Test #2 AMATYC Student Mathematics League Feb/Mar 2011

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Solutions

1. After Ed eats 20% of a pie and Anh eats 40% of a pie, Ed has twice as much pie left as Anh. Find Ed's original amount of pie as a percentage of Anh's original amount

A 120 B 125 C 140 D 150 E 160

Solution. Denote by E, A the Ed's and the Anh's original amounts of pie. Then, by the condition, $0.8E = 2 \cdot 0.6A$. Solve the equation for E to have $E = 1.5A$

Answer - D

2. The expression $a\#b = ab^2 + a$ for integers $a, b > 0$. if $(a\#b)\#3 = 250$, find $a + b$

A 6 B 7 C 8 D 9 E 10

Solution. By the conditions, $250 = (a\#b)3^2 + (a\#b) = 10(a\#b)$. Then $a\#b = 25$. By definition of $a\#b$, we will have $25 = ab^2 + a = a(b^2 + 1)$. Factor $25 = 5 \cdot 5 = 25 \cdot 1$

Case $a = 5, b^2 + 1 = 5$. Then $b = 2$ and $a + b = 7$

Case $a = 25, b^2 + 1 = 1$. Then $b = 0$, which is not greater than 0

Case $a = 1, b^2 + 1 = 25$. Then $b = \sqrt{24}$, which is not an integer.

Answer - B

3. Alicia always climbs steps 1,2, or 4 at a time. For example, she climbs 4 steps by $1-1-1-1$, $1-1-2$, $1-2-1$, $2-1-1$, $2-2$, or 4. In how many ways can she climb 10 steps?

A 81 B 120 C 144 D 150 E 169

Solution. Denote by n_m the number of way for Alicia to reach m^{th} step. Calculate manually $n_1 = 1$, $n_2 = 2$, $n_3 = 3$. And as explained in the problem, $n_4 = 6$

In general, to reach the m^{th} step, Alicia can either climb 1 step at a time from $(m-1)^{th}$ step, or climb 2 steps at a time from $(m-2)^{th}$ step, or climb 4 steps at a time from $(m-4)^{th}$ step. Thus, the following recurrence formula holds $n_m = n_{m-1} + n_{m-2} + n_{m-4}$. Using this formula, calculate $n_5 = n_4 + n_3 + n_1 = 6+3+1 = 10$, $n_6 = n_5+n_4+n_2 = 10+6+2 = 18$, $n_7 = n_6+n_5+n_3 = 18+10+3 = 31$, $n_8 = n_7+n_6+n_4 = 31+18+6 = 55$, $n_9 = n_8+n_7+n_5 = 55+31+10 = 96$, $n_{10} = n_9 + n_8 + n_6 = 96 + 55 + 18 = 169$

Answer - E

4. The sum of six consecutive positive integers beginning at n is a perfect cube. The smallest such n is 2. Find the sum of the next two smallest such n's.

A 679 B 680 C 681 D 682 E 683

Solution. Add six integers starting with n as discribed in the problem to obtain

$\sum_{k=0}^{5}(n+k) = 6n + \sum_{k=0}^{5} k = 6n + \frac{0+5}{2}6 = 6n + 15$. Thus,

$6n + 15 = m^3$ (1)

Since the left side of the last equality is divided by 3, then m^3 is divisible by 3, that is m is divisible by 3. Let $m = 3p$. Then from (1), $6n + 15 = (3p)^3$, or $2n + 5 = 9p^3$. Solving this equation for n, we will have

$n = \frac{9p^3-5}{2}$ (2)

In (2) p must be odd to obtain integer n.

Plug into (2) $p = 1, 3, 5$ to obtain $n = 2$, 119, 560. The sum $119 + 560 = 679$, which is required fo find.

Answer - A

5. The sum of the infinite geometric series S is 6, and the sum of the series whose terms are the squares of the terms of S is 15. Find the sum of the infinite geometric series with the same first term and opposite common ratio as S.

A 2 B 2.5 C 3 D 3.5 E 4

Solution. For the series S, $\sum_{k=0}^{\infty} aq^k = \frac{a}{1-q}$, and

$\frac{a}{1-q} = 6$ (1)

For the series with squared terms $\sum_{k=0}^{\infty}(aq^k)^2 = \sum_{k=0}^{\infty} a^2(q^2)^k = \frac{a^2}{1-q^2}$, and

$\frac{a^2}{1-q^2} = 15$ (2)

Solving the system (1),(2), we obtain $q = \frac{7}{17}$, $a = \frac{60}{17}$. Then the series with the opposite common ratio is $\sum_{k=0}^{\infty} a(-q)^k = \frac{a}{1+q} = \frac{\frac{60}{17}}{1+\frac{7}{17}} = \frac{5}{2}$

Answer - B

6. When 15 is added to a set of 10 numbers, the median changes from 6 to 8. Find the median of the new set if 15 is replaced by 7.

A 4 B 5 C 5.5 D 6 E 7

Solution. Denote the initial set of ten numbers listed in ascending order by $A = \{a_k\}_{k=1}^{10}$. Since $\frac{a_5+a_6}{2} = 6$, then $a_5 \le \frac{a_5+a_6}{2} = 6$, and $a_6 \ge \frac{a_5+a_6}{2} = 6$

After adding 15, denote the set of eleven numbers listed in ascending order by $B = \{b_k\}_{k=1}^{11}$. Number 15 is greater than $a_5 \le 6$.

Case $a_5 < 15 \le a_6$. In this case $15 = b_6$, $a_6 = b_7$, and the median of the set B is $b_6 = 8$. This is impossible since $b_6 = 15$

Case $a_6 < 15$. In this case $a_6 = b_6 = 8$ as the median of the set B.

If number 7 is added to the initial set, then in the new set $C = \{c_k\}_{k=1}^{11}$, $7 = c_7$ is the median of set C.

Answer - E

7. Rectangle $SMLA$ has $SM = 5$ and $ML = 10$. if the unit squares at S and M are removed, leaving 48 squares, how many of the following four sets of rectangles can exactly cover $SMLA$: 24 1x2s, 16 1x3s, 12 1x4s, 8x 2x3s?

A 0 B 1 C 2 D 3 E 4

Solution.

w	b	w	b	w	b	w	b	w	b
b	w	b	w	b	w	b	w	b	w
w	b	w	b	w	b	w	b	w	b
b	w	b	w	b	w	b	w	b	w
w	b	w	b	w	b	w	b	w	b

Color the cells of the rectangle $SMLA$ into white(w) or black(b) in such a way that neighboring vertical and horizontal cells have different colors. Every 1x2 domino will now cover cells of different color. However, after we remove unit squares at S and M, we are left with 25 black cells and 23 white cells. No set of dominos can cover the altered rectangle $SMLA$.

Answer - A. *The answer key indicates the answer B.*

8. In $\triangle SML$, $SM = 17$ and $ML = 12$. If SL is an integer greater than SM or ML, find the smallest value of SL for which $\triangle SML$ has an obtuse angle.

A 12 B 20 C 21 D 22 E 28

Solution. If $\triangle SML$ has an obtuse angle, then the side opposite to this angle is the largest (please draw a picture).

Case $ML < SL \leq SM$. By the Law of Cosines, $\cos(\angle SLM) = (ML^2 + SL^2 - SM^2)/(2ML \cdot SL)$. Assuming that $\angle SLM$ is obtuse, we will get $\cos(\angle SLM) < 0$, that is $ML^2 + SL^2 - SM^2 < 0$. Substituting the values of SM and ML, we obtain $SL < \sqrt{SM^2 - ML^2} = \sqrt{17^2 - 12^2} = 12.042$, so that $SL \leq 12$. On the other side, $SL > ML = 12$. We obtained contradiction.

Case $SL > SM$. By the By the Law of Cosines, $\cos(\angle SML) = (SM^2 + ML^2 - SL^2)/(2SM \cdot ML)$. Assuming that $\angle SML$ is obtuse, we will get $\cos(\angle SML) < 0$, that is $SM^2 + ML^2 - SL^2 < 0$. Substituting the values of SM and ML, we obtain $SL > \sqrt{SM^2 + ML^2} = \sqrt{17^2 + 12^2} > 20.80$, so that $SL \geq 21$. The smallest velue of SL is 21.

Answer - C

9. A polynomial with nonnegative integer coefficients has $P(0) = 3$, $P(1) = 8$, $P(2) = 39$, $p(3) = 144$. Find $P(-2)$.

A -7 B -5 C -3 D -2 E -1

Solution. Denote $P(x) = \sum_{k=0}^{N} a_k x^k$ for some N. Then $3 = P(0) = a_0 + \sum_{k=1}^{N} a_k 0^k = a_0$, so that

$a_0 = 3$ (1)

As coefficients a_k, $k = 1, ..., N$, are nonnegative integers, then $x^N \leq \sum_{k=0}^{N} a_k x^k = P(x)$. Put here $x = 3$ to obtain $3^N \leq P(3) = 144$ and $N \leq \log_3 144 = \frac{\ln 144}{\ln 3} < 4.53$. So that $N \leq 4$.

Write

$P(x) = 3 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$ (2)

In the last equality, set $x = 1, 2, 3$, to obtain the system of equations

$a_1 + a_2 + a_3 + a_4 = 5$

$a_1 2 + a_2 2^2 + a_3 2^3 + a_4 2^4 = 36$ (3)

$a_1 3 + a_2 3^2 + a_3 3^3 + a_4 3^4 = 141$

Solve this system to obtain a_1, a_2, a_3 as functions of a_4. I used the function $rref()$ of my $TI - 84$ calculator to get this system to Gauss-Jordan form

$a_1 + 6a_4 = 8$
$a_2 - 11a_4 = -11$ (4)
$a_3 + 6a_4 = 8$

From the first equation it follows that $a_1 = 8 - 6a_4$. Since a_1, a_4 are nonnegative integers, either $a_4 = 0, a_1 = 8$,or $a_4 = 1, a_1 = 2$. The case $a_4 = 0, a_1 = 8$ is impossible because of the first equation from system (3). Thus

$a_4 = 1,\ a_1 = 2$ (5)

From (5) and the second and third equations of system (3), it follows that

$a_2 = 0,\ a_3 = 2$ (6)

Combining (2), (5), and (6), we get $P(x) = 3 + 2x + 2x^3 + x^4$, and $P(-2) = 3 + 2x + 2x^3 + x^4 = -1$

Answer - E

10. Each digit of a 10-digit number N is either a $1, 2$, or 3. Every 3 consecutive digits of N form a prime number. Find the final two digits of the smallest such N.

A 11 B 13 C 21 D 23 E 31

Solution. Let $N = d_1d_2d_3d_4d_5d_6d_7d_8d_9d_{10,}$ where $d_j \in \{1, 2, 3\}$, $j = 1, ..., 10$. To construct he smallest number with the above properties, let $d_1 = 1, d_2 = 1$, so that $d_1d_2 = 11$. Then check that the smallest prime number $d_1d_2d_3 = 11d_3$ is 113, so that $d_3 = 3$. Similarly, the smallest prime number $d_2d_3d_4 = 13d_4$ is 131, that is $d_4 = 1$. Similarly, the smallest prime number $d_3d_4d_5 = 31d_5$ is 311, so that $d_5 = 1$

So far $d_1d_2d_3d_4d_5 = 11311$. Since $d_4d_5 = 11 = d_1d_2$, then $d_6 = d_3 = 3$. Since $d_5d_6 = 13 = d_2d_3$, then $d_7 = d_4 = 1$. In the same way, we find $d_8 = d_5 = 1$, $d_9 = d_6 = 3$, $d_{10} = d_7 = 1$. Thus $d_9d_{10} = 31$

Answer - E

11. Multiplying the corresponding terms of a geometric and an arithmetic sequence yields $96, 180, 324, 567, ...$ Find the next term of the new sequence.

A 960 B 972 C 980 D 984 E 988

Solution. Denote the arithmetic sequence by $\{a_n\}_{n=0}^{\infty} = \{a_0 + nd\}_{n=0}^{\infty}$, the geometric sequence by $\{b_n\}_{n=0}^{\infty} = \{b_0q^n\}_{n=0}^{\infty}$, and the new sequence $\{c_n\}_{n=0}^{\infty} = \{b_0q^n(a_0 + nd)\}_{n=0}^{\infty} = \{96, 180, 324, 567, ...\}$. We have to find the values of b, q, a_0, d and finally $c_4 = b_0q^4(a_0 + 4d)$

Rewrite $c_n = a_0b_0q^n(1 + n\frac{d}{a_0}) = uq^n(1 + nv)$, where

$u = a_0b_0$ (1)

$v = \frac{d}{a_0}$ (2)

Then $\{c_n\}_{n=0}^{\infty} = \{uq^n(1 + nv)\}_{n=0}^{\infty} = \{96, 180, 324, 567, ...\}$. Find $c_0 = u = 96$, so that

$u = 96$ (3)

Denote $d_n = \frac{c_n}{c_0} = \frac{uq^n(1+nv)}{u} = q^n(1 + nv)$. Then $\{d_n\}_{n=1}^{\infty} = \{q^n(1 + nv)\}_{n=1}^{\infty} = \{\frac{180}{96}, \frac{324}{96}, \frac{567}{96}, ...\}$. Denote $e_n = \frac{d_{n+1}}{d_n d_1} = \frac{q^{n+1}(1+(n+1)v)}{q^n(1+nv)q(1+v)} = \frac{1+(n+1)v}{(1+nv)(1+v)}$, so that $\{e_n\}_{n=1}^{\infty} = \{\frac{1+(n+1)v}{(1+nv)(1+v)}\}_{n=1}^{\infty} = \{\frac{24}{25}, \frac{14}{15}, ...\}$. Then $e_1 = \frac{1+2v}{(1+v)(1+v)} = \frac{24}{25}$, or $24v^2 - 2v - 1 = 0$. Solving this equation, we get $v_1 = \frac{1}{4}$, $v_2 = -\frac{1}{6}$. Also

$e_2 = \frac{1+3v}{(1+2v)(1+v)} = \frac{14}{15}$. This equality for e_2 holds for $v_1 = \frac{1}{4}$, but is not true for $v_2 = -\frac{1}{6}$.

Thus, $v = \frac{1}{4}$ (4)

Find $d_1 = q(1+v) = \frac{180}{96}$,or

$q = \frac{180}{96}/(1+v) = \frac{3}{2}$ (5)

The same value of q satisfies the equations $d_2 = q^2(1+2v) = \frac{324}{96}$, $d_3 = q^3(1+3v) = \frac{567}{96}$

Since the numbers from the sequence $\{c_n\}_{n=0}^{\infty} = \{b_0 q^n(a_0+nd)\}_{n=0}^{\infty} = \{96, 180, 324, 567, ...\}$ are divisible by b_0, and the numbers $96 = 2^5 3$, $180 = 2^2 3^2 5$, $324 = 2^2 3^4$, $567 = 3^4 7$ have no common factors, then $b_0 = \pm 1$

Case $b_0 = 1$. From (1),(3) $a_0 = 96$, and from (2), (4) $d = 24$. Summarizing, $b_0 = 1, q = \frac{3}{2}, a_0 = 96, d = 24$. The next term of the new sequence $c_4 = b_0 q^4(a_0 + 4d) = 1(\frac{3}{2})^4(96 + 4 \cdot 24) = 972$

Case $b_0 = -1$. From (1),(3) $a_0 = -96$, and from (2), (4) $d = -24$. Summarizing, $b_0 = -1, q = \frac{3}{2}, a_0 = -96, d = -24$. The next term of the new sequence $c_4 = b_0 q^4(a_0 + 4d) = (-1)(\frac{3}{2})^4(-96 - 4 \cdot 24) = 972$

Answer - B

12. If $\log_x y + \log_y x = 2.9$, and $xy = 128$, find $x + y$

A 32 B 36 C 40 D 48 E 64

Solution. By the change-of-base formula $\log_y x = \frac{\log_x x}{\log_x y} = \frac{1}{\log_x y}$. So $2.9 = \log_x y + \frac{1}{\log_x y}$. Denoting $u = \log_x y$, we get $u + \frac{1}{u} = 2.9$, or $u^2 - 2.9u + 1 = 0$. Solving this equation, we obtain $u_1 = 2.5$ and $u_2 = 0.4$

Case $u = 2.5$. We have $\log_x y = 2.5 \Rightarrow y = x^{2.5} \Rightarrow xy = x^{3.5}$. Since $xy = 128$, then $x^{3.5} = 128 \Rightarrow x = 4$. Then $y = x^{2.5} = 4^{2.5} = 32$ and $x + y = 36$

Case $u = 0.4$.We have $\log_x y = 0.4 \Rightarrow y = x^{0.4} \Rightarrow xy = x^{1.4}$. Since $xy = 128$, then $x^{1.4} = 128$ and $x = 32$. Then $y = x^{0.4} = 32^{0.4} = 4$ and $x + y = 36$

Answer - B

13. The equation $a^5 + b^2 + c^2 = 2011$ (a, b, c positive integers) has a solution in which two of the three numbers are prime. Find the value of the nonprime number.

A 38 B 40 C 42 D 44 E 46.

Solution. Since $a^5 \le a^5 + b^2 + c^2 = 2011$, then $a \le \sqrt[5]{2011} < 4.6$. Thus $a \in A = \{1, 2, 3, 4\}$

Case a is not prime. Then $a \in \{1, 4\}$.

Subcase $a = 1$. Then $b^2 + c^2 = 2010$. Since $min(b^2, c^2) \le \frac{b^2+c^2}{2} = 1005$, then $\min(b, c) \le \sqrt{1005} < 31.8$, that is $\min(b, c) \le 31$. Let $b \le c$, then a prime $b \in B = \{3, 5, 7, 11, 13, 17, 19, 23, 29, 31\}$. Direct calculations by the formula $c = \sqrt{2010 - b^2}$ shows that for $b \in B$ the value of c is not an integer.

Subcase $a = 4$. In this case the expresion $a^5 + b^2 + c^2$ for prime b, c is even and cannot be equal to 2011

Case a is prime. Then $a \in \{2, 3\}$

Subcase $a = 2$. Then $b^2 + c^2 = 1979$. As above, $\min(b, c) \le \sqrt{\frac{1979}{2}} < 31.5$, that is $\min(b, c) \le 31$. Let $b \le c$. Then $b \in B = \{m\}_{m=1}^{31}$. Direct calculations by the formula $c = \sqrt{1979 - b^2}$ shows that for $b \in B$ the value of c is not an integer.

Subcase $a = 3$. Then $b^2 + c^2 = 1768$. As above $\min(b, c) \le \sqrt{\frac{1768}{2}} < 29.8$, that is $\min(b, c) \le 29$. Let $b \le c$. Then $b \in B = \{m\}_{m=1}^{29}$. Direct calculations by the formula $c = \sqrt{1768 - b^2}$ shows that $(b, c) \in \{(2, 42), (18, 38)\}$. Since among b, c one number is prime and the other is not prime, then the only solution is $(b, c) = (2, 42)$ and the value of the non-prime number is 42.

Answer - C

14. A polindrome is a number like 121 or 1551 which reads the same from right to left and from left to right. How many 4-digit polindromes are divisible by 17.

A 2 B 4 C 5 D 6 E 8

Solution. Denote a 4-digit polindrome by $ABBA$, where digit $A \ge 1$, and digit $B \ge 0$. In extended form $ABBA = 10^3A + 10^2B + 10B + A = 1001A + 110B$. Since $1001 = 59 \cdot 17 - 2$, and $110 = 6 \cdot 17 + 8$, then $ABBA = (59A + 6B)17 - 2A + 8B$. Because $ABBA$ is divisible by 17, then $-2A + 8B$ is divisible by 17. Write $-2A + 8B = -17k$. As the left side here is even, then $k = 2m$ and with dividing by 2, we will have $-A + 4B = -17m$, or $A = 4B + 17m$. Going through the values of $B \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and trying to select, if possible, an appropriate value of m, we will get $(B, m, A) \in \{(1, 0, 4), (2, 0, 8), (5, -1, 3), (6, -1, 7), (9, -2, 2)\}$. So there are five 4-digit polindromes that are divisible by 17, namely $4114, 8228, 3553, 7667, 2992$

Answer - C

15. Six numbers are selected from $0, 1, ..., 6$ and arranged in a $2x3$ grid so that each row is increasing from left to right and each column is increasing from top to bottom. Find the number of such different arrangements.

A 24 B 28 C 30 D 35 E 42

Solution. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ be a $2\ x\ 3$ grid where $a_{i,j} \in \{0, 1, ..., 6\}$, $a_{i,j} < a_{i,j+1}$, $a_{i,j} < a_{i+1,j}$, $i = 1, 2$, $j = 1, ..., 3$.

There is ${}_7C_6 = 7$ selections of $\{0, 1, ..., 6\}$, taken 6 at a time. Let $B = \{b_1, b_2, b_3, b_4, b_5, b_6\}$ be any of these selections written in ascending order. Then $a_{11} = b_1, a_{23} = b_6$. For the rest of the assignments consider the following cases

Case $b_2 = a_{12}$

Subcase $b_3 = a_{21}$

Subsubcase $b_4 = a_{13}$.Then $b_5 = a_{22}$

Subsubcase $b_4 = a_{22}$.Then $b_5 = a_{13}$

Subcase $b_3 = a_{13}$. Then $b_4 = a_{21}$, $b_5 = a_{22}$

Case $b_2 = a_{21}$. Then $b_3 = a_{12}$

Subcase $b_4 = a_{22}$. Then $b_5 = a_{13}$

Subcase $b_4 = a_{13}$. Then $b_5 = a_{22}$

Total number of cases for any selection fixed selection B is 5. Overall number of cases is $7 \cdot 5 = 35$

Answer - D

16. The increasing sequence of positive integers $a_1, a_2, a_3, ...$ satisfies the equation $a_{n+2} = a_n + a_{n+1}$ for all $n \geq 1$. If $a_7 = 160$, find a_8.

A 257 B 258 C 259 D 260 E 261

Solution. Sequentially we find

$a_3 = a_1 + a_2$

$a_4 = a_2 + a_3 = a_1 + 2a_2$

$a_5 = a_3 + a_4 = 2a_1 + 3a_2$

$a_6 = a_4 + a_5 = 3a_1 + 5a_2$

$a_7 = a_5 + a_6 = 5a_1 + 8a_2$ (1)

$a_8 = a_6 + a_7 = 8a_1 + 13a_2$ (2)

From (1) we find

$5a_1 + 8a_2 = 160$ (3)

Since in (3) $8a_2$ and 160 are divisible by 8, then $5a_1$ must be divisible by 8, that is $a_1 = 8m$. Plugging this into (3), we obtain $5(8m) + 8a_2 = 160$, or

$5m + a_2 = 20$ (4)

Since in (4) $5m$ and 20 are divisible by 5, then a_2 must be divisible by 5, that is $a_2 = 5n$. Plugging this into (4), we obtain $5m + 5n = 20$, or

$m + n = 4$ (5)

Since m, n are positive integers, then $(m, n) \in \{(1, 3), (2, 2), (3, 1)\}$, and $(a_1, a_2) = (8m, 5n) \in \{(8, 15), (16, 10), (24, 5)\}$. Since $a_1 \leq a_2$, then the only case is $(a_1, a_2) = (8, 15)$, and by (2) $a_8 = 259$

Answer - C

17. For how many integers $1 \leq n \leq 2011$ is the fraction $\frac{n^2+7}{n+4}$ NOT in lowest terms?

A 85 B 86 C 87 D 88 E 89

Solution. Transform $\frac{n^2+7}{n+4} = n - 4 + \frac{23}{n+4}$. For the initial fraction to be NOT in lowest terms, the fraction $\frac{23}{n+4}$ must be NOT in lowest terms. Since 23 is prime, then $n + 4 = 23m$, $m \geq 1$. As $n \leq 2011$, then $23m \leq 2011 + 4 = 2015$.So $\max m \leq \lfloor \frac{2015}{23} \rfloor = 87$

Answer - C

18. Ten sets of coins each contain one penny, and the kth set has $2k$ dimes for $1 \leq k \leq 10$. If one coin is selected at random from each set, find the probability that the number of pennies in the selection is odd.

A $\frac{10}{21}$ B $\frac{11}{23}$ C $\frac{1}{2}$ D $\frac{11}{21}$ E $\frac{12}{23}$

Solution. Denote $10 = M$. For each A_k of the M sets denote by p_k the set of consisting of one penny and by d_k the set consisting of k dimes. Then $A_k = p_k \cup d_k$, and probabilities $P(p_k) = \frac{1}{2k+1}$, $P(d_k) = \frac{2k}{2k+1}$. The problem describes the binomial selection in M trials, where the probability of success (selecting the penny) changes from trial to trial.

Based on the sets A_k, we will introduce the independent random variables X_k which assign the value of 1 to p_k, and the value of 0 to d_k. Thus

X_k	1	0
P	$\frac{1}{2k+1}$	$\frac{2k}{2k+1}$

Introduce the variables $Y_m = \sum_{k=1}^{m} X_k$, $m = 1, ..., M$. Denote $p_m = P(Y_m$ is odd), $q_m = P(Y_m$ is even). Our objective is to find p_M.

For $m = 1$, $p_1 = P(X_1$ is odd), $q_1 = P(X_1$ is even), so that

$p_1 = \frac{1}{3}, q_1 = \frac{2}{3}$ (1)

. Since $Y_m = Y_{m-1} + X_m$, then $p_m = p_{m-1}P(X_m = 0) + q_m P(X_m = 1) = \frac{2m}{2m+1}p_{m-1} + \frac{1}{2m+1}q_m$ and $q_m = p_m P(X_m = 1) + q_{m-1}P(X_m = 0) = \frac{1}{2m+1}p_{m-1} + \frac{2m}{2m+1}q_m$. Thus

$p_m = \frac{2m}{2m+1}p_{m-1} + \frac{1}{2m+1}q_{m-1}$, $q_m = \frac{1}{2m+1}p_{m-1} + \frac{2m}{2m+1}q_{m-1}$, $m = 2, .., M$ (2)

We can use (1), (2) to sequentially calculate p_m, q_m, $m = 2, .., M$.

Alternatively, using (1) and (2) we can apply induction to obtain

$p_m = \frac{m}{2m+1}$, $q_m = \frac{m+1}{2m+1}$.

For the value $M = 10$, $p_{10} = \frac{10}{21}$

Answer - A

Note. I also ran the following simple Java program on my PC.

```
//————————————————————————
import java.util.*;
class Prob18{
    public static void main(String args[]){
        Random generator = new Random();
        int intNumberOfTrials=100000000;
        int intNumberOfSuccesses=0;
        int intSumOfPennies=0;
        int intSelection=0;
        for (int i=0; i < intNumberOfTrials; i++){
            intSumOfPennies=0;
            for (int j=1;j <= 10;j++){
                intSelection=generator.nextInt(1+2*j);
                if(intSelection==0){
                    intSumOfPennies++;
                }
            }
            if(intSumOfPennies % 2 != 0){
                intNumberOfSuccesses++;
            }
        }
        System.out.println("Number of Trials - "+intNumberOfTrials+";
Number of Successes - "+intNumberOfSuccesses);
    }
}
//————————————————————————
```

After a single computer run of 100,000,000 trials, I obtaind $p_{10} = 0.47619523$, which is pretty close to $\frac{10}{21}$

19. Every set $\{1, 2, 3, ..., n\}$ can be split into sets so that each set sums to the same total. For example, $\{1, ..., 7\} = \{1, 2, 4, 7\} \cup \{3, 5, 6\}$; each set sums to 14. Find the largest number of such equal sum sets into which $\{1, 2, 3..., 15\}$ can be split.

A 4 B 5 C 6 D 8 E 10

Solution. We will solve the problem for any set $A = \{1, 2, 3, ..., n\}$, where n is an odd number.

Denote by S the sum of the numbers in A, by m the number of the equal sum subsets, and by S_m the sum of the numbers in each subset. Because $m = \frac{S}{S_m}$, then the smaller is S_m, the larger is the number m of subsets. Since the number n lies in one of the subsets, then $S_m \geq n$. As $S = \frac{1+n}{2}n, n$ is odd, then $\max m = \frac{S}{n} = \frac{1+n}{2}$. The related subsets are $\{n\}, \{1, n-1\}, ...\{\frac{n-1}{2}, \frac{n+1}{2}\}$. For $n = 15$, $m = \frac{1+15}{2} = 8$

Answer - D

20. If the nine integers 2 through 10 are arranged at random in a row, find the probability that no two prime numbers are next to each other.

A $\frac{1}{14}$ B $\frac{2}{21}$ C $\frac{5}{42}$ D $\frac{1}{7}$ E $\frac{1}{6}$

Solution. The total number of permitations of nine integers is $9! = 362880$.

There are four primes $2, 3, 5, 7$ among the integers 2 through 10. Let denote a place for a prime in the row by S and the place for a nonprime by s. For any permutation under the problem conditions the following configuration holds $G_1SG_2SG_3SG_4SG_5$, where $G_i, i = 1, ..., 5$ are groups of places s. Here G_1, G_2 might be empty, but G_2, G_3, G_4 contain at least one s. As $\min G_i = \{s\}, i = 2, 3, 4$, and two more s can be freely distributed between G_1, G_2, G_3, G_4, G_5, then by standard combinatorial reasoning we find that

$(G_1SG_2SG_3SG_4SG_5) \in$
$\{(ssSsSsSsS), (sSssSsSsS), (sSsSssSsS), (sSsSsSssS), (sSsSsSsSs),$
$(SsssSsSsS), (SssSssSsS), (SssSsSssS), (SssSsSsSs), (SsSsssSsS),$
$(SsSssSssS), (SsSssSsSs), (SsSsSsssS), (SsSsSssSs), (SsSsSsSss)\}$

All-in-all, for any permutation under the problem conditions we have 15 placement configurations of prime and nonprime numbers. So the total number of favorable configurations is $15 \cdot 4! \cdot 5! = 43200$

The probability of a favorable configuration when no two prime numbers are next to each other is $\frac{43200}{362880} = \frac{5}{42}$

Answer - C. *The answer key claims that the answer is E*

Note. I also ran the following simple Java program on my PC.

```
//————————————————————
class Prob20{
    static int[] intIndexes = {0,0,0,0,0,0,0,0,0,0,0};
    static boolean[] boolPrime
        = {false,false,false,false,false,false,false,false,false,false,false,};
```

```
static int[] intPrimes = {2,3,5,7};
public static void main(String args[]){
    int intFavorablesTotal=0;
    for (int i1=2;i1<=10;i1++){
        if (!boolCheckConditions(2,i1)){
            continue;
        }
        for (int i2=2;i2<=10;i2++){
            if (!boolCheckConditions(3,i2)){
                continue;
            }
            for (int i3=2;i3<=10;i3++){
                if (!boolCheckConditions(4,i3)) continue;
                for (int i4=2;i4<=10;i4++){
                    if (!boolCheckConditions(5,i4)) continue;
                    for (int i5=2;i5<=10;i5++){
                        if (!boolCheckConditions(6,i5)) continue;
                        for (int i6=2;i6<=10;i6++){
                            if (!boolCheckConditions(7,i6)) continue;
                            for (int i7=2;i7<=10;i7++){
                                if (!boolCheckConditions(8,i7)) con-
tinue;
                                for (int i8=2;i8<=10;i8++){
                                    if (!boolCheckConditions(9,i8))
continue;
                                    for (int i9=2;i9<=10;i9++){
                                        if (!boolCheckConditions(10,i9))
continue;
                                        intFavorablesTotal++;
                                    }
                                }
                            }
                        }
                    }
                }
            }
        }
    }
    System.out.println("Total number of favorable permutations is "+int-
FavorablesTotal);
}
static boolean boolCheckConditions(int intNo, int i_cur){
    intIndexes[intNo]=i_cur;
    for (int i=2;i<=intNo-1;i++){
        if(i_cur==intIndexes[i]){
            return false;
```

```
        }
    }
    boolPrime[intNo]=false;
    for (int i=0;i<=3;i++){
        if(i_cur==intPrimes[i]){
            boolPrime[intNo]=true;
            break;
        }
    }
    if (boolPrime[intNo-1]&boolPrime[intNo]){
        return false;
    }
    else{
        return true;
    }
  }
}
//————————————————————————
```
The run confirmed that the total number of favorable configurations is 43200.