Test #2 AMATYC Student Mathematics League Feb/Mar 2012

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Solutions

Problem #17 has been worked out together with Marta Hidegkuti

1. The probability that the product of the numbers rolled on three fair six-sided dice is prime is A. $\frac{1}{36}$ B. $\frac{1}{24}$ C. $\frac{1}{16}$ D. $\frac{1}{12}$ E. $\frac{1}{8}$

Solution. Consider the set S consisting of triples (l, m, n) of numbers showing up on three fair six-sided dice. Since rolling each die results in one of *six* outcomes, and the rolls are independent, then by the Fundamental Counting Principle, the size of S is $|S| = 6^3 = 216$.

Let P be the subset of S such that $P = \{(p, q, r)|p \cdot q \cdot r = prime\}$ Then for any $(p, q, r) \in P$, one of the p, q, r should be *prime* and the other two should be *one*.

Since among $1, 2, 3, 4, 5, 6$ the primes are $2, 3, 5$, then
$P = \{(2, 1, 1), (1, 2, 1), (1, 1, 2), (3, 1, 1), (1, 3, 1), (1, 1, 3), (5, 1, 1), (1, 5, 1), (1, 1, 5)\}$.
As the size of P is $|P| = 9$. The probability of the event P is $\frac{|P|}{|S|} = \frac{9}{216} = \frac{1}{24}$

Answer - B

2. If $x^2 + 1$ is a factor of $6x^3 + 5x^2 + Px + Q$, then $P + Q =$
A. 10 B. 11 C. 12 D. 13 E. 14

Solution. Method 1. Since $x^2 + 1$ is a factor of $6x^3 + 5x^2 + Px + Q$ and since $x - i$ is a factor of $x^2 + 1$, then $x - i$ is a factor of $f(x) = 6x^3 + 5x^2 + Px + Q$. Thus $f(i) = 0$. Substituting, $0 = f(i) = 6i^3 + 5i^2 + Pi + Q = -6i - 5 + Pi + Q$. Separating here real from imaginary parts, obtain $-6 + P = 0$, $-5 + Q = 0$, or $P = 6$, $Q = 5$, $P + Q = 11$

Method 2. Applying Long Division to divide $6x^3 + 5x^2 + Px + Q$ by $x^2 + 1$, we will get $6x^3 + 5x^2 + Px + Q = (6x + 5)(x^2 + 1) + [(P - 6)x + (Q - 5)]$. Since $x^2 + 1$ is a factor of $6x^3 + 5x^2 + Px + Q$, than $(P - 6)x + (Q - 5) \equiv 0$. That yields $P - 6 = 0$, $Q - 5$, and $P + Q = 11$

Answer - B

3. Call a date $mm/dd/yy$ *magical* if $mm \cdot dd = yy$. For example, 12/02/24 is magical, but 02/05/11 and 7/15/05 are not. How many of the following dates can NEVER be magical? *January* 31 *February* 29 *March* 31 *April* 30
A. 0 B. 1 C. 2 D. 3 E. 4

Solution. January $31 = 01/31/yy$. We have $mm \cdot dd = 01 \cdot 31 = 31$. *January* 31 can be *magical date for 31st year of each century*

February 29 = 02/29/*yy*. We have $mm \cdot dd = 02 \cdot 29 = 58$. Years ending with 58 are not divided by 4, and thus cannot be leap years. Dates 02/29/*yy* can never be *magical.*

March 31 = 03/31/*yy*. We have $mm \cdot dd = 03 \cdot 31 = 93$. *March* 31 can be *magical for 93*rd *year of each century.*

April 30 = 04/30/*yy*. We have $mm \cdot dd = 04 \cdot 30 = 120 > 99$. *April* 30 can never be magical.

Answer - C

4. Suppose $a^2 - b^2 = 91$ (a, b integers). If $n = a^2 + b^2 < 1000$, find the units digit of n.

A. 1 B. 3 C. 5 D. 7 E. 9

Solution. Factoring $a^2 - b^2 = |a|^2 - |b|^2 = (|a| - |b|)(|a| + |b|)$, and factoring,uniquely, $91 = 7 \cdot 13$, we have $(|a| - |b|)(|a| + |b|) = 7 \cdot 13$. Since $0 < |a| - |b| < |a| + |b|$, then $|a| - |b| = 7$, $|a| + |b| = 13$. Solving this system, we find $|a| = 10$, $|b| = 3$, and $n = a^2 + b^2 = |a|^2 + |b|^2 = 10^2 + 3^2 = 109$. The unit digit is 9.

Answer - E

5. Let S be the set of all lines with equation $y = mx + b$ for which $m + b = 36$. For how many of the elements of S are both the x- and y-intercepts integers?

A. 8 B. 9 C. 12 D. 15 E. 18

Solution. Denote by $(a, 0), (0, b)$ correspondingly $x-$intercept, and $y-$intercept of a line. Adding the equalities given in the problem and simplifying the result, we will come to the equation of the set of the lines $y = (m-1)x + 36$. Thus for all lines in the problem set there is a unique $y-$intercept $(0, b) = (0, 36)$. To find $x-$intercept, set up in the last equation $y = 0$ and solve it for x to get $x = -\frac{36}{m-1}$, to obtain $(a, 0) = (-\frac{36}{m-1}, 0)$. To have $a = -\frac{36}{m-1}$ being an integer, the value of $m - 1$ should be a factor of 36. All possible factors of 36 are $\pm1, \pm2, \pm3, \pm4, \pm6, \pm9, \pm12, \pm18, \pm36$. So there are 18 lines satisfying the conditions of the problem.

Answer - E

6. A bridge charges 2-axled vehicles a \$5 toll and 3-axled vehicles an \$8 toll. In an hour the bridge collected \$741 from 120 vehicles. If tolls were \$1 higher for 2-axled and \$2 higher for 3-axled vehicles, how much would the bridge have collected?

A. \$888 B. \$908 C. \$926 D. \$934 E. \$1012

Solution. Let x and y be correspondingly the number of 2-axled vehicles and the number 3-axled vehicles passing through the bridge. Then $x + y = 120$, and $5x + 8y = 741$. Solving this system, we will got $x = 73$, $y = 47$. For increased toll charges, the collected amount would be $A = (5+1)x + (8+2)y = 6 \cdot 73 + 10 \cdot 47 = 908$

Answer - B

7. Let a, b, and c be positive integers which satisfy $a^3 + b^3 + c^2 = 2012$. Find $a + b + c$.

A. 28 B. 30 C. 32 D. 34 E. 36

Solution. Resolve the equation for c to have $c = \sqrt{2012 - a^3 - b^3}$. From the condition $2012 - a^3 - b^3 > 0$ we get $\max(a, b) < \sqrt[3]{2012} < 12.7$, or $\max(a, b) \leq 12$

We will utilize graphing calculator $TI - 83/84$ in the following way to numerically find the values of a, b, c.

The described below procedure is neither the best, nor the worst, possible.

Clicking $MODE$ button, set $REAL$ for type of numbers.

Denote $X = c$. Consider b as a parameter that will sequentially take the values of $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$.

In $\overline{TI - 83/84, \text{ clear}}$ the list of functions $Y =$, and write the function $\backslash Y_1 = \sqrt{2012 - X\hat{}3 - 0\hat{}3}$, where 0 is an initial trial value of b. Press the button $TBLSET$, and set up $TblStart = 0, \triangle TBL = 1$. Press the button $TABLE$ and view the column results from $X = 0$ until $Y_1 = ERROR$. Write down (b, X, Y_1) with integer Y_1, if any. In this trial, with $b = 0$, there is no integer value of Y_1

Go to the list of functions and change the function to $\backslash Y_1 = \sqrt{2012 - X\hat{}3 - 1\hat{}3}$, where 1 is the next trial value of b. Continue with the procedure. With $b = 1$, there is no integer value of Y_1

Sequentially repeat trials for $b = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$.

We find two symmetric integer solutions: $(b, X, Y_1) = (8, 11, 13)$ and $(b, X, Y_1) = (11, 8, 13)$.

In our initial notations the solutions are $(a, b, c) = (11, 8, 13)$ and $(a, b, c) = (8, 11, 13)$

In both cases $a + b + c = 32$

Answer - C

8. Tom, Dick, and Harry each have children. The sum of the number of Tom's children and the average of Dick's and Harry's children is 5, while the sum of the number of Harry's children and the average of Tom's and Dick's children is 7. Find the total number of children in the three families.

A. 7 B. 8 C. 9 D. 10 E. 11

Solution. Denote by t, d, h the number of children in Tom's, Dick's and Harry's families. Then the following two equations hold $t + \frac{d+h}{2} = 5, h + \frac{t+d}{2} = 7$, or

$2t + d + h = 10$ (1),

$t + d + 2h = 14$ (2).

Denote $S = t + d + h$. Then (1), (2) can be written in the form $t + S = 10$, $h + S = 14$,or

$t = 10 - S$ (3),

$h = 14 - S$ (4).

Adding (3) and (4), we will have $t + h = 24 - 2S$, which yields $d = S - (t + h) = S - (24 - 2S)$, or

$d = 3S - 24$ (5)

Since t, h, d are positive integers, then from (3) $S < 10$, and from (5) $S > 8$. Thus $S = 9$

Answer - C

9. In the 5x5 grid below, each cell contains one of the digits 1 to 5 so that each row and each column has exactly one of each digit. Find the entry in row 3, column 4.

1	2			
2				
			?	
				5
			5	4

A. 1 B. 2 C. 3 D. 4 E. 5

Solution. The required entry in row 3, column 4 is 1. In the following table the sequence of filled entries is marked by the entries subscripts.

1	2		4_4	3_1
2			3_5	1_2
			1_6	2_3
				5
			5	4

Answer - A

10. When the 2-digit number aa is multiplied by the 1-digit number $b \neq a$, the result is the 3-digit number cba. Find the sum of all possible values of cba.

A. 264 B. 528 C. 759 D. 891 E. 1045

Solution. If $a \cdot b = a$, then $aa \cdot b = aa$, which contradicts the fact that $b \neq a$. Thus, $a \cdot b = da$, with $d \geq 1$. Subtract a on both side of the last equation, to obtain $a(b-1) = 10 \cdot d$. Since the right side is divisible by 10, then either $a = 5, b-1 = 2m$, or $a = 2m, b-1 = 5$, where $m \in \{1, 2, 3, 4\}$

Case $a = 5, b-1 = 2m$. Then $b = 2m+1$, and

$m = 1, b = 3, aa \cdot b = 55 \cdot 3 = 165$ - not of the form cba

$m = 2, b = 5, aa \cdot b = 55 \cdot 5 = 275$ - not of the form cba

$m = 3, b = 7, aa \cdot b = 55 \cdot 7 = 385$ - not of the form cba

$m = 4, b = 9, aa \cdot b = 55 \cdot 9 = 495$ - of the form cba

Case $a = 2m, b-1 = 5$. Then $b = 6$

$m = 1, a = 2, aa \cdot b = 22 \cdot 6 = 132$ - not of the form cba

$m = 2, a = 4, aa \cdot b = 44 \cdot 6 = 264$ - of the form cba

$m = 3, a = 6, aa \cdot b = 66 \cdot 6 = 396$ - not of the form cba

$m = 4, a = 8, aa \cdot b = 88 \cdot 6 = 528$ - not of the form cba

The sum of all possible values of cba is $495 + 264 = 759$

Answer - C

11. In the equation $x^2 - \frac{10}{9}x + c = 0$, one solution is the square of the other solution. If $c > 0$ is the rational number $\frac{m}{n}$ in simplest terms, find $m + n$.

A. 25 B. 28 C. 32 D. 35 E. 38

Solution. Denote the "base" solution x_1 of the equation by a, the other solution being $x_2 = a^2$. Since $x_1 \cdot x_2 = c > 0$, than $c = a \cdot a^2 \equiv a^3 > 0$ and $a > 0$. Because $x_1 + x_2 = \frac{10}{9}$, then $a + a^2 = \frac{10}{9}, a > 0$. Solving this quadratic equation, we obtain $a_1 = \frac{2}{3}, a_2 = -\frac{5}{3}$. We select only positive $a = a_1 = \frac{2}{3}$. Finally, $c = a^3 = (\frac{2}{3})^3 = \frac{8}{27}$, and $m + n = 8 + 27 = 35$

Answer - D

12. Which of the following best describes the graph of $(x + y)^2 = x^2 + y^2$?

A. The empty set B. A single point C. two intersecting lines

D. two parallel lines E. A circle

Solution. Expanding the square on the left side of the equation and simplifying the result, we will come to the equality $xy = 0$. The left side here vanishes when either $x = 0$, the equation of the $y - axis$, or when $y = 0$, the equation of the $x - axis$. Thus, any point on the coordinate axes satisfies the equality $xy = 0$, and, backwardly, to the equation $(x + y)^2 = x^2 + y^2$.

Answer C

13. Sue and Thai are asked to multiply a positive integer a by a positive integer b and then add a positive integer c to the result. Sue mistakenly first multiplies by c and then adds b, while Thai mistakenly adds b and then multiplies by c. The correct answer was 29, Sue got 59, and Thai got 80. Find $a + b + c$.

A. 12 B. 14 C. 16 D. 18 E. 20

Solution. Translating the correct procedure and Sue, and Thai calculations, we will get

$a \cdot b + c = 29$ (1)

$a \cdot c + b = 59$ (2)

$(a + b)c = 80$ (3)

Subtract (1) from (2) to obtain $(a - 1)(c - b) = 30$, from which it follows that $b < c$.Similarly subtract (2) from (3) to obtain $b(c - 1) = 21$. Factoring $21 = 1 \cdot 21 = 3 \cdot 7$, we will consider the following cases, where $b < c$,

Case $b = 1, c - 1 = 21$. Then $c = 22$, and from (1) $a \cdot 1 + 22 = 29$, or $a = 7$. Since for $(a, b, c) = (7, 1, 22)$ the equation (2) is not satisfied, we dismiss this case.

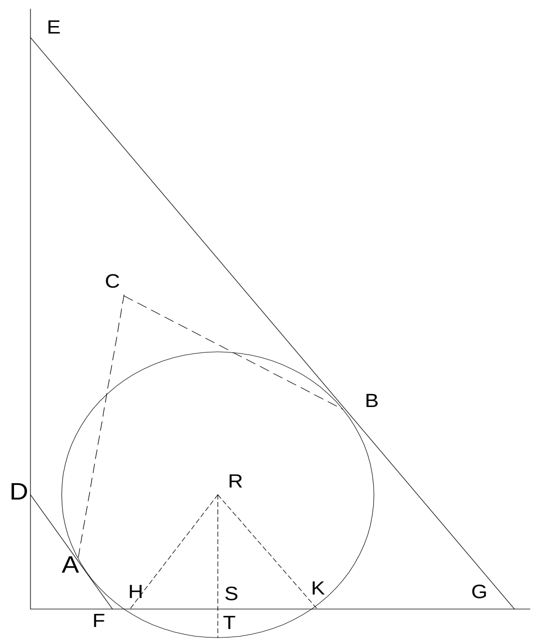
Case $b = 3, c - 1 = 7$. Then $c = 8$, and from (1) $a \cdot 3 + 8 = 29$, or $a = 7$. Verify that $(a, b, c) = (7, 3, 8)$ satisfies equations (1), (2), (3). The value $a + b + c = 7 + 3 + 8 = 18$

Answer D

14. Let $A(2, 1)$ and $B(10, 7)$ be points in the xy-plane. Let R be the region in the first quadrant consisting of all points C for which $\triangle ABC$ has three acute angles. Find the area of R rounded to the nearest integer.

A. 11 B. 61 C. 71 D. 121 E. the area of R is infinite

Solution. The region R is the portion of the trapezoid $FDEG$ outside the circle with diameter AB.



The requested area $A = A_T - A_C + A_S$, where

A_T is the area of trapezoid $FDEG$,

A_C is the area of the circle,

A_S is the area of the circle segment outside the first quadrant.

1. The trapezoidal area A_T

By the distance formula, the diameter of the circle $d = |AB| = \sqrt{(10-2)^2 + (7-1)^2} = 10$.

The slope m of the line AB is $m = \frac{7-1}{10-2} = \frac{6}{8} = \frac{3}{4}$.

The slope m_1 of the lines DF and EG, which are perpendicular to AB, is $m_1 = -\frac{1}{m} = -\frac{4}{3}$.

The equation of the line DF is $y - 1 = -\frac{4}{3}(x-2)$, or $y = -\frac{4}{3}x + \frac{11}{3}$. The x-intercept is $F(\frac{11}{4}, 0)$, the y-intercept is $D(0, \frac{11}{3})$. The distance $|FD| = \sqrt{\left(\frac{11}{4}\right)^2 + \left(\frac{11}{3}\right)^2} = \frac{55}{12}$

The equation of the line EG is $y - 7 = -\frac{4}{3}(x-10)$, or $y = -\frac{4}{3}x + \frac{61}{3}$. The x-intercept is $G(\frac{61}{4}, 0)$, the y-intercept is $D(0, \frac{61}{3})$. The distance $|EG| = \sqrt{\left(\frac{61}{4}\right)^2 + \left(\frac{61}{3}\right)^2} = \frac{305}{12}$

The area A_T of the trapezoid $FDEG$ is $A_T = \frac{|FD|+|EG|}{2}|AB| = \frac{1}{2}\left(\frac{55}{12} + \frac{305}{12}\right)10 = 150$

2. The circle area A_C

The area of the circle is $A_C = \pi\left(\frac{d}{2}\right)^2 = 25\pi$

3. The segment of a circle area A_S

From now on in this problem all angles are measured in radians.

The the center of the circle has coordinats $x = \frac{2+10}{2} = 6$, $y = \frac{1+7}{2} = 4$. As the diameter of the circle is $d = 10$,i.e. $r = 5$, then the equation of the circle is $(x-6)^2+(y-4)^2=25$. To find x-intercepts H, K of the circle, plug into the last equation $y=0$. We will get $x=6\pm 3$, and $H(3,0)$, $K(9,0)$. The distance $|HK|=9-3=6$.Also $|HS|=\frac{1}{2}|HK|=3$, $|RS|=\sqrt{|RH|^2-|HS|^2}=\sqrt{5^2-3^2}=4$

We have $\angle HRK=2\angle HRS=2\sin^{-1}\frac{|HS|}{|HR|}=2\sin^{-1}\frac{3}{5}$.

Area of $\triangle HRK$ is $A(\triangle HRK)=2A(\triangle HRS)=2\left(\frac{1}{2}|HS|\cdot|RS|\right)=|HS|\cdot|RS|=3\cdot 4=12$

Area of sector $HRKT$ is $A(HRKT)=\frac{r^2}{2}\cdot\angle HRK=\frac{5^2}{2}\cdot 2\sin^{-1}\frac{3}{5}=25\sin^{-1}\frac{3}{5}$

The area of segment FKT is $A_S=A(HRKT)-A(\triangle HRK)=25\sin^{-1}\frac{3}{5}-12$

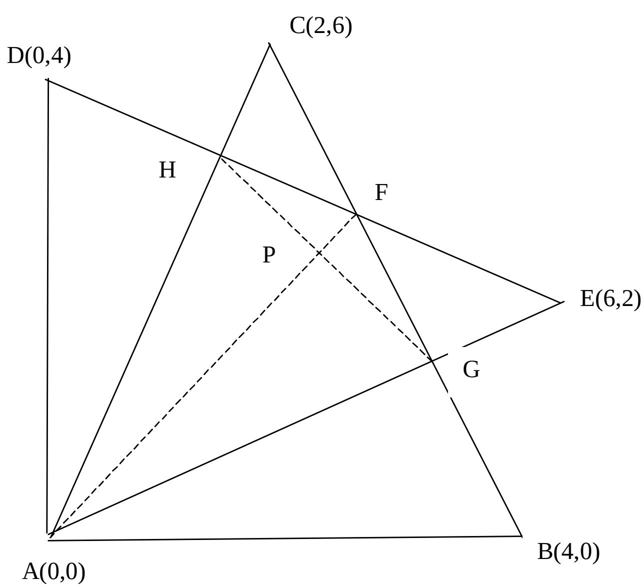
4. The area of region R is $A_T-A_C+A_S=150-25\pi+(25\sin^{-1}\frac{3}{5}-12)=138+25(\sin^{-1}\frac{3}{5}-\pi)\approx 76$

Answer - 76, not in answer options

15. Isosceles $\triangle ABC$ has base $AB=4$ and altitude $CP=6$. Choose point D with $\overleftrightarrow{AD}\perp\overleftrightarrow{AB}$, $AD=AB$, and $\overline{BD}$ intersecting $\overline{AC}$. Choose point E so that $\triangle ADE\cong\triangle ABC$ and $\overline{AE}$ intersects $\overline{BC}$. Find the area common to the two triangles.

A. 7 B. 7.2 C. 7.6 D. 8.0 E. 8.4

Solution. Introduce the coordinate system as shown on the picture. We have to calculate the area of the quadrilateral $AHFG$. It is clear that the picture is symmetric with respect to the lines $y=x$. The area S of $AHFG$ will be the sum of the areas of $\triangle AGH$ and $\triangle HFG$. Since AF is perpendicular to HG, then $S=\frac{1}{2}|AF|\cdot|HG|$. Find the values $|AF|$, $|HG|$.



The equation of the line AE is $y = \frac{1}{3}x$. The equation of the line BC is $y = -3x+12$. Solving the system of these equations, we will find $G(3.6, 1.2)$. By the symmetry, $H(1.2, 3.6)$. The distance $|HG| = \sqrt{(1.2-3/6)^2 + (3.6-1.2)^2} = 2.4\sqrt{2}$.

To find the coordinates of point F, plug $y = x$ into the equation of line BC. We find $F(3,3)$. The distance $|AF| = 3\sqrt{2}$. The area of the common region $S = \frac{1}{2}|AF| \cdot |HG| = \frac{1}{2}(3\sqrt{2})(2.4\sqrt{2}) = 7.2$

Answer - B

16. How many positive integers not divisible by 6 have a base 6 representation which is the reverse of their base 9 representation?

A. 5 B. 6 C. 7 D. 8 E. more than 8

Solution. Let a number A be represented in both forms $A = \sum_{k=0}^{n} a_k 6^k = \sum_{k=0}^{n} a_{n-k} 9^k$, where $a_k \in \{0,1,2,3,4,5\}$ and $a_0 \geq 1$. We will show that $n \leq 4$, and then discuss separate cases of n.

We have $9^n \leq a_0 9^n \leq \sum_{k=0}^{n} a_{n-k} 9^k = \sum_{k=0}^{n} a_k 6^k \leq 5\sum_{k=0}^{n} 6^k = 5\frac{6^{n+1}-1}{6-1} = 6^{n+1} - 1 < 6^{n+1}$

Solve the inequality $9^n < 6^{n+1}$ to obtain $n < \log(6)/\log(\frac{3}{2}) < 4.42$, which uields $n \leq 4$

Consider the cases $n \in \{0,1,2,3,4\}$

Case $n = 0$. Then $A = a_0 6^0 = a_0 9^0 = a_0$. The possible values of A are $1,2,3,4,5$

Case $n = 1$. Then $A = a_1 6 + a_0 = a_0 9 + a_1$. Rewrite this equality as $5a_1 = 8a_0$. Since both sides here should be divided by 8, then a_1 should be divided by 8, which contradicts to the fact that $a_1 \leq 5$

Case $n = 2$. Then $A = a_2 6^2 + a_1 6 + a_0 = a_0 9^2 + a_1 9 + a_2$. Rewrite this equality as $35a_2 = 80a_0 + 3a_1$. Since both sides of this equality are divisible by 5, then a_1 should be either 0, or 5

Subcase $a_1 = 0$. Then $35a_2 = 80a_0$, or $7a_2 = 16a_0$. Both sides here should be divided by 16, that is a_2 should be divided by 16, which contradicts to the fact that $a_2 \leq 5$

Subcase $a_1 = 5$. In this case $35a_2 = 80a_0 + 15$, or $7a_2 = 16a_0 + 3$, and $a_0 = \frac{1}{16}(7a_2 - 3)$. Trying here all $a_2 \in \{0,1,2,3,4,5\}$, we find that the only possible $(a_2, a_0) = (5,2)$. The appropriate value of A is $A = 5 \cdot 6^2 + 5 \cdot 6 + 2 = 2 \cdot 9^2 + 5 \cdot 9 + 5 = 212$

Case $n = 3$. Then $A = a_3 6^3 + a_2 6^2 + a_1 6 + a_0 = a_0 9^3 + a_1 9^2 + a_2 9 + a_3$. Rewrite this equality as $728a_0 = 215a_3 + 27a_2 - 75a_1$. We have $a_0 \leq \frac{1}{728}(215a_3 + 27a_2) \leq \frac{1}{728}(215 \cdot 5 + 27 \cdot 5) \leq \frac{1}{728} \cdot 1210 < 2$. Thus $a_0 = 1$ and the last equality can be written as $215a_3 = 75a_1 - 27a_2 + 728$. We have $a_3 \geq \frac{1}{215}(-27a_2 + 728) \geq \frac{1}{215}(-27 \cdot 5 + 728) > 2.76$, so that $a_3 \geq 3$. Rewrite the latest equality in the form $215a_3 - 728 = 75a_1 - 27a_2$. As the right side here is divisible by 3, then so is the left side, which is possible only for $a_3 = 4 \geq 3$. Then the last equality will take the form $75a_1 - 27a_2 = 132$, or $a_1 = \frac{44+9a_2}{25}$. Trying $a_2 = 0,1,2,3,4,5$, we fail to receive $a_1 \in \{0,1,2,3,4,5\}$. The case is contradictory

Case $n = 4$. Then $A = a_4 6^4 + a_3 6^3 + a_2 6^2 + a_1 6 + a_0 = a_0 9^4 + a_1 9^3 + a_2 9^2 + a_3 9 + a_4$. Rewrite this equality as $6560a_0 = 1295a_4 + 207a_3 - 45a_2 - 723a_1$. We have $a_0 \le \frac{1}{6560}(1295a_4 + 207a_3) \le \frac{1}{6560}(1295 \cdot 5 + 207 \cdot 5) < 1.15$. Thus $a_0 = 1$ and the last equality can be written as $1295a_4 = -207a_3 + 45a_2 + 723a_1 + 6560$. From here $a_4 \ge \frac{1}{1295}(-207a_3 + 6560) \ge \frac{1}{1295}(-207 \cdot 5 + 6560) > 4$. Thus $a_4 = 5$. Rewrite the last equality in the form $-207a_3 + 45a_2 + 723a_1 = -85$. Since the left side is divisible by 3, but not so is the right side, then the case is contradictory.

In summary, the positive integers under the problem conditions are $1, 2, 3, 4, 5, 212$

Answer - B.

17. Let S be the set of all ordered triples (p, q, r) of prime numbers for which the equation $px^2 + qx + r = 0$ has at least one rational solution. How many primes appear in S at least seven times?

A. 0 B. 1 C. 2 D. 3 E. An infinite number

Solution. If a is a solution of an equation $px^2 + qx + r = 0$, then $1/a$ is a solution of the equation $rx^2 + qx + p = 0$. Thus if (p, q, r) satisfies the problem conditions, then the same is true for the symmetric triple (r, q, p).

Note that the only even prime number is 2.

Let (p, q, r) be a triple under the conditions of the problem. Let a value $x = -\frac{m}{n}$, where m, n are relatively prime positive integers, is a solution of the equation $px^2 + qx + r = 0$. Plugging the above expression for x into the equation, we will come to the equivalent equation

$pm^2 + rn^2 = qmn$ (1)

Since the right side and the first term on the left in (1) are divisible by m, then r is divisible by m. Since r is a prime, then $m = 1$ or $m = r$. Similarly, we find that $n = 1$, or $n = p$

Case $m = 1, n = 1$,or $m = r, n = p$. From (1). after simplifications we will have

$p + r = q$ (2)

If both p, r are odd, then q must be even, which yields $p + r = 2$, and this is impossible.

If one of the p, r, say p, is even, then from (2) we will have $q = r + 2$, and (r, q) are twin primes. The related generated sequence of (p, q, r) is

$\{(2, 5, 3), (2, 7, 5), (2, 13, 11), (2, 19, 17), (2, 31, 29), (2, 43, 41), (2, 49, 47), ...\}$.

In this sequence the value of 2 repeats at least seven times, the value of 5 repeats twice, in triples $(2, 5, 3), (2, 7, 5)$, but neither of other values repeats more than once. Because of triples symmetry, the value of 5 also appears in $(3, 5, 2), (5, 7, 2)$. Totally for this case the value of 5 appears four times.

Case $m = 1, n = p$, or $m = r, n = 1$.From (1), after simplifications we will have $1 + pr = q$. If both p, r are odd, then q is even and $1 + pr = 2$, which is impossible. If one of p, r, say p, is even, then $q = 2r + 1$. The generated sequence of (p, q, r) is $\{(2, 5, 2), (2, 7, 3).(2, 11, 5), (2, 23, 11), (2, 39, 19), (2, 47, 23), (2, 59, 29), ...\}$.

In this sequence the value of 2 repeats at least seven times, the value of 5 repeats twice in triples $(2, 5, 2), (2, 11, 5)$, but neither of other values repeats more than once. Because of triples symmetry, the value of 5 also appears in $(5, 11, 2)$. Totally for this case the value of 5 appears three times.

Conclusion. In the triples under the problem conditions the value of 2 repeated at least seven times, and the value of 5 repeates exactly seven times.

Answer - C

18. If $cos\theta = \frac{7}{8}$and $0^\circ < \theta < 90^\circ$, the value of $\tan\frac{1}{4}\theta$ can be expressed as $\frac{\sqrt{m}}{n}$ where m has no perfect square factors > 1. Find $m+n$.

A. 20 B. 22 C. 24 D. 26 E. 28

Solution. With double-angle formulas in mind, we obtain $\cos 4x = 2\cos^2 2x - 1 = 2(2\cos^2 x - 1)^2 - 1$.

Taking in the last equality $x = \frac{\theta}{4}$, we will have

$\frac{7}{18} = 2(2\cos^2 x - 1)^2 - 1 \iff \frac{25}{18} = 2(2\cos^2 x - 1)^2$

$\iff \frac{25}{36} = (2\cos^2 x - 1)^2 \iff \frac{5}{6} = 2\cos^2 x - 1 \iff \frac{11}{6} = 2\cos^2 x$

$\iff \frac{11}{12} = \cos^2 x$

Then $\sin^2 x = 1 - \cos^2 x = \frac{1}{12}$, $\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1}{11}$ and $\tan x = \frac{\sqrt{11}}{11}$. Here $m = 11$, $n = 11$, $m + n = 22$

Answer - B

19. The Robotics Club has 12 members, 3 each who are freshmen, sophomores, juniors, and seniors. In each of the freshman and junior classes, 1 member is an engineer and two are in CS; in each of the sophomore and senior classes, 2 are engineers and 1 is in CS. Find the number of ways to have a committee of 6 members so that each class and each major is represented on the committee.

A. 492 B. 502 C. 540 D. 592 E. 594

Solution. The data in the problem is summarized into the table

	1^{st} year student	2^{nd} year student	3^{rd} year student	4^{th} year student
engineers	1	2	1	2
CS	2	1	2	1

For any set C, let $|C|$ be the number of elements in a set C, and $\overline{C}$ be the complement of C in some universal set..

Denote by M the set of all members of the Robotics Club, $|M| = 12$.

Define the sets

$a_k = \{m \in M \mid m \text{ is a } k^{th} \text{ year student}\}, k = 1, 2, 3, 4;$

$a_5 = \{m \in M \mid m \text{ is a engineering major}\}, a_6 = \{m \in M \mid m \text{ is a CS major}\}$

Let P be the universal set of all subsets of M with *six* elements,

Define the sets

$A_k = \{p \in P \mid p \cap a_k \neq \varnothing\}$, $k = 1, ..., 6$. and

$I = \bigcap_{k=1}^{6} A_k$

The problem requires to find $|I|$

Introduce complements $\overline{A_k} = \{p \in P \mid p \cap a_k = \varnothing\}, k = 1, ..., 6.$

We have $\overline{I} = \overline{\bigcap_{k=1}^{6} A_k} = \bigcup_{k=1}^{6} \overline{A_k}$.

Applying inclusion–exclusion principle to $\bigcup_{k=1}^{6} \overline{A_k}$, we will have

$$\left|\overline{I}\right| = \left|\bigcup_{k=1}^{6} \overline{A_k}\right| = \sum_{k=1}^{6} \left|\overline{A_k}\right| - \sum_{j,k:1\le j<k\le 6} \left|A_j \bigcap A_k\right| + \sum_{i,j,k:1\le i<j<k\le 6} \left|A_i \bigcap A_j \bigcap A_k\right| -$$
$$... + (-1)^6 \left|\bigcap_{k=1}^{6} \overline{A_k}\right|$$

Here on the right side, using the data table,

$A_j \bigcap A_k = \varnothing, k = 5, 6; 1 \le j \le 4;$

$A_i \bigcap A_j \bigcap A_k = \varnothing, 1 \le i < j < k \le 6;$ ... ; $\bigcap_{k=1}^{6} \overline{A_k} = \varnothing$

Then the expression for $\left|\overline{I}\right|$ is simplified as

$$\left|\overline{I}\right| = \sum_{k=1}^{6} \left|\overline{A_k}\right| - \sum_{j,k:1\le j<k\le 4} \left|A_j \bigcap A_k\right|$$

Since, again using the data table,

$\left|\overline{A_k}\right| = \binom{9}{6} = 84.\ k = 1, ..., 4;$

$\left|\overline{A_k}\right| = 1,\ k = 5, 6;$

$\left|A_j \bigcap A_k\right| = 1,\ j, k : 1 \le j < k \le 4,$

then

$$\left|\overline{I}\right| = (4 \cdot 84 + 2 \cdot 1) - 1 \cdot \sum_{j,k:1\le j<k\le 4} 1 = (4 \cdot 84 + 2 \cdot 1) - 1 \cdot 6 = 332$$

Finally, $|I| = |P| - \left|\overline{I}\right| = \binom{12}{6} - 332 = 924 - 332 = 592$

Answer - D

Note. The method can be extended to numerically solve similar problems with any number of rows, columns in the data table, and any size of selected subsets.

20. The equation $x^2 - 11y^2 + 23 = 10xy$ has four solutions (x_i, y_i) in which both coordinates are integers. Find $|x_1| + |x_2| + |x_3| + |x_4|$.

A. 8 B. 40 C. 44 D. 46 E. 52

Solution. Transform the equation to the form $x^2 - y^2 - 10y^2 - 10xy = -23 \iff (x + y)(x - y) - 10y(x + y) = -23 \iff (x + y)(x - 11y) = -23.$

Factors on the left in the last equality divide into -23. Consider the cases

Case $x + y = 1, x - 11y = -23$. Solution is $(x_1, y_1) = (-1, 2)$

Case $x + y = -1, x - 11y = 23$. Solution is $(x_2, y_2) = (1, -2)$

Case $x + y = 23, x - 11y = -1$. Solution is $(x_3, y_3) = (21, 2)$

Case $x + y = -23, x - 11y = 1$. Solution is $(x_4, y_4) = (-21, -2)$

The required $|x_1| + |x_2| + |x_3| + |x_4| = |-1| + |1| + |21| + |-21| = 44$

Answer - C