Test #2 AMATYC Student Mathematics League Feb/Mar 2009

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Solutions

1. If the coordinates of one endpoint of a line segment are $(3, -3)$ and the coordinates of the midpoint are $(7, 5)$, what are the coordinates of the other endpoint ?

A $(11, 13)$ B $(13, 11)$ C $(17, 7)$ D $(7, 17)$ E $(5, 1)$

Solution. Denoting the midpoint coordinates by (x, y), we will have $\frac{3+x}{2} = 7$, $\frac{-3+y}{2} = 5$. Thus, $x = 11$, $y = 13$

Answer - A

2. Let the operation Δ be defined for positive integers a and b by $a\Delta b = ab + b$. If $x\Delta(x-1) = 323$, find $x\Delta(x+1)$

A 324 B 325 C 342 D 360 E 361

Solution. $323 = x\Delta(x-1) \equiv x(x-1)+(x-1) \equiv x^2-1$. The positive solution of the equation $x^2-1 = 323$ is $x = 18$. So $x\Delta(x+1) = 18\nabla 19 = 18\cdot 19+19 = 361$

Answer - E

3. The perimeter of a rectangle is 36 ft and a diagonal is $\sqrt{170}$ ft. Its area in ft^2 is

A 70 B 72 C 75 D 77 E 80 2

Solution. Denote by x and y the lengths of the sides of the rectangle. We have to find xy. By the conditions, $2x+2y = 36$ and $x^2+y^2 = (\sqrt{170})^2$, that is

$x + y = 18$

$x^2 + y^2 = 170$

Squaring both sides of the first equation above and subtracting from the result the second equation, we will get $2xy = 18^2 - 170$, or $xy = 77$

Answer - D

4. Which of the following functions satisfies the equation $f(x + f(x)) = f(f(x)) + f(x)$ for all real values of x and y ?

A $f(x) = x$ B $f(x) = 2x$ C $f(x) = \ln(x)$ D A and B E all of them

Solution. Function $f(x) = \ln(x)$ is not defined at $x = 0$. So it cannot the a solution of the given functional equation.

Let us show that the equation holds for functions $f(x) = kx$ for any real value of k.

Really, $f(x+f(x)) = k(x+kx) \equiv kx+k^2x$, $f(f(x))+f(x) = k(kx)+kx = kx+k^2x$.Thus $f(x+f(x)) = f(f(x)) + f(x)$ for any real k.

Answer - D

5. For what values of k will the equation $x\sqrt{14}+7=kx^2$ have exactly 2 real values?

A $k>2$ B $k>-\frac{1}{2}$ C $k>-2$ D $k<-2$ E $k<-\frac{1}{2}$

Solution. Rewrite the equation in the form $kx^2-(\sqrt{14})x-7=0$. The solutions of this equation are $x=\frac{\sqrt{14}\pm\sqrt{14-4k(-7)}}{2k}=\frac{\sqrt{14}\pm\sqrt{14}\cdot\sqrt{1+2k}}{2k}$. Two real solutions occur when $1+2k>0$, or $k>-\frac{1}{2}$

Answer - B

6. If x and n are positive integers with $x>n$ and $x^n-x^{n-1}-x^{n-2}=2009$, find $x+n$.

A 10 B 11 C 12 D 13 E 14

Solution. In the equality $x^n-x^{n-1}-x^{n-2}=2009$. Factoring both sides of the initial equality, we get

$x^{n-2}(x^2-x-1)=7^2\cdot 41$ (1)

Show that $n\geq 4$. Otherwise,

if $n=1$, then the left side of the initial equality is not an integer;

if $n=2$,then the initial equation $x^2-x-1=2009$ has no integer solutions.

if $n=3$, then (1) turns to be $x(x^2-x-1)=7^2\cdot 41$. Show that $x^2-x-1>x$. Really, $x^2-x-1=[x(x-2)+x+x]-x-1=x(x-2)+x-1$. As $x>n=3$, $x(x-2)+x-1>x$. Then in the equality $x(x^2-x-1)=7^2\cdot 41$, x is the smaller factor of $7^2\cdot 41$ than (x^2-x-1),and should not exceed $\sqrt{7^2\cdot 41}<44.9$. So x can be either 1, or 7, or 41.neither of these values satisfies the equality $x(x^2-x-1)=7^2\cdot 41$.

Since $n-2\geq 2$, then x^{n-2} is a perfect power, and in (1) $x^{n-2}=7^2$, so that $x=7$ and $n=4$.These values satisfy the initial equation $x^n-x^{n-1}-x^{n-2}=2009$. The requested value $x+n=7+4=11$

Answer - B

7. In a tournament, $\frac{3}{7}$ of the women are matched against half of the men. What fraction of all players is matched against someone of the other gender?

A $\frac{2}{5}$ B $\frac{3}{7}$ C $\frac{4}{9}$ D $\frac{6}{13}$ E $\frac{13}{28}$

Solution. Let us denote by m the number of men in the tournament, and by w the number of women in the tournament. So that total number of players is $m+w$. By the condition of the problem $\frac{3}{7}w=\frac{1}{2}m$, or $\frac{6}{7}w=m$. Adding w to both sides of the last equation, we get $\frac{6}{7}w+w=m+w$, or $\frac{13}{7}w=m+w$, or $w=\frac{7}{13}(m+w)$. So the number of matched against the men women is $\frac{3}{7}w=\frac{3}{7}\left(\frac{7}{13}(m+w)\right)=\frac{3}{13}(m+w)$.Since the same number of men matched against the women, the total,number of players matched against the opposite gander is $2\left(\frac{3}{13}(m+w)\right)=\frac{6}{13}(m+w)$

Answer - D

8. Four points A,B,C, and D on a given circle are chosen. If the diagonals of the quadrilateral $ABCD$ intersect at the center of the circle, then $ABCD$ must be a

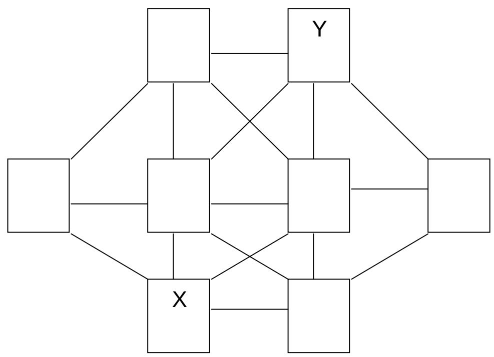
A trapezoid B square C rectangle D kite E none of these

Solution. Since the diagonals of the quadrilateral are at the same time diameters of the circle, then all the angles of the quadrilateral are of 90°. Thus, the quadrilateral is a rectangle.

Answer - C

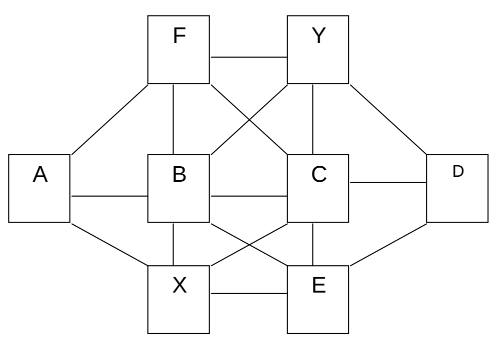
9. In the diagram shown, the boxes are to be filled with the digits 1 through 8 (each used exactly once). If no two boxes connected directly by a line segment can contain consecutive digits, find $X + Y$.

A 7 B 8 C 9 D 10 E 11



Solution . Denote by A, B, C, D, E, F the digits in the rest of the boxes, as shown at the picture below.

Write down the consecutive digits 12345678. Digits B and C have 6 nonconsecutive connections each. These digits can be only 1 or 8.Case 1) $B = 1, C = 8$. Case 2)$B = 8, C = 1$



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Case 1. $B=1, C=8$. The remaining digits are among 234567. Digits F and X have connection to $B=1$ and $C=8$, so that they are of the digits 3456. They also have two other nonconsecutive connections. So F and X can only be 3 or 6. Case 11) $F=3$, $X=6$.Case 12) $F=6$, $X=3$.

Case 11 $B=1, C=8$, $F=3$, $X=6$. The remaining digits are among 2457. Digit A cannot be 2 because of connection with $B=1$,cannot be 4 because of connection with $F=3$,cannot be 5 because of connection with $X=6$, cannot be 7 because of connection with $X=6$.Contradictory case.

Case 12 $B=1, C=8$, $F=6$, $X=3$ The remaining digits are among 2457. Digit A cannot be 2 because of connection with $B=1$, cannot be 4 because of connection with $X=3$,cannot be 5 because of connection with $F=6$, cannot be 7 because of connection with $F=6$.Contradictory case.

Case 2. $B=8, C=1$. The remaining digits are among 234567. Digits Y and E have connection to $C=1$ and $B=8$, so that they are of the digits 3456. They also have two other nonconsecutive connections. So Y and E can only be 3 or 6. Case 21) $Y=3$, $E=6$.Case 22) $Y=6$, $E=3$.

Case 21. $B=8, C=1$, $Y=3$, $E=6$. The remaining digits are among 2457. Digit D cannot be 2 because of connection with $C=1$,cannot be 4 because of connection with $Y=3$,cannot be 5 because of connection with $E=6$, cannot be 7 because of connection with $E=6$.Contradictory case.

. *Case 22.* $B=8, C=1$, $Y=6$, $E=3$. The remaining digits are among 2457. Digit D cannot be 2 because of connection with $C=1$,cannot be 4 because of connection with $E=3$,cannot be 5 because of connection with $Y=6$, cannot be 7 because of connection with $Y=6$.Contradictory case.

Answer - No solution. The Answer key says that the answer is C)

10. A cone has a circular base with a radius of 4 cm. A slice is made parallel to the base of the cone so that the new cone formed has half the volume of the original cone. What is the radius in centimeters of the base of the new cone ?

A $2\sqrt[3]{4}$ B $2\sqrt[3]{2}$ C $2\sqrt{2}$ D 2 E 1

Solution. Denote by R, H and V $(r, h,$ and $v)$ the radius, the height and the volume of the original (new) cone.

$V = kR^2H$, $v = kr^2h$, where $k = \frac{1}{3}\pi^2$. By the condition, $\frac{1}{2} = \frac{v}{V} = \frac{kr^2h}{kR^2H} = \frac{r^2h}{R^2H}$. Denoting $\frac{h}{H} = \frac{r}{R} = \alpha$, we will have $\frac{1}{2} = \frac{(\alpha R)^2(\alpha H)}{R^2H} = \alpha^3$. Thus $\alpha = \sqrt[3]{1/2} = 1/\sqrt[3]{2}$. So the radius of the new cone $r = \alpha R = 4/\sqrt[3]{2} = 2\sqrt[3]{4}$

Answer - A

11. At one point as Elena climbs a ladder, she finds that the number of rungs above her is twice the number below her (not counting the rung she is on). After climbing 5 more rungs, she finds that the number of rungs above and below her are equal. How many more rungs must she climb to have the number below her be four times the number above her ?

A 5 B 6 C 7 D 8 E 9

Solution. Denote by x and y the number of rungs below and above Elena at the start of climbing. Then $y = 2x$ and $y - 5 = x + 5$. solving this system, we will find that $x = 10$ and $y = 20$. Denote by m the required number of the additional rungs. Then $10 + 5 + m = 4(20 - 5 - m)$.Solving this equation, we will find that $m = 9$

Answer - E

12. If $\sin\theta - \cos\theta = 0.2$ and $\sin 2\theta = 0.96$, find $\sin^3\theta - \cos^3\theta$.

A 0.25 B 0.276 C 0.28 D 0.296 E 0.3

Solution. $\sin^3\theta - \cos^3\theta = (\sin\theta - \cos\theta)(\sin^2\theta + \sin\theta\cdot\cos\theta + \cos^2\theta) = 0.2(1 + \frac{1}{2}\sin 2\theta) = 0.2\cdot(1 + \frac{1}{2}\cdot 0.96) = 0.296$

Answer - D

13. How many asymptotes does the function $g(x) = \frac{x}{10\sqrt{100x^2-1}}$ have ?

A 0 B 1 C 2 D 3 E 4

Solution. Solve the equation $100x^2 - 1 = 0$, to get $x = \pm\frac{1}{10}$. These are two vertical asymptotes. $\lim\limits_{x\to\pm\infty} \frac{x}{10\sqrt{100x^2-1}} = \pm\frac{1}{100}$. $y = \pm\frac{1}{100}$ are two horizontal asymptotes

Answer - E

14. For how many solutions of the equation $x^2 + 4x + 6 = y^2$ are both x and y integers ?

A 0 B 1 C 2 D 3 E *an infinite number*

Solution. $x^2 + 4x + 6 = y^2 \Leftrightarrow (x+2)^2 + 2 = y^2 \Leftrightarrow y^2 - (x+2)^2 = 2 \Leftrightarrow (y - x + 2)(y + x - 2) = 2$

Consider four cases

1). $y - x + 2 = 1$, $y + x - 2 = 2$ 2) $y - x + 2 = 2$, $y + x - 2 = 1$ 3) $y - x + 2 = -1$, $y + x - 2 = -2$ 4) $y - x + 2 = -2$, $y + x - 2 = -1$

Solve in general the system

$y - x + 2 = a$

$y + x - 2 = b$

to obtain $x = \frac{b-a}{2} + 2$, $y = \frac{a+b}{2}$

Case 1. $a = 1, b = 2 \Leftrightarrow x = \frac{2-1}{2} + 2$ - not an integer

Case 2 $a = 2, b = 1 \Leftrightarrow x = \frac{1-2}{2} + 2$ - not an integer

Case 3 $a = -1, b = -2 \Leftrightarrow x = \frac{-2-(-1)}{2} + 2$ - not an integer

Case 4 $a = -2, b = -1 \Leftrightarrow x = \frac{-1-(-2)}{2} + 2$ - not an integer

The equation has no integer solutions

Answer - A

15. The sum of the squares of the four integers r, s, t and u is 685, and the product of r and s is the opposite of the product of t and u. Find $|r|+|s|+|t|+|u|$

A 39 B 41 C 43 D 45 E 47

Solution. By the conditions, the following system of equations holds

$r^2 + s^2 + t^2 + u^2 = 685 \Leftrightarrow (r+s)^2 + (t+u)^2 - 2(rs + tu) = 685$

$rs = -tu \Leftrightarrow rs + tu = 0$

Simplifying the first transformed equation with the aid of the second transformed equation, we get $(r+s)^2 + (t+u)^2 = 685$

Numerical calculations show that 685 can be represented as the sum of two squares only in the following two forms

$685 = 3^2 + 26^2 = 18^2 + 19^2$. So the following is true

$(r+s)+(t+u) = \pm 3 \pm 26$, that is $r+s+t+u = -29, -23, 23, 29$ or

$(r+s)+(t+u) = \pm 18 \pm 19$, that is $r+s+t+u = -37, -1, 1, 37$

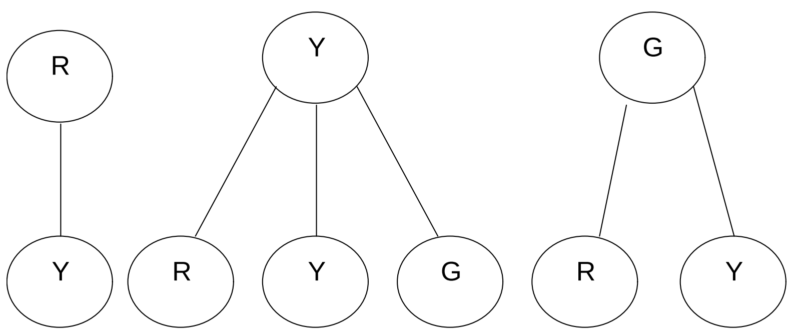
Answer- The answer does not correspond to listed choices

16. You pass through five traffic signals on your way to work. each is either red, yellow. or green. A red is always immediately followed by a yellow; a green is never followed immediately by a green. How many different sequences of color are possible for the five signals ?

A 42 B 48 C 54 D 60 E 66

Solution. Let us denote by $n_r^j . n_y^j, n_g^j$ the number of sequences that end at step j by corresponding colors *red*, *yellow*. *green*.

By the conditions of the problem, the immediate sequence of colors follows the next diagram, where R, Y, G correspond to red, yellow, green colors



This establishes the following relationships between number of sequences on steps j and $j+1$

$n_r^{j+1} = n_y^j + n_g^j$

$n_y^{j+1} = n_r^j + n_y^j + n_g^j$

$n_g^{j+1} = n_y^j$

The numerical calculations are shown in the table below

Step	$n_r^{j+1} = n_y^j + n_g^j$	$n_y^{j+1} = n_r^j + n_y^j + n_g^j$	$n_g^{j+1} = n_y^j$	Total number of sequences
1	1	1	1	3
2	2	3	1	6
3	4	6	3	13
4	9	13	6	32
5	19	28	13	60

Total number of sequences after five steps is 60

Answer - D

17. How many different ordered pairs of integers with $y \neq 0$ are solutions for the system of equations $6x^2y + y^3 + 10xy = 0$ and $2x^2y + 2xy = 0$?

A 1 B 2 C 3 D 4 E 5

Solution. Factor the first equation $y(6x^2 + y^2 + 10x) = 0$. Since $y \neq 0$, the equation can be written in the form $6x^2 + y^2 + 10x = 0$, from which $y = \pm\sqrt{-6x^2 - 10x}$. Factor the second initial equation $2xy(x+1) = 0$..Since $y \neq 0$, then $x(x+1) = 0$ and either $x = 0$ or $x = -1$. For $x = 0$, $y = \sqrt{-60^2 - 100} = 0$, which is not acceptable by the conditions of the problem. For $x = -1$, $y = \pm\sqrt{-6(-1)^2 - 10(-1)} = \pm 2$. There are two solutions of the problem: $(x, y) = (-1, 2), (x, y) = (-1, -2)$

Answer - B

18. The graph of the equation $x + y = x^3 + y^3$ is the union of a

A *line and an ellipse* B *line and a parabola* C *parabola and an ellipse* D *pair of lines* E *line and a hyperbola*

Solution. Transform the equation $x+y = x^3+y^3 \Leftrightarrow x+y-(x^3+y^3) = 0 \Leftrightarrow (x+y) - (x+y)(x^2 - xy + y^2) = 0 \Leftrightarrow (x+y)(x^2 - xy + y^2 - 1) = 0$. Equating each of the factors in the last equation to 0, we get

1. $x + y = 0$, which is an equation of the line $y = -x$

2. $x^2 - xy + y^2 - 1 = 0 \Leftrightarrow x^2 - xy + y^2 = 1$. Change the variables $x = u + v, y = u - v$ and substitute into the last equation. $x^2 - xy + y^2 = 1 \Leftrightarrow (u+v)^2-(u+v)(u-v)+(u-v)^2 = 1 \Leftrightarrow (u^2+2uv+v^2)-(u^2-v^2)+(u^2-2uv+v^2) = 1 \Leftrightarrow u^2 + 3v^2 = 1$. The last equation is an equation of an ellipse.

Answer - A

19. A four-digit number each of whose digits is 1, or 5, or 9 is divisible by 37. If the digits add up to 16, find the sum of the last two digits.

A 2 B 6 C 10 D 12 E 14

Solution. Denote the number by $A = a10^3 + b10^2 + c10 + d$, where a, b, c, d are digits from $1, 5, 9$. We have $A = a(27\cdot37+1)+b(2\cdot37+26)+c10+d = (27a+2b)37+a+26b+10c+d$. Since A is divided by 37, then $B = a+26b+10c+d = (a+d)+26b+10c$ is divided by 37. As $a+b+c+d = 16$, then $a+d = 16-b-c$. Replace in B $a+d$ with $16-b-c$. we have $B = (16-b-c)+26b+10c = 25b+9c+16$. The following table shows the values of B for various values of b and c

b	1	1	1	5	5	5	9	9	9
c	1	5	9	1	5	9	1	5	9
$B = 25b + 9c + 16$	50	86	122	150	186	222	250	286	322
Divisible by 37?	No	No	No	No	No	Yes	No	No	No

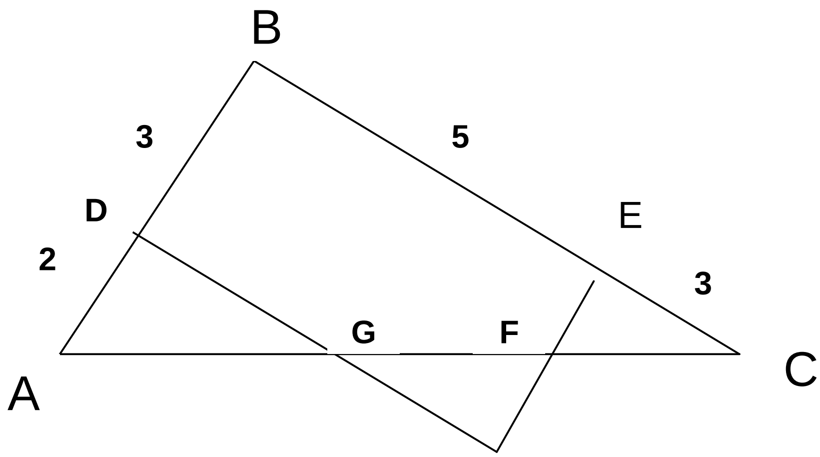
As follows from the table, $b = 5, c = 9$. Since $a + d = 16 - b - c$, then $a+d = 16-5-9 = 2$. So $a = d = 1$. The requested value of $c+d = 9 = 1 = 10$

Answer - C

20. In $\triangle\ ABC$, $AB = 5, BC = 8$, and $\angle B = 90°$ Choose D on $\overline{AB}$ and E on $\overline{BC}$ such that $BD = 3$ and $BE = 5$. Find the area common to the interiors of $\triangle\ ABC$ and the rectangle determined by $\overline{BD}$ and $\overline{BE}$

A $\frac{1111}{80}$ B $\frac{1113}{80}$ C $\frac{1117}{80}$ D $\frac{1119}{80}$ E $\frac{1121}{80}$

Solution.



In the picture, the requested area is the area of the polygon $DBEFG$, that is $Area(\triangle\ ABC) - Area(\triangle\ FEC) - Area(\triangle\ ADG)$.

$Area(\triangle\ ABC) = \frac{5 \cdot 8}{2} = 20$. $\triangle\ FEC$ is similar to $\triangle\ ABC$ and $Area(\triangle\ FEC) = \left(\frac{3}{8}\right)^2 Area(\triangle\ ABC) = \frac{9}{64}20 = \frac{45}{16}$. $\triangle\ ADG$ is similar to $\triangle\ ABC$ and $Area(\triangle\ ADG) = \left(\frac{2}{5}\right)^2 Area(\triangle\ ABC) = \frac{4}{25}20 = \frac{16}{5}$. The area of the polygon $DBEFG = 20 - \frac{45}{16} - \frac{16}{5} = \frac{1119}{80}$

Answer - D