1. The triangles $\triangle\ ABC$ and $\triangle\ DEF$ are not isosceles, not congruent, and have integer-length sides. If they have the same perimeter, what is the smallest such perimeter they could share?

   A. 10 B. 11 C. 12 D. 13 E. 14

   Solution: Recall the the triangle inequality: for all triangles with sides $a$, $b$, and $c$,

   $$a+b>c \text{ and } a+c>b \quad\text{and}\quad b+c>a$$

   Let us agree to organize all triangles by listing their sides in an increasing order, and let us systematically look for the triangles with the lowest possible values for perimeter.

   First notice that the shortest side can not be 1 unit long. This is because if the shortest side is 1 unit long and the second shortest side is $n$, then the longest side must be shorter than $n+1$ and there isn't an integer between $n$ and $n+1$. So the shortest side possible is 2 units long.

   If the smallest side is 2: For example, $2, 3, 4$ is possible as $2+3>4$. But if the shortest two sides are 2 and 3, 4 is the only integer possibility.

   | If the shortest side is 2 | If the shortest side is 3 |
   |---|---|
   | $2,3,4 \implies P=9$ | $3,4,5 \implies P=12$ |
   | $2,4,5 \implies P=11$ | $3,4,6 \implies P=13$ |
   | $2,5,6 \implies P=13$ | $3,5,7 \implies P=15$ |
   | $2,6,7 \implies P=15$ | |
   | $2,7,8 \implies P=17$ | |

   The smallest repeating value is 13, which is choice $\boxed{\text{D}}$.

2. At the intergalactic trading station, several different currencies are used. Today, 15 gleeks = 11 zorks, 7 gleeks = 3 zeffs, and 5 zeffs = 2 gems. A certain merchant lists prices in gleeks, but only gives change in zorks. Can someone afford to buy an item that costs 45 gleeks if s/he has 8 gems? If so, how much change will be given (to the nearest tenth)?

   A. No B. Yes, 3.1 zorks C. Yes, 1.2 zorks D. Yes, 0.3 zorks E. Yes, 1.8 zorks

   Solution: We can use several unit conversion factors to switch from gems to gleeks. She has 8 gems, which is

   $$8 \text{ gems} = \frac{8 \text{ gems}}{1}\cdot\frac{5 \text{ zeffs}}{2 \text{ gems}}\cdot\frac{7 \text{ gleeks}}{3 \text{ zeffs}} = \frac{8\cdot 5\cdot 7}{6}\text{ gleeks } = \frac{140}{3} = 46\frac{2}{3}\text{ gleeks}$$

   Thus she has enough money to buy the item, and the change is $1\frac{2}{3}=\frac{5}{3}$ gleeks. Convert the change to zorks:

   $$\frac{5}{3}\text{ gleeks} = \frac{5 \text{ gleeks}}{3}\cdot\frac{11 \text{ zork}}{15 \text{ gleeks}} = \frac{11}{9}\text{ zork} = 1.222 \text{ zork}$$

   So the answer is $\boxed{\text{C}}$.

3. Suppose $a$ and $b$ are integers such that $(a, b)$ is a solution of $a^2 + b^2 + 2ab + 16a + 16b = 36$. Let $c$ be the average of $a$ and $b$. Find the sum of all possible values of $c$.

A. 2 B. $-8$ C. $-18$ D. $-2$ E. 8

Solution:

$$\begin{aligned} a^2 + b^2 + 2ab + 16a + 16b &= 36 \\ (a+b)^2 + 16(a+b) &= 36 \qquad \text{factor out } a+b \\ (a+b)(a+b+16) &= 36 \end{aligned}$$

There are only finitely many ways we can express 36 as a product of two integers. Among those, we collect the ones where the difference is 16.

| $a+b$ | $a+b+16$ | difference | |
|---|---|---|---|
| 1 | 36 | 35 | |
| 2 | 18 | 16 | This one works! |
| 3 | 12 | 9 | |
| 4 | 9 | 5 | |
| 6 | 6 | 0 | |
| $-36$ | $-1$ | 35 | |
| $-18$ | $-2$ | 16 | This one works! |
| $-12$ | $-3$ | 9 | |
| $-9$ | $-4$ | 5 | |
| $-6$ | $-6$ | 0 | |

The value of $c = \dfrac{a+b}{2}$ is either $\dfrac{2}{2} = 1$ or $\dfrac{-18}{2} = -9$. The sum of the possible values is $1 + (-9) = -8$. The answer is $\boxed{\text{B}}$.

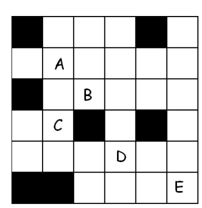
4. If cars hold 5 passengers and charge for \$29 a trip to the airport, and vans hold 7 passengers and charge \$41, find the minimum cost to transport 49 people to the airport.

A. \$290 B. \$285 C. \$287 D. \$280 E. \$282

Solution: Let us start with all cars and no vans. The lowest answer we could find is $\boxed{\text{B}}$.

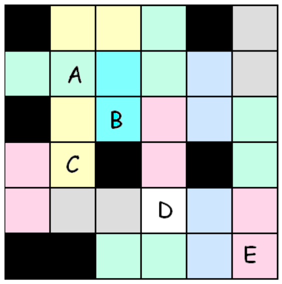
| Number of vans | Number of cars | Capacity | Cost |
|---|---|---|---|
| 0 | 10 | 50 | \$290 |
| 1 | 9 | 52 | \$302 |
| 2 | 7 | 49 | \$285 |
| 3 | 6 | 51 | \$333 |
| 4 | 5 | 53 | \$309 |
| 5 | 3 | 50 | \$292 |
| 6 | 2 | 52 | \$304 |
| 7 | 0 | 49 | \$287 |

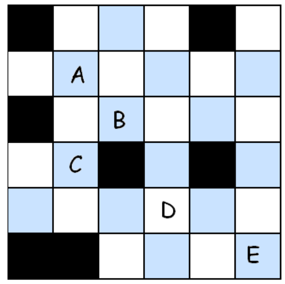
5. In the grid (made up of $1 \times 1$ squares) on the right, which of the squares A, B, C, D, or E, when shaded, will allow the unshaded squares to be covered by exactly 14 dominos ($1 \times 2$ rectangles) with no overlaps or gaps?



Solution 1: I found a covering that skipped D.

Solution 2: While we are at coloring; consider the following. Suppose we color the fields to be covered so that two fields with a side common have different colors. Then A, B, C, and E are all blue and only E is white. The 29 fields are: 14 blue and 15 white. Since each domino can cover exactly one white and one blue field, the covering is impossible if we skip a blue field as we would have 14 dominos to cover 13 blue and 15 white fields. Either way, the correct answer is $\boxed{\text{D}}$.




6. The graph of $x^2 + xy + x + 3y = 6$ is

A. an ellipse B. a parabola C. a hyperbola D. 2 parallel lines E. two intersecting lines.

Solution: We reduce one side to zero, factor, and apply the zero product rule.

$$\begin{aligned}
x^2 + xy + x + 3y &= 6 \\
x^2 + xy + x + 3y - 6 &= 0 \\
x^2 + xy + x + 3y - 6 &= 0 \\
x^2 + x - 6 + 3y + xy &= 0 \\
(x+3)(x-2) + 3y(x+3) &= 0 \\
(x+3)(x-2+3y) &= 0
\end{aligned}$$

$$x_1 = -3 \quad \text{or} \quad y_2 = \frac{-x+2}{3}$$

This are the equations of two non-parallel lines, so the correct answer is $\boxed{\text{E}}$.

7. Let a and b be positive integers such that $(a, b)$ is a solution to $\sqrt[3]{a + 4\sqrt{b}} + \sqrt[3]{a - 4\sqrt{b}} = 3$. Find the smallest possible value of $a + b$.

A. 14 B. 18 C. 22 D. 26 E. 30

$$\sqrt[3]{a + 4\sqrt{b}} + \sqrt[3]{a - 4\sqrt{b}} = 3 \qquad \text{raise to third power}$$

$$\left(\sqrt[3]{a + 4\sqrt{b}}\right)^3 + 3\left(\sqrt[3]{a + 4\sqrt{b}}\right)^2\left(\sqrt[3]{a - 4\sqrt{b}}\right) + 3\left(\sqrt[3]{a + 4\sqrt{b}}\right)\left(\sqrt[3]{a - 4\sqrt{b}}\right)^2 + \left(\sqrt[3]{a - 4\sqrt{b}}\right)^3 = 27$$

$$a + 4\sqrt{b} + a - 4\sqrt{b} + 3\left(\sqrt[3]{a + 4\sqrt{b}}\right)\left(\sqrt[3]{a - 4\sqrt{b}}\right)\left(\underbrace{\sqrt[3]{a + 4\sqrt{b}} + \sqrt[3]{a - 4\sqrt{b}}}_{=3}\right) = 27$$

$$\begin{aligned}
2a + 3\sqrt[3]{a^2 - 16b} \cdot 3 &= 27 \\
2a + 9\sqrt[3]{a^2 - 16b} &= 27 \\
2a &= 27 - 9\sqrt[3]{a^2 - 16b} \\
2a &= 9\left(3 - \sqrt[3]{a^2 - 16b}\right)
\end{aligned}$$

Thus $a$ is divisible by 9. We are looking for the smallest value of $a+b$, so let us try the smallest positive value, $a=9$.

$$\begin{aligned} 2\cdot 9 &= 9\left(3-\sqrt[3]{81-16b}\right) \\ 2 &= 3-\sqrt[3]{81-16b} \\ \sqrt[3]{81-16b} &= 1 \\ 81-16b &= 1 \\ 80 &= 16b \\ 5 &= b \end{aligned}$$

Thus $a+b=9+5=14$. We didn't prove yet that this is the smallest value for $a+b$, but it is the smallest value offered. So, if the problem is well designed, the answer is $\boxed{\text{A}}$.

8. The matrix $A=\begin{bmatrix} a & 8 \\ -3 & b \end{bmatrix}$ is its own inverse (that is, $A$ times $A$ equals the identity matrix). Find $|a-b|$.

A. 4 B. 6 C. 8 D. 10 E. 12

Solution:

$$\begin{aligned} A^2 &= I \\ \begin{bmatrix} a & 8 \\ -3 & b \end{bmatrix}\begin{bmatrix} a & 8 \\ -3 & b \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} a^2-24 & 8a+8b \\ -3a-3b & -24-b^2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

So $a^2-24=1 \Longrightarrow a^2=25 \Longrightarrow a=\pm 5$ and $a+b=0$ so $b=-(\pm 5)$. Either way, $|a-b|=|(5-(-5))|=10$ or $|-5-5|=10$. The correct answer is $\boxed{\text{D}}$.

9. Let $f(x)=x^2+bx+c$. If $f(4)=f(2)+11$, find $f(4)-f(0)$.

A. $-6$ B. $-8$ C. 8 D. 10 E. 14

Solution:

$$\begin{aligned} f(4) &= f(2)+11 \\ 4^2+b\cdot 4+c &= 2^2+b\cdot 2+c+11 \\ 16+4b+c &= 4+2b+c+11 \\ 16+4b &= 2b+15 \\ 2b &= -1 \\ b &= -\frac{1}{2} \end{aligned}$$

So $f(x)=x^2-\frac{1}{2}x+c$ Thus

$$f(4)-f(0)=\left(4^2-\frac{1}{2}\cdot 4+c\right)-\left(0^2-\frac{1}{2}\cdot 0+c\right)=16-2=14$$

The answer is $\boxed{\text{E}}$.

10. Three people $(X, Y, Z)$ are in a room with you. One is a Knight (Knights always tell the truth), one is a Knave (Knaves always lie), and the other is a Spy (Spies may either lie or tell the truth), but you don't know who is which. Each person makes exactly one statement. Which of the following sets of three statements is NOT possible?

| A | B | C | D | E |
|---|---|---|---|---|
| X: I am a Knight | X: I am not a Spy | X: I am a Spy | X: I am a Knight | X: I am not a Knave |
| Y: I am a Knave | Y: I am not a Spy | Y: I am a Spy | Y: I am a Knave | Y: I am not a Knave |
| Z: X is a Spy | Z: X is not a Knight | Z: I am a Knight | Z: X is a Knight | Z: I am not a Knave |

Solution: The only things impossible here are 3 true or 3 false statements. 1 true because of the Knight, one false because of the Knave and the Spy can go either way.

Consider A. X could be telling the truth –> X is knight. Y could be the spy and lying that he is a knave. Then Z is the knave and X being a Spy is false. So, A is possible.

Consider B. X could be Spy and lying. Then Y must be telling the truth –> he is a Knight. Then Z must be a knave but then he is lying and so X is a knight. So if X is the knight and he is telling the truth, then Y could be a Spy and be lying. then Z must be a knave and X not a knight is then a lie. So, B is possible.

Consider C. X could be a spy and telling the truth. Then Y lies and so he must be a knave. Then Z must be the knight. So, C is possible.

Consider D. The statement "I am knave" can not be true. Thus, Y is lying and he is thus a Spy. So X and Z are the knight and knave but they literally agree! So their statements aare both true or both false. They are both impossible. So, D is impossible.

Consider E. X might be a knave and lying,Y might be a knight and telling the truth and Z a spy and telling the truth. So, this is quite possible.

So the correct answer is $\boxed{\text{D}}$.

11. For a positive integer $n$, let $S(n)$ be the sum of the first $n$ positive integers (for example, $S(5) = 15$). For how many positive integers, $n$, less than 2017, will all digits of $S(n)$ be 1s?

A. 0 B. 1 C. 2 D. 3 E. 4

Solution: $S(1) = 1$ $\quad S(2) = 3$ $\quad S(n) = \dfrac{n(n+1)}{2}$

$$\begin{aligned} S(n) &= \frac{n(n+1)}{2} \\ 1 + 2 + \ldots + n &= \underbrace{11....1}_{k \text{ times}} \\ \frac{n(n+1)}{2} &= \underbrace{11....1}_{k \text{ times}} \\ n(n+1) &= \underbrace{22....2}_{k \text{ times}} \\ n^2 + n - \underbrace{22....2}_{k \text{ times}} &= 0 \\ n_{1,2} &= \frac{-1 \pm \sqrt{1 + 4(22....2)}}{2} \end{aligned}$$

The smaller solution is clearly negative, so we focus on the larger, positive one.

If $k = 1$ $\quad n = \dfrac{-1 + \sqrt{1 + 4(2)}}{2} = 1$ $\quad$ Indeed, $S(1) = 1$

If $k = 2$ $\quad n = \dfrac{-1 + \sqrt{1 + 4(22)}}{2} = \dfrac{\sqrt{89} - 1}{2}$ $\quad$ not an integer

If $k = 3$ $\quad n = \dfrac{-1 + \sqrt{1 + 4(222)}}{2} = \dfrac{\sqrt{889} - 1}{2}$ not an integer

If $k = 4$ $\quad n = \dfrac{-1 + \sqrt{1 + 4(2222)}}{2} = \dfrac{\sqrt{8889} - 1}{2}$ not an integer

If $k = 5$ $\quad n = \dfrac{-1 + \sqrt{1 + 4(22222)}}{2} = \dfrac{\sqrt{88889} - 1}{2} \approx 148.5713$ not an integer

If $k = 6$ $\quad n = \dfrac{-1 + \sqrt{1 + 4(222222)}}{2} = \dfrac{\sqrt{888889} - 1}{2} \approx 470.905$ not an integer

If $k = 7$ $\quad n = \dfrac{-1 + \sqrt{1 + 4(2222222)}}{2} = \dfrac{\sqrt{8888889} - 1}{2} \approx 1490.212$ not an integer

If $k = 8$ $\quad n = \dfrac{-1 + \sqrt{1 + 4(22222222)}}{2} = \dfrac{\sqrt{88888889} - 1}{2} \approx 4713.55$ not an integer. Now we are beyond 2017 so we can stop looking. $S(1) = 1$ is the ony solution so the answer is $\boxed{\text{B}}$.

12. Ed filled $\frac{2}{3}$ of his radiator with antifreeze and then added 4 more quarts (a gallon) of antifreeze. After draining half the antifreeze, he needed 11 quarts of antifreeze to fill the radiator to capacity. How many gallons of antifreeze can the radiator hold?

    A. 4.65 B. 4.875 C. 18.6 D. 19.5 E. 78

    Solution: Let $x$ be the capacity of the radiator.

$$\begin{aligned}
\frac{\frac{2}{3}x + 4}{2} + 11 &= x \\
\frac{2}{3}x + 4 + 22 &= 2x \\
\frac{2}{3}x + 26 &= 2x \\
26 &= \frac{4}{3}x \\
\frac{3 \cdot 26}{4} &= x \\
x &= \frac{39}{2} = 19.5
\end{aligned}$$

    So the answer is 19.5 quarts, but we need to present the answer in gallons: $\dfrac{19.5 \text{ quarts}}{4} = 4.875$ gallons so the correct answer is $\boxed{\text{B}}$.

13. On a game show, the final contestant each day can win $\$1,000,000$ by correctly guessing an integer between 1 and 100 inclusive (which is chosen randomly each day). Before guessing the contestant can ask one yes/no question of his or her choice. Monday's contestant asked "Is the number 57?" and Tuesday's contestant asked "Is the number greater than 50?". Let $P(M)$ be the probability of Monday's contestant winning and let $P(T)$ be the probability of Tuesday's contestants winning (assume each contestant properly uses the information gained from the question). Which of the following is true?

    A. $\dfrac{P(M)}{P(T)} \leq .1$ B. $0.1 < \dfrac{P(M)}{P(T)} < 0.9$ C. $0.9 \leq \dfrac{P(M)}{P(T)} \leq 1.1$ D. $1.1 < \dfrac{P(M)}{P(T)} < 2$ E. $\dfrac{P(M)}{P(T)} \geq 2$

    Solution: $P(M) = \dfrac{1}{100}(1) + \dfrac{99}{100}\left(\dfrac{1}{99}\right) = \dfrac{2}{100} = \dfrac{4}{200} = 0.02$

$$P(T) = \frac{1}{2}\left(\frac{50}{100}\right) + \frac{1}{2}\left(\frac{49}{100}\right) = \frac{99}{200} = 0.495 \qquad \frac{P(M)}{P(T)} = \frac{\frac{4}{200}}{\frac{99}{200}} = \frac{4}{99} \approx 0.040404 \quad \text{so the answer is } \boxed{\text{A}}.$$

(Caution! This is in conflict with the official AMATYC answer....)

14. Let $P(x)$ be a degree 5 polynomial with rational coefficients and $P(0) = -53\,040$. Suppose $x = 12$, $x = 3 + 5i$, and $x = 4 - 7i$ are zeros of $P(x)$. In which interval does the coefficient of $x^3$ lie?

A. $(-\infty, -500]$ B. $(-500, -100]$ C. $(-100, 100)$ D. $[100, 500)$ E. $[500, \infty)$

Solution: $x = 12$ is a zero $\Rightarrow$ $(x - 12)$ is a linear factor by the remainder theorem.

$x = 3 + 5i$ is a zero and $P$ has rational (thus real) coefficients $\Rightarrow$ $(x - (3 + 5i))(x - (3 - 5i))$ is a factor

$x = 4 - 7i$ is a zero and $P$ has rational coefficients $\Rightarrow$ $(x - (4 - 7i))(x - (4 + 7i))$ is a factor

$$\begin{aligned} P(x) &= A(x-12)\left((x-(3+5i))(x-(3-5i))\right)(x-(4-7i))(x-(4+7i)) \\ &= A(x-12)\left(x^2-6x+34\right)\left(x^2-8x+65\right) \end{aligned}$$

The degree is correct. We will find the leading coefficient $A$ using $P(0)$.

$$\begin{aligned} P(0) &= A(-12)(34)(65) = -53\,040 \\ -26\,520A &= -53\,040 \\ A &= 2 \end{aligned}$$

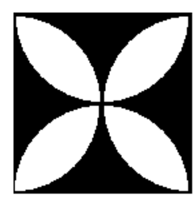
So $P(x) = 2(x-12)\left(x^2-6x+34\right)\left(x^2-8x+65\right)$. Now for the cubic term:

$$\begin{aligned} a_3x^3 &= 2\left(x\left(x^2\right)65 + x(-6x)(-8x) + x(34)x^2 - 12x^2(-8x) - 12(-6x)x^2\right) \\ &= 2x^3(65+48+34+96+72) = 630x^3 \end{aligned}$$

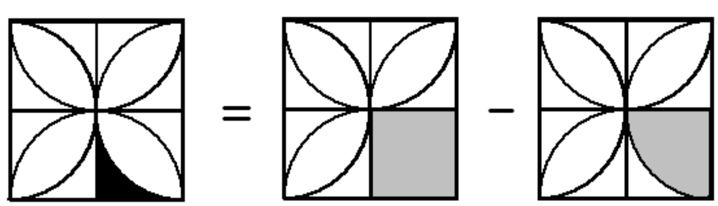
Since $630 \in [500, \infty)$, the answer is $\boxed{\text{E}}$.

15. A company designed a new logo by constructing semicircles inside of a unit square (side length = 1) as shown on the right. Which of the following is closest to the area of the shaded region?

A. 0.4 B. 0.45 C. 0.5 D. 0.55 E. 0.6



Solution: Consider one-eighth of the shaded region as shown. Clearly it is the difference between a square of sides $\frac{1}{2}$ unit and a quarter of a circle with radius $\frac{1}{2}$.



So, the area we must find is

$$A = 8\left(\left(\frac{1}{2}\right)^2 - \frac{1}{4}\pi\left(\frac{1}{2}\right)^2\right) = 8\left(\frac{1}{4} - \frac{1}{16}\pi\right) = 2 - \frac{\pi}{2} \approx 0.4292037$$

To find the correct answer, we need to find which number given is closest to our answer.

A: $0.4292037 - 0.4 \approx 0.03$ C: $0.5 - 0.4292037 \approx 0.071$ E: $0.6 - 0.4292037 \approx 0.17$

B: $0.45 - 0.4292037 \approx 0.021$ D: $0.55 - 0.4292037 \approx 0.12$

So the correct answer is $\boxed{\text{B}}$.

16. How many positive integers less than 1000 are divisible by exactly one of 7 or 11?

A. 196 B. 208 C. 220 D. 232 E. 244

Solution: Recall that if $|S|$ denotes the size or cardinality of a set $S$, then for all sets $A$, $B$,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Define $E = \{n : n \text{ is a positive integer, } n < 1000, n \text{ is divisible by } 11\}$ and
$S = \{n : n \text{ is a positive integer, } n < 1000, n \text{ is divisible by } 7\}$.

Then $E \cap S = \{n : n \text{ is a positive integer, } n < 1000, n \text{ is divisible by } 77\}$. In terms of these sets, what we are looking for is

$$X = |E| + |S| - 2\,|E \cap S|$$

Let us compute the size of these sets. $1000 \div 7 = 142 \text{ R } 6 \quad 142 \cdot 7 = 994$

$$S = \left\{ \begin{array}{ccccccc} 7 & , & 14 & , & 21 & ,\ldots, & 994 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 \cdot 7 & & 2 \cdot 7 & & 3 \cdot 7 & & 142 \cdot 7 \end{array} \right\} \implies |S| = 142$$

Similarly, $1000 \div 11 = 90 \text{ R } 10 \quad 90 \cdot 11 = 990$

$$E = \left\{ \begin{array}{ccccccc} 11 & , & 22 & , & 33 & ,\ldots, & 990 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 \cdot 11 & & 2 \cdot 11 & & 3 \cdot 11 & & 90 \cdot 11 \end{array} \right\} \implies |E| = 90$$

Now for $|E \cap S|$: $1000 \div 77 = 12 \text{ R } 15 \quad 12 \cdot 77 = 924 \text{ R } 76$

$$E \cap S = \left\{ \begin{array}{ccccccc} 77 & , & 154 & , & 231 & ,\ldots, & 924 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 \cdot 77 & & 2 \cdot 77 & & 3 \cdot 77 & & 12 \cdot 77 \end{array} \right\} \implies |E \cap S| = 12$$

Now our solution is

$$X = |E| + |S| - 2\,|E \cap S| = 142 + 90 - 2 \cdot 12 = 208$$

which is choice $\boxed{\text{B}}$.

17. A neon light is failing. When the switch is flipped, it lights for a second, then goes off for a second; lights for a second, then goes off for 2 seconds; lights for a second, then goes off for 3 seconds, etc. Exactly two minutes after the switch is flipped, how long (in seconds) will it stay off before it goes on again?

A. 12 B. 13 C. 14 D. 15 E.16

Solution:

| | | |
|---|---|---|
| 1 second on<br>1 second off $\Longrightarrow$ $k=1$ | 1 second on<br>2 seconds off $\Longrightarrow$ $k=2$ | 1 second on<br>3 seconds off $\Longrightarrow$ $k=3$ |

So for general $k$:

$$\begin{aligned} (1+1)+(1+2)+(1+3)+...+(1+k) &\geq 120 \\ k+\frac{k(k+1)}{2} &= 120 \\ 2k+k^2+k-240 &= 0 \qquad k_{1,2}=\frac{-3\pm\sqrt{9+4\cdot 240}}{2} \end{aligned}$$

The positive root is about $14.064$. This we are talking about between $k=14$ and $k=15$

$k=14 \quad\Longrightarrow\quad 14+\dfrac{14(15)}{2}=119$

That is: as $k=14$, the light goes on for a second and then is off for $14$ seconds. That is the $119$th second. So, $1$ more second markes exactly two minutes after. It just got dark a second ago. So, it will be dark for another 13 seconds before it lights up again. This is answer $\boxed{\text{B}}$.

18. Let $N$ be the smallest positive integer such that ALL $N$-digit numbers of the form $aa...a$ are divisible by 7. Let $M$ be the smallest positive integer such that $10^M$ does NOT have a factorization $ab$ in which neither factor has any 0 digits. Find $M+N$.

A. 18 B. 17 C. 16 D. 15 E. 14

Solution: Clearly $10^n = 2^n\cdot 5^n$. If a 2 paired with a 5, in the prime factorization of a factor, then it will end in a zero. Thus, we need to look at factorizations of the form of $2^n\cdot 5^n$. Eventually, these factors will have a zero among their digits, although not as the last digit.

| | | | |
|---|---|---|---|
| $10^1=2\cdot 5$ | $10^3=8\cdot 125$ | $10^5=32\cdot 3125$ | $10^7=128\cdot 78\,125$ |
| $10^2=4\cdot 25$ | $10^4=16\cdot 625$ | $10^6=64\cdot 15\,625$ | $10^8=256\cdot 390\,625 \Longrightarrow$ Thus $M=8$ |

Now for $N$: 11 and 111 and 1111 are not divisible by 7. 11111 aren't either.

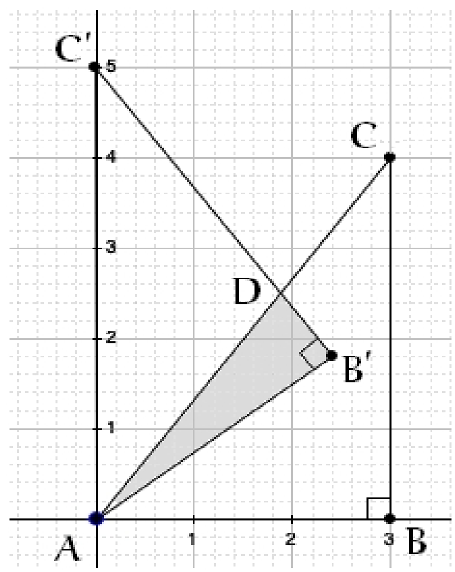
But then $111111 = 7\cdot 15\,873$ and so all six-digit numbers $aaaaaa$ are divisible by 7 as $aaaaaa = a(111111) = 7(15\,873a)$. So, $N=6$

$M+N=8+6=14$ which is choice $\boxed{\text{E}}$.

19. A triangle has vertices $A(0,0)$, $B(3,0)$, and $C(3,4)$. If the triangle is rotated counterclockwise around the origin until $C$ lies on the positive $y$-axis, find the area of the intersection of the region bounded by the original triangle and the region bounded by the rotated triangle.

A. $\frac{21}{16}$ B. $\frac{25}{16}$ C. $\frac{29}{16}$ D. $\frac{35}{16}$ E. $\frac{75}{16}$

Solution:




Let $\alpha$ denote angle $CAB$. Clearly, $\alpha = \tan^{-1}\left(\frac{4}{3}\right)$. Then angle $CAC' = \beta = 90° - \alpha = \tan^{-1}\left(\frac{3}{4}\right)$.

$$\begin{aligned} \theta &= \alpha - \beta \\ \tan\theta &= \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{\frac{4}{3}-\frac{3}{4}}{1+\frac{4}{3}\cdot\frac{3}{4}} = \frac{\frac{7}{12}}{2} = \frac{7}{24} \end{aligned}$$

The area of triangle $AB'D$ is $\frac{1}{2}(AB')(AB'\tan\theta) = \frac{1}{2}(3)\left(3\cdot\frac{7}{24}\right) = \frac{21}{16}$ which is answer $\boxed{\text{A}}$.

20. Consider a game where a player bets $\$X$ and then flips a biased coin where the probability of flipping heads is $0.4$. If the result is heads, she wins $\$X$; if it is tails, she loses $\$X$. Suppose she starts with \$25 and her first bet is \$5. Every time she wins, she will bet double what she won on the next flip. Whenever she loses, she will bet \$5 on the following flip. If she has \$100 or more at any point, she will quit. What is the probability (rounded to the nearest thousandth) that she will quit with \$100 or more in 7 flips or less?

A. 0.026 B. 0.035 C. 0.038 D. 0.052 E. 0.070

Solution: As we investigate the situation, it occurs to us that the stakes are higher and higher in a winning streak and so a loss takes one down drasticallly. Therefore, she can win this game with either all wins, or, if losses occur, they should be at the early flips.

Let us organize our cases of winning within 7 flips by the number of lost games.

All wins:

| | |
|---|---|
| before 1st flip | has \$25, bets \$5 |
| 1st flip | win $\Longrightarrow$ has \$30, bets \$10 |
| 2nd flip | win $\Longrightarrow$ has \$40, bets \$20 |
| 3rd flip | win $\Longrightarrow$ has \$60, bets \$40 |
| 4th flip | win $\Longrightarrow$ has \$100 DING! DING! $\Longrightarrow$ $P = 0.4^4$ |

All wins and one loss: if we lose only in the first flip:

| | |
|---|---|
| before 1st flip | has \$25, bets \$5 |
| 1st flip | lose $\Longrightarrow$ has \$20, bets \$5 |
| 2nd flip | win $\Longrightarrow$ has \$25, bets \$10 |
| 3rd flip | win $\Longrightarrow$ has \$35, bets \$20 |
| 4th flip | win $\Longrightarrow$ has \$55, bets \$40 |
| 5th flip | win $\Longrightarrow$ has \$95, bets \$80 |
| 6th flip | win $\Longrightarrow$ has \$175 DING! DING! $\Longrightarrow$ $P = 0.6 \cdot 0.4^5$ |

or if we lose first in the second flip:

| | |
|---|---|
| before 1st flip | has \$25, bets \$5 |
| 1st flip | win $\Longrightarrow$ has \$30, bets \$10 |
| 2nd flip | lose $\Longrightarrow$ has \$20, bets \$5 |
| 3rd flip | win $\Longrightarrow$ has \$25, bets \$10 |
| 4th flip | win $\Longrightarrow$ has \$35, bets \$20 |
| 5th flip | win $\Longrightarrow$ has \$55, bets \$40 |
| 6th flip | win $\Longrightarrow$ has \$95, bets \$80 |
| 7th flip | win $\Longrightarrow$ has \$175 DING! DING! $\Longrightarrow$ $P = 0.6 \cdot 0.4^6$ |

if we lose first in the third flip, we will not make to 100 in seven steps:

| | |
|---|---|
| before 1st flip | has \$25, bets \$5 |
| 1st flip | win $\Longrightarrow$ has \$30, bets \$10 |
| 2nd flip | win $\Longrightarrow$ has \$40, bets \$20 |
| 3rd flip | lose $\Longrightarrow$ has \$20, bets \$5 |
| 4th flip | win $\Longrightarrow$ has \$25, bets \$10 |
| 5th flip | win $\Longrightarrow$ has \$35, bets \$20 |
| 6th flip | win $\Longrightarrow$ has \$55, bets \$40 |
| 7th flip | win $\Longrightarrow$ has \$95 almost... |

We can imagine it is even worse if the first loss occurs in the fourth flip.

What if losing first and second is better than losing just once but in the third flip?

| | |
|---|---|
| before 1st flip | has \$25, bets \$5 |
| 1st flip | lose $\Longrightarrow$ has \$20, bets \$5 |
| 2nd flip | lose $\Longrightarrow$ has \$15, bets \$5 |
| 3rd flip | win $\Longrightarrow$ has \$20, bets \$10 |
| 4th flip | win $\Longrightarrow$ has \$30, bets \$20 |
| 5th flip | win $\Longrightarrow$ has \$50, bets \$40 |
| 6th flip | win $\Longrightarrow$ has \$90, bets \$80 |
| 7th flip | win $\Longrightarrow$ has \$170 DING! DING! $\Longrightarrow$ $P = 0.6^2 \cdot 0.4^5$ |

There are no other ways feasable to win within seven flips. So our probability is

$$P\,(\text{win in } \leq 7 \text{ steps}) = 0.4^4 + 0.6 \cdot 0.4^5 + 0.6 \cdot 0.4^6 + 0.6^2 \cdot 0.4^5 \approx 0.037888$$

So the answer is 0.038, which is $\boxed{\text{C}}$.