Test #1 AMATYC Student Mathematics League Oct/Nov 2011

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Solutions

1. If the standard order of operations is reversed (that is, additions and subtractions are done first and exponentiation is done last), what is the value of $2 \cdot 3\hat{}2 + 3$?

A 21 B 24 C 39 D 486 E 7776

Solution. Using the grouping symbols, we have $(2 \cdot 3)\hat{}(2+3) = 6\hat{}5 = 7776$

Answer - E

2. The price of a stock rose 20% on Monday, fell 10% on Tuesday, and increased by 1/6 on Wednesday. By what percent did the price rise from Monday to Wednesday?

A 24 B 26 C 28 D 30 E 32

Solution. Denote by P the stock on Monday morning. Then the stock price A on Wednesday evening will be $A = P(1.2)(0.9)(1+\frac{1}{6}) = P\frac{63}{50}$. The price percent increase is $\frac{A-P}{P}100\% = (\frac{A}{P} - 1)100\% = (\frac{63}{50} - 1)100\% = 26\%$

Answer - B

3. The system of equations $ax - by = 8$ and $ax + by = 20$ has the solution $(x, y) = (2.3)$. Find $a + b$

A 6 B 7 C 8 D 9 E 10

Solution. Substituting the values of $x = 2, y = 3$, into the initial system, we obtain the system of equations $2a - 3b = 8, 2a + 3b = 20$. Solving this system for a, b, we get $a = 7, b = 2$, and $a + b = 9$

Answer - D

4. The positive integers a, b, and c satisfy $a^6 + b^2 + c^2 = 2011$. Find $a + b + c$.

A 50 B 51 C 52 D 53 E 54

Solution. Since $a^6 < a^6 + b^2 + c^2 = 2011$, then $a \le \lfloor \sqrt[6]{2011} \rfloor \le \lfloor 3.56 \rfloor = 3$. So $a \in \{1, 2, 3\}$

Case $a = 1$. In this case $b^2 + c^2 = 2010$. Factor $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. Represent $b = 3m + p, c = 3n + q$, where $p \in \{0, 1, 2\}, q \in \{0, 1, 2\}$. Thus, $2 \cdot 3 \cdot 5 \cdot 67 = (3m + p)^2 + (3n + q)^2 = 3(3m^2 + 2p + 3n^2 + 2q) + (p^2 + q^2)$ and $p^2 + q^2$ is divisible by 3. But $p^2 \in \{0, 1, 4\}, q^2 \in \{0, 1, 4\}$, and $p^2 + q^2$ is divisible by 3 only when $p = q = 0$. Consequently, $2 \cdot 3 \cdot 5 \cdot 67 = (3m)^2 + (3n)^2 = 9(m^2 + n^2)$, and $3(m^2 + n^2) = 2 \cdot 5 \cdot 67$. This is a contradiction since the right side is not divisible by 3.

Case $a = 2$. In this case $b^2 + c^2 = 1947$. Factor $1947 = 3 \cdot 11 \cdot 59$. Repeating the reasoning for the previous case, we find that this case also leads to contradiction.

Case $a = 3$. In this case $b^2 + c^2 = 1282$. It is easy to see that b, c should be both odd numbers. Let $b \ge c$. Estimate $b = \sqrt{1282 - c^2} \le \sqrt{1282 - 1^2} \le 35$. On the other hand, $b^2 \geq \frac{1}{2}(b^2+c^2) = \frac{1}{2}1282 = 641$, and $b \geq \sqrt{641} \geq 27$. Calculating $c = \sqrt{1282 - b^2}$ for $b = 27, 29, 31, 33, 35$, we find that the only solution values are $b = 29, c = 21$.

Finally, the value $a + b + c = 3 + 29 + 21 = 53$

Answer - D

5. Different shades of pink, red, and white can be made by mixing whole numbers of quarts of red and white paint. Shades are different if the ratio of red to white paint is different. Find the number of different possible shades that can be made from at most 4 quarts of red and 5 quarts of white paint.

A 15 B 16 C 17 D 18 E 19

Solution. Any shade is defined by a whole number r of red paint quarts and a whole number w of white paint quarts , where $0 \leq r \leq 4, 0 \leq w \leq 5$, and r, w have no common factors. The possible pairs of (r, w), listed in ascending order of 4, are $(0,1)$, $(1,0)$, $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(2,1)$, $(2,3)$, $(2,5)$, $(3,1)$, $(3,2)$, $(3,4)$, $(3,5)$, $(4,1)$, $(4,3)$, $(4,5)$. Totally, there is 17 shades

Answer - C

6. The function $y = f(x)$ has zeros -2, and 6.Find the zeros of $y = -3f(2-2x)$

A 2, -2 B 5, 1 C 4, -1 D -1, -5 E 1, -3

Solution. Let $g(x) = -3f(2-2x)$, so that $f(2-2x) = -\frac{1}{3}g(x)$. Denote $u = 2-2x$, thus $x = \frac{2-u}{2}$. Then $f(u) = -\frac{1}{3}g(\frac{2-u}{2})$.

For $u = -2$, $0 = f(-2) = -\frac{1}{3}g(\frac{2-(-2)}{2}) = -\frac{1}{3}g(2)$

For $u = 6$, $0 = f(6) = -\frac{1}{3}g(\frac{2-6)}{2}) = -\frac{1}{3}g(-2)$

Zeros of $g(x) = -3f(2-2x)$ are numbers 2 and -2

Answer - A

7. One population $P_1(t)$ grows exponentially at the same rate that another population $P_2(t)$ decays exponentially. If the populations were both equal to P on Jan. 1, 2009, how will the populations be related on Jan. 1, 2012?

A $P_1(t)P_2(t) = P$ B $P_1(t)P_2(t) = P^2$ C $P_1(t)/P_2(t) = P$
D $P_1(t) + P_2(t) = P$ E $P_1(t) + P_2(t) = 2$

Solution. Let the populations changes be $P_1(t) = A\exp(kt)$ and $P_2(\tau) = B\exp(-k\tau\}$, and let Jan. 1, 2009 corresponds to $t = t_0$ and $\tau = \tau_0$. Then $P = P_1(t_0) = P_2(\tau_0)$, or $A = P\exp(-kt_0), B = P\exp(k\tau_0)$. Thus, $P_1(t) = P\exp\{k(t-t_0)\}, P_2(\tau) = P\exp\{-k(\tau-\tau_0)\}$. Since the date of Jan. 1, 2012 corresponds to $t_1 = t_0 + 3$ and $\tau_1 = \tau_0 + 3$, then $P_1(t_1) = P\exp\{k(t_1 - t_0)\} = P\exp\{3k)\}, P_2(\tau_1) = P\exp\{-k(\tau_1 - \tau_0)\} = P\exp\{-3k\}$. Consequently, $P_1(t_1)P_2(\tau_1) = P^2$

Answer - B

8. For $b > c > 0$, both $x^2 + bx + 8$ and $x^2 + cx + 8$ factor over the integers. Find $b - c$

A 1 B 2 C 3 D 4 E 5

Solution. Factor $8 = 1 \cdot 8 = 2 \cdot 4$. Since both trinomials factor over the integers, then b, c are one of the numbers $1 + 8 = 9, 2 + 4 = 6$. As $b > c$, then $b = 9, c = 6$. Finally, $b - c = 9 - 6 = 3$

Answer - C

9. Ed drives from San Mateo to Atascadero, a distance of 197.5 mi. He starts driving at a constant speed and reduces his speed by 5 mph after each half hour of driving. If the trip takes 3hr 20 min, how far did he travel in the first 2 hours?

A 127 B 132 C 137 D 142 E 147

Solution. Denote by v mph the Ed's initial speed. Then the total driving distance of 197.5 mi can be represented as $197.5 = v\frac{1}{2} + (v-5)\frac{1}{2} + (v-10)\frac{1}{2} + (v-15)\frac{1}{2} + (v-20)\frac{1}{2} + (v-25)\frac{1}{2} + (v-30)\frac{1}{3}$. Simplify this equation to obtain $197.5 = \frac{10}{3}v - \frac{95}{2}$.Solve this equation for v to get $v = 73.5$ mph. The distance traveled for the first two hours is $v\frac{1}{2} + (v-5)\frac{1}{2} + (v-10)\frac{1}{2} + (v-15)\frac{1}{2} = 132$

Answer - B

10. Sun fills her 10 liter radiator with 20% antifreeze and 80% water. She removes some of the mixture and replaces it with antifreeze. If the radiator is now one quarter antifreeze, how many liters of the original mixture did she removed?

A 0.25 B 0.375 C 0.5 D 0.625 E 0.75

Solution. Denote by x the amount of the original mixture Sun removed. The amount of antifreeze just after the removal is $(10-x)0.2$. After the replacement with pure antifreeze, the total amount of antifreeze is $(10-x)0.2+x$. This quantity is now one quarter of the initial volume, that is $(10-x)0.2+x = 0.25 \cdot 10$. Solving the last equation, we get $x = 0.625$

Answer - D

11. How many numbers with no more than six digits can be formed using only the digits 1 through 7, with no digit used more than once in a given number?

A 879 B 1956 C 3619 D 5040 E 8659

Solution. Denote N_m, $m = 1, ...6$, the the count of m-digit numbers formed from digits $1, 2, 3, 4, 5, 6, 7$ without digits repetition. We use the Fundamental Counting Principle to calculate $N_1 = 7$, $N_2 = 7 \cdot 6 = 42$, $N_3 = 7 \cdot 6 \cdot 5 = 210$, $N_4 = 7 \cdot 6 \cdot 5 \cdot 4 = 840$, $N_5 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$, $N_6 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 5040$. The total count $N_1 + N_2 + N_3 + N_4 + N_5 + N_6 = 8659$

Answer - E

12. The lines with equations $2x + 3y = 24$ and $3x + 2y = 6$ are symmetric with respect to a line with equation $y = mx + b$ with $m > 0$. Find $m + b$.

A 5 B 12 C 17 D 19 E 20

Solution. Solving the system formed by the given equations, we find the point of intersection $A = (-6, 12)$. Introduce new coordinates (X, Y) by $x = X - 6, y = Y + 12$. Then the equations of the given lines are translated into $2X + 3Y = 0$, $3X + 2Y = 0$, or $Y = -\frac{2}{3}X, Y = -\frac{3}{2}X$. The functions $f(X) =$

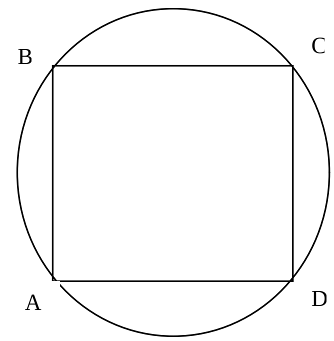
$-\frac{2}{3}X$, $g(X) = -\frac{3}{2}X$ are inverses of one another, and their graphs are symmetric with respect to the line $Y = X$. In the original coordinate system the line $Y = X$ has the form $y - 12 = x + 6$, or $y = x + 18$. For this line, $m + b = 1 + 18 = 19$

Answer - D

13. A square of area 45 is inscribed in circle C. Find the area of a square inscribed in a semicircle of circle C. (Inscribed means having all 4 vertices on the given figure).

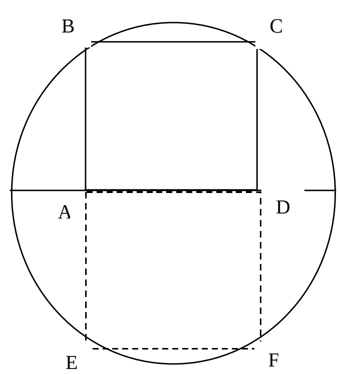
A $5\sqrt{5}$ B 18 C $9\sqrt{5}$ D 20 E 25

Solution. In the figure below a square is inscribed into a circle of diameter d



By the Pythagorean theorem, $|AB|^2+|BC|^2 = |AC|^2 = d^2$, so that $2|AB|^2 == d^2$, and the area of the square $A = |AB|^2 = \frac{d^2}{2}$

In the figure below a square $ABCD$ is inscribed into a semicircle of diameter d.



By the Pythagorean theorem, $(|AB|+|AE|)^2+|BC|^2 = |EC|^2 = d^2$, so that $(2|AB|)^2 + |BC|^2 = d^2$, or $5|AB|^2 = d^2$ and $|AB|^2 = \frac{d^2}{5}$.The area of the square $ABCD$ $a = |AB|^2 = \frac{d^2}{5}$

Comparing the expression A for the area of a square inscribed in a circle with that of the area a of a square inscribed into semicircle, we see that $a = \frac{2}{5}A$.

If $A = 45$, then $a = 18$

Answer - B

14. The left edge of a dollar bill is folded against the bottom edge to form an isosceles right triangle at the left end. The new left edge is again folded against the bottom edge. A vertex of a new triangle is the upper right corner of the bill. If a dollar bill is 157 mm long, find its width to the nearest millimeter.

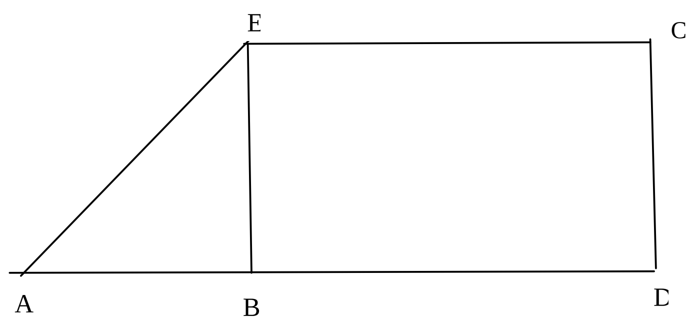
A 63 B 64 C 65 D 66 E 67

Solution. Thew figures below show the sequential stages for the bill folding

Picture below shows the initial state of the dollar bill.

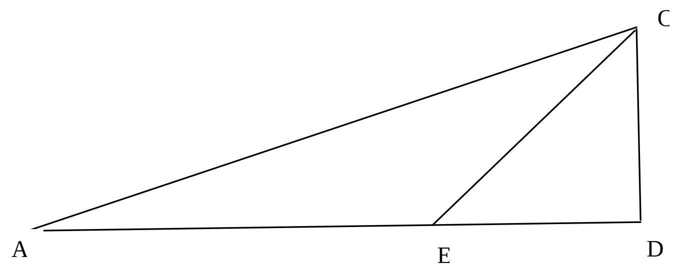
$\angle BAC = \pi/2$

The next picture shows the bill after the first folding.



$\angle EAB = \pi/4$

The last picture shows the bill after the second folding. $\angle EAB = \pi/4$



$\angle CAD = \pi/8$
The length $|CD| = |AD| \tan \angle CAD = 157 \tan \pi/8 \approx 65$
Answer - C

15. Five boxes are placed inside an empty box. Each of the 5 new boxes is either left empty or has 5 new boxes placed inside it. This process is repeated until there are 18 boxes containing other boxes. Find the number of empty boxes.

A 73 B 75 C 77 D 79 E 81

Solution. Denote by E_n and F_n the count of corresponding empty and filled boxes after the n_{th} step of the procedure. At the next procedure step one empty box gets filled by five empty boxes, so that $F_{n+1} = F_n + 1$ and $E_{n+1} = E_n - 1 + 5 = E_n + 4$. Sequences $\{E_n\}$ and $\{F_n\}$ are arithmetic with the differences 1 and 4. Thus, $F_n = F_0 + n$ and $E_n = E_0 + 4n$. As $E_0 = 1$, $F_0 = 0$, then $F_n = n$, $E_n = 1 + 4n$. For $F_n = 18$, $n = 18$, and $E_{18} = 1 + 4 \cdot 18 = 73$
Answer - A

16. Al, Bo, Cy, and Di are to receive math, physics, chem, and bio awards. Al thinks Di will win bio, Bo thinks Cy will win chem, Cy thinks Al won't win math, and Di thinks Bo will win physics. The math and bio winners are both right, and the other winners are both wrong. Who wins the math award?

A Al B Bo C Cy D Di E not enough information given

Solution. Numerate the problem statements
Al thinks Di will win bio (1)
Bo thinks Cy will win chem (2)
Cy thinks Al won't win math (3)
Di thinks Bo will win physics (4)
The math and bio winners are both right (5)
The chem and phys winners are both wrong (6)

Assume Al is right
From (1)
Di is bio winner (7)
From (7),(5),(6)
Al is math winner (8)

From (7),(5),(4)
Bo is phys winner (9)
From *(7),(8),(9)*
Cy is chem winner (10)
From (9),(6),(2)
Cy is not chem winner (11)
(10) and (11) are contradictory

Assume Al is wrong
From (1)
Di is math or chem or phys winner (12)
From (6)
Al is chem or phys winner (13)
From (13),(3)
Cy is right (14)
From (14),(5)
Cy is math or bio winner (15)
From (15),(2)
Bo is wrong (16)
From (16), (6)
Bo is chem or phys winner (17)
From (12),(15),(17)
Di is math winner (18)
From (18),(15)
Cy is bio winner (19)
From (18),(5),(4)
Bo is phys winner (20)
From (13), (20)
Al is chem winner (21)
Answer - D

17. The digits 1 through 9 are separated into 3 groups of three digits, and the product of each group is found. Let P be the largest of 3 products. Find the smallest possible value of P.

A 70 B 71 C 72 D 73 E 74

Solution. Let P_1, P_2, P_3 be the products of numbers in the groups, and let $P = \max(P_1, P_2, P_3)$. Then $P^3 \geq P_1 \cdot P_2 \cdot P_3 = 9! = 362880$, and $P \geq \sqrt[3]{362880} \geq 71.3$. As P is an integer, then $P \geq 72$. Show that $P = 72$ is attainable. Factor $72 = 9 \cdot 8$ and set $P_1 = 9 \cdot 8 \cdot 1$. By trials and checks, put $P_2 = 7 \cdot 5 \cdot 2 = 70$, and $P_3 = 6 \cdot 4 \cdot 3 = 72$

Answer - C

18. Out of 10 red chips and 15 green chips, 6 are placed into a bag, 10 into a box, and 9 into a bowl. In how many ways can the chips be distributed, if only the number of red and green chips in each container matters?

A 45 B 49 C 50 D 55 E 56

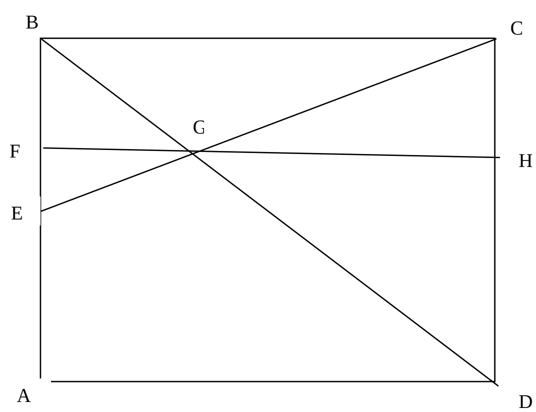
Solution. Let r_1, r_2, r_3 and g_1, g_2, g_3 be the number of red and green chips in the bag, the box, and the bowl. Then $r_1 + g_1 = 6$, $r_2 + g_2 = 10$, $r_3 + g_3 = 9$. In these equations, moving the red variables to the right and taking into account that $r_3 = 10 - (r_1 + r_2)$, we get $g_1 = 6 - r_1$, $g_2 = 10 - r_2$, $g_3 = r_1 + r_2 - 1$. We are looking for the number of nonnegative integer solutions of this system. Geometrically, we are looking for the number of nonnegative integer lattice points of r_1, r_2 plane in the region $0 \le r_1 \le 6$, $0 \le r_2 \le 10$, $r_1 + r_2 \ge 1$, $r_1 + r_2 \le 10$,

$r_1 = 0$, $r_2 \ge 1$, $r_2 \le 10$ 10 points
$r_1 = 1$, $r_2 \ge 0$, $r_2 \le 9$ 10 points
$r_1 = 2$, $r_2 \ge 0$, $r_2 \le 8$ 9 points
$r_1 = 3$, $r_2 \ge 0$, $r_2 \le 7$ 8 points
$r_1 = 4$, $r_2 \ge 0$, $r_2 \le 6$ 7 points
$r_1 = 5$, $r_2 \ge 0$, $r_2 \le 5$ 6 points
$r_1 = 6$, $r_2 \ge 0$, $r_2 \le 4$ 5 points
Total number of points is 55
Answer - D

19. Square $ABCD$ has side length 72. Let E be the midpoint of side AB, and let $\overline{BD}$ and $\overline{CE}$ intersect at G. Find the length of the altitude to $\overline{BE}$ in $\triangle GEB$.

A 12 B 18 C 21 D 24 E 27

Solution. Consider a square of side $a = 72$, see below. We have to find the length $|FG|$



Here $|AB| = a, |BE| = \frac{a}{2}$

The triangles $\triangle EBG$, $\triangle CDG$ are similar as having equal angles. Then $\frac{|FG|}{|GH|} = \frac{|BE|}{|CD|} = \frac{1}{2}$. By the properties of proportions, $\frac{|FG|}{|GH|+|FG|} = \frac{1}{2+1}$. Since $|GH| + |FG| = a$, then $|FG| = \frac{1}{3}a$.

For $a = 72$, $|FG| = 24$

Answer - D

20. Let r be the positive real zero of $P(x) = 9x^5 + 7x^2 - 9$. The sum $r^4 + 2r^9 + ... + kr^{5k-1} + ...$ can be represented as the rational number a/b in lowest terms. Find $a + c$.

A 110 B 115 C 120 D 125 E 130

Solution. The polynomial cannot have real zeros $x \geq 1$, since $9x^5+7x^2-9 = (9x^5 - 9) + 7x^2 > 0$ for $x \geq 1$. Thus the sum in the problem description is convergent.

For $0 \leq r < 1$, rewrite the sum as $S(r) = r^4(1 + 2r^5 + ... + kr^{5(k-1)} + ...) = r^4 \sum_{k=1}^{\infty} kr^{5(k-1)} = r^4 \sum_{k=1}^{\infty} ku^{k-1}$, where $u = r^5$. To find the finite expression for $\sum_{k=0}^{\infty} ku^{k-1}$, write the geometrical series $\sum_{k=0}^{\infty} u^k = \frac{1}{1-u}$, and square here both sides.

Then $\frac{1}{(1-u)^2} = \left(\sum_{k=0}^{\infty} u^k\right)^2 = \left(\sum_{m=0}^{\infty} u^m\right)\left(\sum_{n=0}^{\infty} u^n\right) = \sum\limits_{m\geq 0, n\geq 0} u^{m+n}$

$$= \sum_{k=1}^{\infty}\Big(\sum\limits_{\forall(m\geq 0, n\geq 0): m+n=k-1} 1\Big)u^{k-1} = \sum_{k=1}^{\infty}\Big(\sum\limits_{m=0}^{k-1} 1\Big)u^{k-1} =$$

$\sum_{k=0}^{\infty} ku^{k-1}$.

Thus, with $u = r^5$, $S(r) = r^4 \sum_{k=1}^{\infty} ku^{k-1} = r^4 \frac{1}{(1-u)^2} = \frac{r^4}{(1-r^5)^2} = \left(\frac{r^2}{r^5-1}\right)^2$. Denote $t = \sqrt{S(r)} = \frac{r^2}{r^5-1}$. Then $\frac{1}{t} = \frac{r^5-1}{r^2} \implies \frac{9}{7t} = \frac{9r^5-9}{7r^2} \implies \frac{9+7t}{7t} = \frac{9r^5+7r-9}{7r^2} = 0$ since r is a zero for $P(x) = 9x^5 + 7x^2 - 9$. Then $9 + 7t = 0 \implies t = -\frac{9}{7} \implies S(r) \equiv t^2 = \frac{81}{49} \implies 81 + 49 = 130$

Answer - E