prime, and each term after the second is the product of the previous two terms. If seventh term is $12,500,000$, find the eighth term divided by the seventh term

A 1000 B 2500 C 5000 D 10000 E 25000

Solution. In the sequence $\{a_m\}_{m=1}^{\infty}$, $a_m = a_{m-2}a_{m-1}$, $m = 3, 4, ...$ Denote $a_1 = p, a_2 = q$. Using the recurrence relation, we can find the first several terms of the sequence: $a_3 = a_1a_2 = pq$, $a_4 = a_2a_3 = pq^2$, $a_5 = a_3a_4 = p^2q^3$, and similarly, $a_6 = p^3q^5$, $a_7 = p^5q^8$, $a_8 = p^8q^{13}$. Since $a_7 = 12,500,000$, then $p^5q^8 = 12,500,000$. Factoring $12,500,000 = 2^5 5^8$, we find $p^5q^8 = 2^5 5^8$, and $p = 2$, $q = 5$. Thus, $a_8/a_7 = p^3q^5 = 2^3 5^5 = 25000$

Answer - E

19. Let $P(x)$ be a polynomial with nonnegative integer coefficients. If $P(2) = 77$ and $P(P(2)) = 1874027$, find the sum of its coefficients.

A 11 B 13 C 15 D 17 E 19

Solution. In all estimates for coefficients of $P(x)$ below, I liberally use the fact that coefficients of $P(x)$ are nonnegative integers.

Let the degree of $P(x)$ be n. Since $P(77) = P(P(2)) = 1874027$, then $77^n \le P(77) = 1874027$, and $n \le \log_{77} 1874027 < 3.4$. That is $n \le 3$.

For $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, $P(77) = a_3 77^3 + a_2 77^2 + a_1 77 + a_0 = 1874027$, and

$P(2) = a_3 2^3 + a_2 2^2 + a_1 2 + a_0 = 77$ (1)

Then $1873950 = P(77) - P(2) = a_3(77^3 - 2^3) + a_2(77^2 - 2^2) + a_1(77 - 2)$, or $a_3 75 \cdot 6087 + a_2 75 \cdot 79 + a_1 75 = 1873950$. Dividing both sides by 75, we get

$6087a_3 + 79a_2 + a_1 = 24986$ (2)

From (2), $6087a_3 \le 6087a_3 + 79a_2 + a_1 = 24986$. Thus $a_3 \le \frac{24986}{6087} < 4.11$ and

$0 \le a_3 \le 4$ (3)

From (2), $79a_2 + a_1 = 24986 - 6087a_3$, $79a_2 \le 79a_2 + a_1 = 24986 - 6087a_3$, and

$a_2 \le \frac{24986-6087a_3}{79}$ (4)

From (1), if follows that $a_1 + a_2 = \frac{1}{2}(2a_1 + 2a_2) \le \frac{1}{2}(2a_1 + 4a_2) = \frac{1}{2}(77 - 8a_3 - a_0) \le \frac{1}{2}77 = 38.5$. So that

$a_1 + a_2 \le 38$ (5)

From (2), it follows that $a_1 = 24986 - (6087a_3 + 79a_2)$ and from (5) $38 \ge 24986 - (6087a_3 + 79a_2) + a_2 = 24986 - 6087a_3 - 78a_2$, or $78a_2 \ge 24986 - 38 - 6087a_3 = 24948 - 6087a_3$, or

$a_2 \ge \frac{24948-6087a_3}{78}$ (6)

Comparing (4) and (6), we find

$\frac{24948-6087a_3}{78} \le a_2 \le \frac{24986-6087a_3}{79}$ (7)

From (7), $\frac{24948-6087a_3}{78} \le \frac{24986-6087a_3}{79}$, or $a_3 \ge 3.61$. Comparing this with (3), we find that

$a_3 = 4$ (8)

From (7) and (8), $\frac{24948-6087\cdot4}{78} \le a_2 \le \frac{24986-6087\cdot4}{79}$, or $7.6923 \le a_2 \le 8.0760$. So

$a_2 = 8$ (9)

From (2), $a_1 = 24986 - (6087a_3 + 79a_2) = 24986 - (6087 \cdot 4 + 79 \cdot 8)$. Calculating, find

$a_1 = 6$ (10)

From (1), $a_0 = 77 - (a_3 2^3 + a_2 2^2 + a_1 2) = 77 - (4 \cdot 8 + 8 \cdot 4 + 6 \cdot 2)$, and

$a_0 = 1$ (11)

The total $a_3 + a_2 + a_1 + a_0 = 4 + 8 + 6 + 1 = 19$

Answer - E

20. If $|x-1| + |x-2| + ... + |x-2010| \ge m$ for every real number x, find the maximum possible value for m.

A $1004{\cdot}1005$ B 1005^2 C $1004{\cdot}1006$ D 1006^2 E $1005 \cdot 1006$

Solution. Denote $f(x) = \sum_{k=1}^{2010} |x-k|$. We have to find the value x_0 such that $f(x) \ge f(x_0)$ for all real x. Then $f(x_0) = m$.

Let $x = y + 1005.5$. Then

$g(y) = f(x) = f(y + 1005.5) = \sum_{k=1}^{2010} |y + 0.5 - (k - 1005)|$
$= \sum_{k=1}^{1005} |y + 0.5 - (k - 1005)| + \sum_{k=1006}^{2010} |y + 0.5 - (k - 1005)|$
$= \sum_{m=-1004}^{0} |y + 0.5 - m| + \sum_{m=1}^{1005} |y + 0.5 - m|$
$= \sum_{m=-1005}^{-1} |y - 0.5 - m| + \sum_{m=1}^{1005} |y + 0.5 - m|$
$= \sum_{m=1}^{1005} |y-0.5+m| + \sum_{m=1}^{1005} |y+0.5-m| = \sum_{m=1}^{1005} (|y - 0.5 + m| + |y + 0.5 - m|)$
Show that the function $g(y)$ is even.
Really,
$g(-y) = \sum_{m=1}^{1005} (| - y - 0.5 + m| + | - y + 0.5 - m|)$
$= \sum_{m=1}^{1005} (|y + 0.5 - m| + |y - 0.5 + m|)$
$= g(y)$.
Thus, $\min g(y)$ can be located at $y \geq 0$.
For $y \geq 0$ rewrite
$g(y) = \sum_{m=1}^{1005} (y - 0.5 + m + |y + 0.5 - m|)$
$= \sum_{m \leq y+0.5} (y-0.5+m+y+0.5-m) + \sum_{m>y+0.5} ((y - 0.5 + m) - (y + 0.5 - m))$

$= \sum_{m \leq y+0.5} 2y + \sum_{m>y+0.5} (2m - 1)$, $y \geq 0$.

For a fixed $y \neq$ integer $+0.5$ and in a small neighborhood of y the sums on the right in the last equation have a fixed number of terms and are non-decreasing functions of y. Since $g(y)$ is a continuous function, then $g(y)$ is non-decreasing for $y \geq 0$, and attains its minimum value at $y = 0$. Thus $\min g(y) = g(0) = \sum_{m>0.5} (2m - 1) = \sum_{m=1}^{1005} (2m - 1) = \frac{1+2009}{2} 1005 = 1005^2$

Answer - B