Test #2 AMATYC Student Mathematics League Feb/Mar 2008

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Solutions

Contributors
Michael Maltenfort - simplification of #14

1. If $g(x-1) = x^2 + 1$, find $g(2)$.
A 1, B 2, C 5, D 9, E 10

Solution. Let $x - 1 = y$, so $x = y + 1$. Then $g(y) = (y+1)^2 + 1$, and $g(2) = (2+1)^2 + 1 = 10$.
Answer - E

2. Airport runways are labeled by two numbers giving the nonnegative clockwise angles less that $360°$ of the runway's direction measured from north to the nearest $10°$,divided by 10. Thus a runway with heading $223°$ is labeled 22. Which is the other number on this runway?
A 4, B 14, C 16, D 32, E 40

Solution. The other angle, which corresponds to the opposite direction of the runway, is $223° - 180° = 43°$. The label is $\left\lfloor \frac{43}{10} \right\rfloor = 4$.
Answer - A

3. The equation $a^3 + b^3 + c^3 = 2008$ has a solution in which a, b, and c are distinct even positive integers. Find $a + b + c$
A 20, B 22, C 24, D 26, E 28

Solution. Since the numbers a, b, and c are all even, let us denote $a = 2\alpha$, $b = 2\beta$, $c = 2\gamma$. Then $(2\alpha)^3 + (2\beta)^3 + (2\gamma)^3 = 2008$, or $\alpha^3 + \beta^3 + \gamma^3 = 251$

Let $\alpha = \max(\alpha, \beta, \gamma)$. Then $3\alpha^3 \geq \alpha^3 + \beta^3 + \gamma^3 = 251$, or $\alpha \geq 4.7$. Also, $\alpha^3 \leq \alpha^3 + \beta^3 + \gamma^3 = 251$, or $\alpha \leq 6.31$. As α is an integer, then $\alpha = 5$ or 6.

Since $\alpha^3 + \beta^3 + \gamma^3 (\equiv 251)$ is an odd number, then either all of α, β, γ are odd numbers, or else only one of them is odd.

The following table summarized the potential cases of α, β, γ

α, β, γ	5,3,1	5,4,2	6,5,4	6,5,2	6,4,3	6,4,1	6,3,2	6,2,1
$\alpha^3 + \beta^3 + \gamma^3$	153	198	405	349	307	281	251	225

It follows from the table, that the only triple (α, β, γ) for which $\alpha^3 + \beta^3 + \gamma^3 = 251$ is (6,3,2).

Thus, $a + b + c = (2\alpha) + (2\beta) + (2\gamma) = 2(6 + 3 + 2) = 22$.
Answer - B

4. For how many different integers b is the polynomial $x^2 + bx + 16$ factorable over the integers?
A 2, B 3, C 4, D 5, E 6

Solution. Let $x^2 + bx + 16 = (x + \alpha)(x + \beta)$, where α, β are integers. Then $\alpha \cdot \beta = 16$, $\alpha + \beta = b$

The following table shows possible factors of 16 and related values of b

Factors α, β of 16	$\pm 1, \pm 16$	$\pm 2, \pm 8$	$\pm 4, \pm 4$
Sum $b = \alpha + \beta$	± 17	± 10	± 8

Total number of distinct values of b is 2+2+2=6.

Answer - E

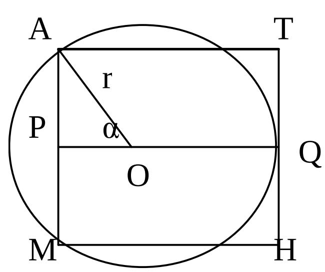
5. Let $f(x) = x^2 - 2x + 4$. Which of the following is a factor of $f(x) - f(2y)$?
A $x + 2y$, B $x + 2y + 2$, C $x - 2y + 2$, D $x + 2y - 2$, E none of these

Solution. $f(x) - f(2y) = (x^2 - 2x + 4) - ((2y)^2 - (2x) + 4)$
$= x^2 - 2x + 4 - 4y^2 + 4y - 4 = x^2 - 2x - 4y^2 + 4y$
$= (x^2 - 4y^2) - 2(x - 2y) = (x - 2y)(x + 2y) - 2(x - 2y)$
$= (x - 2y)(x + 2y - 2)$.
Answer - D

6. In square MATH, M and A lie on a circle of radius 20, and the circle is tangent to side $\overline{TH}$ at the middle of $\overline{TH}$. Find the lengths of the sides of the square.

A 24, B 26, C 28, D 30, E 32

Solution.



Denote the side of the square by l. From $\triangle AOP$, $AO^2 = AP^2 + OP^2$, or $r^2 = \left(\frac{l}{2}\right)^2 + (l - r)^2$. Solving for l, we get $l = \frac{8}{5}r$. Since $r = 20$, then $l = 32$

Answer - E

7. A fair coin is labeled A on one side and M on the other; a fair die has two sides labeled T, two labeled Y, and two labeled C. The coin and die are each tossed three times. Find the probability that the six letters can be arranged to spell $AMATYC$.

A $\frac{1}{60}$, B $\frac{1}{48}$, C $\frac{1}{36}$, D $\frac{1}{24}$, E $\frac{1}{12}$

Solution.

a) Let $C_1 = \{A, M\}$ be the sample space of equally likely outcomes for coin tossing.

Let C_3 be the sample space of possible triples randomly selected from C_1.

By the counting principle, the number of elements in C_3 is $2 \cdot 2 \cdot 2 = 8$. The event $\{two\ "A"s\ and\ one"M"are\ selected\} = \{MAA, AMA, AAM\}$. Thus, the probability $p(two\ "A"s\ and\ one"M"are\ selected\} = \frac{3}{8}$

b) Let $D_1 = \{T, Y, C\}$ be the sample space of equally likely outcomes for die rolling.

Let D_3 be the sample space of possible triples randomly selected from D_1.

By the counting principle, the number of elements in D_3 is $3 \cdot 3 \cdot 3 = 27$. The event $\{one\ "T", one\ "Y"\ and\ one\ "C"are\ selected\} = \{TYC, TCY, YTC, YCT, CTY, CYT\}$. Thus, the probability $p(one"T", one"Y"and\ one"C"are\ selected\} = \frac{6}{27} = \frac{2}{9}$

c) Since tossing a coin and rolling a die are independent procedures, then

$P(two\ "A"s,\ one"M",\ one"T",\ one"Y"and\ one"C"are\ selected\} = p(two\ "A"s\ and\ one"M"are\ selected\} \cdot p(one"T", one"Y"and\ one"C"are\ selected\} = \frac{3}{8} \cdot \frac{2}{9} = \frac{1}{12}$

Answer - E

8. What is the value of $(\log_{624} 625)(\log_{623} 624)(\log_{622} 623)\ldots(\log_6 7)(\log_5 6)$?

A 2, B 2.5, C 4, D 5, E 6

Solution. Let $A = \prod_{n=5}^{624} \log_n(n+1)$. By change-of-base formula, $\log_n(n+1) = (\log_a(n+1)) / (\log_a n)$ for any convenient $a \neq 1$. Then $A = \prod_{n=5}^{624} (\log_a(n+1)) / (\log_a n) = \left(\prod_{n=5}^{624} \log_a(n+1)\right) / \left(\prod_{n=5}^{624} \log_a n\right)$

$= \left(\prod_{n=6}^{625} \log_a n\right) / \left(\prod_{n=5}^{624} \log_a n\right) = (\log_a 625) / (\log_a 5)$.

Taking $a = 5$, we get $A = (\log_5 625) / (\log_5 5) = \log_5 5^4 = 4$

Answer - C

9. The letters $AMATYC$ are written in order, one letter to a square of graph paper, to fill 100 squares. If three squares are chosen at random without replacement, find the probability to the nearest 1/10 of a percent of getting three A's.

A 3.3%, B 3.7%, C 4.0%, D 7.3%, E 11.1%

Solution. As $\frac{100}{6} = 16.6...$, then 100 consecutive squares will be filled with 16 strings of $AMATYC$ and the ending string of $AMAT$. Thus the total number of $A's$ in 100 squares is $16 \cdot 2 + 2 = 34$.

By formulas of conditional probability.

$P(A_1A_2A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2) = \frac{34}{100} \cdot \frac{33}{99} \cdot \frac{32}{98} \approx 0.370 = 3.7\%$

Answer - B

10. A student committee must consist of two seniors and three juniors. Five seniors are able to serve on the committee. What is the least number of junior volunteers needed if the selectors want at least 600 different possible ways to pick the committee?

A 6, B 7, C 8, D 9, E 10

Solution. The standard notation ${}_pC_q = \frac{p(p-1)\ldots(p-q+1)}{q!}$ defines the number of possible combinations when q objects are selected from p objects.

Denote by n the number of junior volunteers. The number of possible senior selections is ${}_5C_2$ and number of possible junior selections is ${}_nC_3$. The counting principle shows that the total number of possible committee selections is $M = \left({}_5C_2\right)\left({}_nC_3\right)$. The condition $M \geqslant 600$ leads to the inequality $\left({}_5C_2\right)\left({}_nC_3\right) \geqslant 600$, or $(10)\left(\frac{n(n-1)(n-2)}{3!}\right) \geqslant 600$, which simplifies to $n(n-1)(n-2) \geqslant 360$.To solve this inequality, let us first estimate the value of n.

Start with $(n-1)^2 = n^2 - 2n + 1 > n^2 - 2n = n(n-2)$. From here, $(n-1)^3 = (n-1)(n-1)^2 > (n-1)n(n-2) \geqslant 360$. Thus $n - 1 > \sqrt[3]{360} \approx 7.11$. That is $n > 8.11$. Since n is an integer, $n \geqslant 9$.

For $n = 9$ we have $n(n-1)(n-2)|_{n=9} = 9 \cdot 8 \cdot 7 = 504 > 360$. For $n = 8$ we have $n(n-1)(n-2)|_{n=8} = 8 \cdot 7 \cdot 6 = 336 < 360$.Thus, minimum value of n is 9

Answer - D

11. Ed drives to work at a constant speed S. One day he is halfway to work when he immediately turns around, speeds up by 8 mph, and drives home. As soon as he is home, he turns around and drives to work at 6 mph faster that he drove home. His total driving time is exactly 67% greater than usual. Find S in mph and write the answer in the corresponding blank on the answer sheet.

Solution. Denote by $2L$ the distance between Ed's home and work. His regular time from home to work is $t = \frac{2L}{S}$. His this day time from home to work is $\tau = \frac{L}{S} + \frac{L}{S+8} + \frac{2L}{S+8+6}$. Since $\tau = 1.67t$, then $\frac{L}{S} + \frac{L}{S+8} + \frac{2L}{S+8+6} = 1.67\frac{2L}{S}$. Cancelling L on both sides, we get $\frac{1}{S} + \frac{1}{S+8} + \frac{2}{S+8+6} = 1.67\frac{2}{S}$. Solve this equation.

Multiply both sides by 50, we get $50\frac{1}{S} + 50(\frac{1}{S+8} + \frac{2}{S+14}) = 167 \cdot \frac{1}{S} \Longrightarrow 50(\frac{1}{S+8} + \frac{2}{S+14}) = 117 \cdot \frac{1}{S} \Longrightarrow 150(\frac{S+10}{(S+8)(S+14)}) = 117 \cdot \frac{1}{S} \Longrightarrow 50(S+10)S = 39(S+8)(S+14) \Longrightarrow 50(S^2+10S) = 39(S^2+22S+112) \Longrightarrow 11S^2 - 358S - 4368 = 0 \Longrightarrow S = \frac{358+\sqrt{358^2+4\cdot 11\cdot 4368}}{2\cdot 11} = \frac{358+\sqrt{320356}}{22} = \frac{358+566}{22} = \frac{924}{22} = 42 \Longrightarrow$

$S = 42$ mph

Answer - 42 mph

12. Each bag to be loaded onto a plane weighs either 12, 18, or 22 lb. If the plane is carrying exactly 1000 lb of luggage, what is the largest number of bags it could be carrying?

A 80, B 81, C 82, D 83, E 84

Solution. Denote by m, n, p the numbers of bags of correspondingly 12, 18, or 22 lb. We have to find $\max(m+n+p)$ under restriction $12m+18n+22p = 1000$, $m \geq 0, n \geq 0, p \geq 0; m, n, p$ are integers

Since $3(22) = 3(18) + 1(12)$, then every three of 22 lb bags can be replaced with three of 18 lb bags and one 12 lb bags, which increases the total number of bags. Thus $p = 0, 1$, or 2.

Similarly as $2(18) = 3(12)$, then every two of 18 lb bags can be replaced with three of 12 lb bags, which increases the total number of bags. Thus $n = 0$ or 1.

For fixed integer values n and p, the integer value of m is defined from the restriction above as

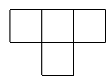
$m = \frac{1000-18n-22p}{12}$, or after simplification $m = \frac{500-9n-11p}{6}$

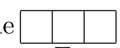
The potential cases of m, n, p are summarized in the following table

(n,p)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
$m = \frac{500-9n-11p}{6}$	63.3	81.5	79.7	81.8	80	78.2

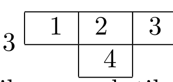
The table shows that the only feasible case is $(m, n, p) = (80, 1, 1)$ and $\max(m + n + p) = 80 + 1 + 1 = 82$

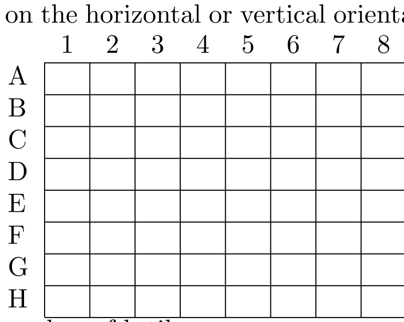
Answer - C

13. An 8x8 checkerboard is exactly covered by 16  shaped tiles.

What is the least possible number of tiles for which the  is horizontal?

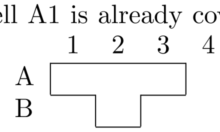
A 0, B 2, C 4, D 6, E 8

Solution. In a tile, let us refer to three cells $\left(1, 2, 3\right.$ 

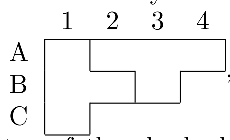
$\left.\right)$ lying on the same line as the tile base. We will refer to a tile as an h-tile or a v-tile depending on the horizontal or vertical orientation of its base. Consider 8x8 checkerboard 

and find the covering with the least possible number of h-tiles.

We will initially show that any covering of cells A1-A4 includes the base of an h-tile.

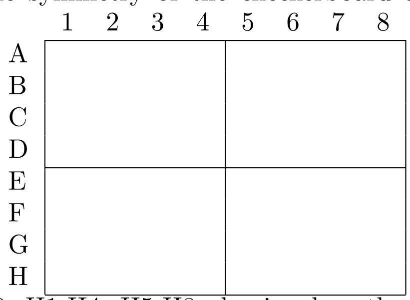
If cell A1 is already covered by the base of an h-tile, then the statement is proven 

If cell A1 is not covered by the base of an h-tile, then the only way of covering cells A1-A4 is



, which involves the base of an h-tile.

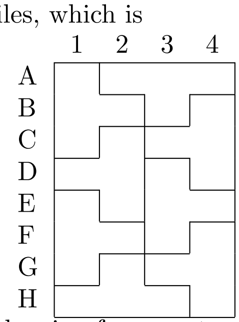
The symmetry of the checkerboard about horizontal and vertical central lines



shows that any covering of half-raws A5-A8, H1-H4, H5-H8 also involves the base of a h-tile. Thus, to cover the whole checkerboard, we need at least four h-tiles.

We now produce a covering of checkerboard that involves exactly four h-tiles.

We will initially produce a covering of columns 1-4 that will involve two h-tiles, which is



Reflecting this covering about the checkerboard's vertical axis of symmetry onto columns 5-8, we will get the required complete covering.

Thus, four h-tile are enough to cover checkerboard.

Answer - C

14. Call a positive integer *biprime* if it is the product of exactly two distinct primes (thus 6 and 15 are biprime, but 9 and 12 are not), If N is the smallest number such that $N, N+1$, and $N+2$ are all biprime, find the largest prime factor of $N(N+1)(N+2)$.

A 13, B 17, C 29, D 43, E 47

Note. A prime number p is not considered biprime because $p = 1 \cdot p$, and 1 is not treated as a prime.

Solution 1 (Michael Maltenfort)

Of three consecutive numbers $N, N+1, N+2$ one or two are even.

Neither of numbers $N, N+2$ is even. Otherwise, both $N, N+2$ must be divisible by 2. Then one of them must be divided by 4 and thus cannot be a biprime. As a result, $N+1$ is an even number. Denote $N+1=2p$, where p is prime. In the following table we assign to p prime values in search for biprime consecutive triples. Empty cells indicate that the cell number is not biprime

p	3	5	7	11	13	17
N				21		33=3·11
$N+1=2p$	6	10	14	22	26	34=2·17
$N+2$			15			35=5·7

As the table shows, the smallest biprime triple is $(N, N+1, N+2) = (33, 34, 35)$. The product $33 \cdot 34 \cdot 35 = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 17$. Thus, the largest factor in question is 17

Answer - B

Solution 2. As in solution 1, show that $N+1$ is even. Of three numbers $N, N+1, N+2$, one is divisible by 3. It cannot be $N+1$, because otherwise $N+1=2\cdot 3$ and is not biprime. Thus either N, or $N+2$ are divisible by 3. Changing as needed notations, consider triples $N-2, N-1, N$ or $N, N+1, N+2$, where $N=3p$. In the following table we assign to p prime values in search for biprime consecutive triples. Empty cells indicate that the cell number is not biprime

p	$N-2$	$N-1$	$N=3p$	$N+1$	$N+2$
3			9	10	
5		14	15		
7			21	22	
11			33=3·11	34=2·17	35=5·7

As the table shows, the smallest biprime triple is $(N, N+1, N+2) = (33, 34, 35)$. The product $33 \cdot 34 \cdot 35 = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 17$. Thus, the largest factor in question is 17

Answer - B

Further development. Let us find the some next triples of biprime numbers. We will continue the last table.

p	$N-2$	$N-1$	$N=3p$	$N+1$	$N+2$
13		38	39		
17			51		
19	55		57	58	
23			63		
29	85=5·17	86=2·43	87=3·29		
31			93=3·31	94=2·47	95=5·19
37			111		
41			123		
47			141=3·47	142=2·71	143=11·13
53		158	159		161
59			177	178	
61			183		185
67			201=3·67	202=2·101	203=7·29
71			213=3·71	214=2·107	215=5·43
73	217=7·31	218=2·109	219=3·73		221
79	235		237		
83	247		249		
89	265		267		
97			291		
101			303		305

Numbers in three consecutive cells of the same row represent biprime triples

15. You have 8 identical red counters and n identical green counters. You find that you can line them up in a single row in such a way that the number of counters whose right-hand neighbor is the same color equals the number of counters whose right-hand neighbor is the other color. What is largest possible

value of n?

A 17, B 19, C 21, D 25, E 27

Solution. Let us denote by r a red counter, and by g a green counter. We will consider only the strings of $r's$ and $g's$ for which the number of counters whose right-hand neighbor is the same color equals the number of counters whoseright-hand neighbor is the other color.

In a string with the largest value of n two r cannot be adjacent. Otherwise, we can replace the substring rr with the substring $rggggr$, which preserves the equality between the same and different color neighboring pairs, but increases the number of $g's$.

Also, red counters cannot start or end the string with the largest value of n. Otherwise, the starting (ending) r can be replaced with substring ggr (rgg) which preserves the equality between the same and different color neighboring pairs, but increases the number of $g's$.

Thus, a required string of r and g should have a form

$\cdots grg\cdots grg\cdots grg\cdots grg\cdots grg\cdots grg\cdots grg\cdots grg\cdots$, where $\cdots$ represent gaps in which additional $g's$ might be found.

In the last string, every red counter forms two pairs (one on the left, one on the right) of different color neighbors. Thus the total number of different color pairs is $2(8) = 16$. By the condition, the same is the total number of same color pairs

Every internal gap $g\cdots g$ forms at least one pair of the same color neighbors, which yields at least 7 same color pairs. So the green counters inside the gaps account for $16 - 7 = 9$ pairs. As each g inside a gap forms one additional same color pair, then the total number of inside green counters must be 9. The total number of green counters is 16+9=25.

Answer - D

16. If a and b are positive integers such that $\frac{b}{11}, \frac{c}{b}$, and $\frac{c}{15}$ all lie in the interval $(1.5, 1.8)$, find $b + c$.

A 43, B 44, C 45, D 46, E 47

Solution.

a) $\frac{b}{11} > 1.5 \Longrightarrow b > 16.5$. As b is an integer, then $b \geq 17$

b) $\frac{c}{15} < 1.8 \Longrightarrow c < 27$. As c is an integer, then $c \leq 26$

c) Using a) we will get $\frac{c}{b} > 1.5 \Longrightarrow c > 1.5b \geq 1.5 \cdot 17 = 25.5$. As c is an integer, then $c \geq 26$

d) Combining b) and c), obtain $c = 26$

e) From a) and d) $b + c \geq 17 + 26 = 43$

f) Using d), we will get $\frac{c}{b} > 1.5 \Longrightarrow \frac{b}{c} < \frac{1}{1.5} \Longrightarrow b < \frac{1}{1.5}c \Longrightarrow b + c < (\frac{1}{1.5} + 1)c = (\frac{1}{1.5} + 1)(26) < 43.4$. As $b + c$ is an integer, then $b + c \leq 43$

g) By combining e) and f), it follows that $b + c = 43$

Answer - A

17. Let r, s and t be nonnegative integers. For how many such triples (r, s, t) satisfying the system $\begin{cases} rs + t = 24 \\ r + st = 24 \end{cases}$ is it true that $r + s + t = 25$?

A 23, B 24, C 25, D 26, E 27

Solution. r and t cannot be both zero, because otherwise the system is not satisfied.

Case A: $r \neq 0, t \neq 0$. From the first equation of the system multiplied by t subtract the second equation of the system multiplied by r.The result is $t^2 - r^2 = 24t - 24r$ or $(t - r)(t + r - 24) = 0$. Equate the factors on the left to 0

Case A1: $t + r - 24 = 0$ or $t + r = 24$. Subtracting this equation from the constriction $r + s + t = 25$, we will get $s = 1$. The triples satisfying the conditions of the problem and the case are $(r, s, t) = (r, 1, 24 - r)$, $r = 1, \ldots, 23$. Totally, there are 23 triples in this case.

Case A2: $t - r = 0$ or $t = r$. Replacing t with r in the restriction $r + s + t = 25$, we get $s = 25 - 2r$. In the first equation of the system, substitute the last expression for s and substitute r for t. We will have $r(25 - 2r) + r = 24$, or after simplifications $r^2 - 13r + 12 = 0$.The solutions of the this equation are $r = 12, 1$. If $r = 12$, then $t \equiv r = 12$ and $s \equiv 25 - 2r = 1$. The corresponding triple $(r, s, t) = (12, 1, 12)$ has already been found in case A1. If $r = 1$, then $t \equiv r = 1$ and $s \equiv 25 - 2r = 23$.The solution $(r, s, t) = (1, 23, 1)$ is new. Totally, there is 1 new triple in this case

Case B: r, or t, but not both, are 0. If $r = 0$, then the first equation of the system yields $t = 24$, and the second equation yields $s = 1$.The triple $(0, 1, 24)$ is a new solution. If $t = 0$, then similarly the triple $(24, 1, 0)$ is a new solution. Totally, there are 2 new solutions in this case.

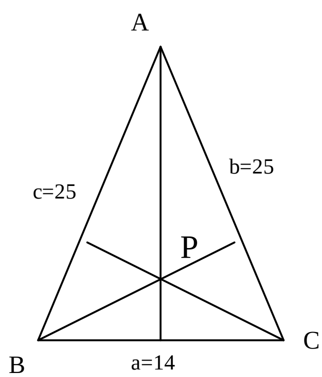
The grand total of the triples satisfying the conditions of the problem is $23 + 1 + 2 = 26$

Answer - D

18. In $\triangle ABC$, $AB = AC = 25$ and $BC = 14$. The perpendicular distances from a point P in the interior of $\triangle ABC$ to each of the three sides are equal. Find this distance.

A $\frac{9}{2}$, B $\frac{19}{4}$, C 5, D $\frac{21}{4}$, E $\frac{11}{2}$

Solution. Let us denote the lengths $AB = c$, $AC = b$, $BC = a$, so $b = c = 25$, $a = 14$. Denote the equal distances from P to the sides of the triangle as d. Let $s = \frac{a+b+c}{2}$



By Heron's formula, the area of $\triangle ABC$ is $S_{\triangle ABC} = \sqrt{s(s-a)(s-b)(s-c)}$

On the other hand, $S_{\triangle ABC} = S_{\triangle BPC} + S_{\triangle APC} + S_{\triangle APB} = \frac{1}{2}ad + \frac{1}{2}bd + \frac{1}{2}cd = \frac{a+b+c}{2}d = sd$

Equating the two expressions for the area of $\triangle ABC$, we obtain

$sd = \sqrt{s(s-a)(s-b)(s-c)}$, or $d = \sqrt{s(s-a)(s-b)(s-c)} \diagup s$

Numerically, $s = \frac{14+25+25}{2} = 32$,

$d = \sqrt{32(32-14)(32-25)(32-25)} \diagup 32 = 5.25 = \frac{21}{4}$

Answer - D

19. The digits 1 to 9 can be separated into 3 disjoint sets of 3 digits each so that the digits in each set can be arranged to form a 3-digit perfect square. Find the last two digits of the sum of these three perfect squares.

A 26, B 29, C 34, D 46, E 74

Solution. Since 3-digit perfect squares a^2 range between 123 and 987, then the bases a range between 12 and 31. The table below shows the squares with distinct digits of integers between 12 and 31.

a	13	14	16	17	18	19	23	24	25	27	28	29	31
a^2	169	196	256	289	324	361	529	576	625	729	784	841	961

The next table catches the triple of disjoint perfect squares from the above table.

a^2	$b^2 > a^2$ and disjoint with a^2	Disjoint pairs among b^2
169	324, 784	No
196	324, 784	No
256	784, 841	No
289	361, 576	No
324	196, 576, 961	No
361	529, 729, 784	(529, 784)
529	784, 841	No
576	841	No
625	729, 784	No
729	841	No
784	961	No
841		No
961		No

The table shows that the only disjoint triple of perfect squares is $(361, 529, 784)$, with the sum $361 + 529 + 784 = 1674$. The last two digits of the sum are 74.

Answer - E

20. The sequence $\{a_n\}$ is defined by $a_o = a_1 = a_2 = 1$, and $a_{n-3}a_n - a_{n-2}a_{n-1} = (n-3)!$ for $n \geq 3$. If 5^k is the largest power of 5 that is a factor of $a_{100}a_{101}$, find k.

A 20, B 22, C 24, D 25, E 26

Solution. From the recurrence equation it follows that $a_n = \left((n-3)! + a_{n-2}a_{n-1}\right)/a_{n-3}$. Using this formula, calculate several first values of a_n

n	0	1	2	3	4	5	6
$a_n = \left((n-3)! + a_{n-2}a_{n-1}\right)/a_{n-3}$	1-given	1-given	1-given	2	3	8	15
a_n/a_{n-2}			1	2	3	4	5

Denote $n!! = n(n-2)(n-4)\ldots\delta_n$ where $\delta_n = \begin{cases} 1 \text{ if } n \text{ is odd} \\ 2 \text{ if } n \text{ is even} \end{cases}$ Based on the table data, we start with working assumption that $a_m/a_{m-2} = m-1$ or $a_m = (m-1)a_{m-2}$. Then recursively, $a_m = (m-1)(m-3)a_{m-4} = \cdots = (m-1)(m-3)\ldots\delta_{m-1} = (m-1)!!$

It is true that $a_2 = 1 = 1!!$, $a_3 = 2 = 2!!$, $a_4 = 3 = 3!!$. Now assume that for any fixed positive integer n the relation $a_m = (m-1)!!$ is true for all $m \leq n-1$. Show that it is also true for $m = n$

$a_n = \frac{(n-3)!+a_{n-2}a_{n-1}}{a_{n-3}} = \frac{(n-3)!+[(n-3)!!][(n-2)!!]}{(n-4)!!}$. It is clear that $[(n-3)!!]\,[(n-2)!!] = (n-2)!$. Thus $a_n = \frac{(n-3)!+(n-2)!}{(n-4)!!} = \frac{[(n-3)!](1+n-2)}{(n-4)!!} = (n-1)\frac{(n-3)!}{(n-4)!!}$. Since

$\frac{(n-3)!}{(n-4)!!} = (n-3)!!$, then $a_n = (n-1)(n-3)!! = (n-1)!!$. We proved that $a_n = (n-1)!!$ for all $n \geq 2$

Returning to the problem question, $a_{100}a_{101} = [99!!]\,[100!!] = 100!$. Among 100 factors of 100!, $\frac{100}{5} = 20$ factors contribute factor 5. Numbers 25,50, 75, and 100 contribute 5^2. The rest of $20-4 = 16$ factors contribute 5^1. Thus the largest power of 5 in $a_{100}a_{101}$is $2(4) + 1(16) = 24$

Answer - C